Set-Theoretic Methods in General Topology

K. P. Hart

Contents

Chapter 1

Preliminaries

1. Well-orderings

A *well-ordering* of a set is a linear order $\prec$ of it such that every nonempty subset has a $\prec$-minimum. The natural ordering of $\mathbb{N}$ is a well-ordering; it is the shortest of all well-orderings of $\mathbb{N}$. To get an example of a longer well-ordering define $n \prec m$ iff 1) n is even and m is odd or 2) n and m are both even or both odd and $n < m$: we put all odd numbers after the even numbers but keep all even and all odd numbers in their natural order. This example illustrates one important difference between the natural order on $\mathbb{N}$ and most other well-orderings on $\mathbb{N}$: there will be elements, other than the minimum, without immediate predecessor. Indeed, 1 is not the $\prec$-minimum of $\mathbb{N}$, because $1024 \prec 1$, but if $n \prec 1$ then also $n + 2 \prec 1$ and $n \prec n + 2$. An element like 1 in the above example will be called a *limit*; elements with a direct predecessor will be called *successors*. Observe, however, that every element (other than the maximum) of a well-ordered set does have a direct successor. We shall denote the direct successor of an element x by $x + 1$.

▶ **1**. Every compact subset of the Sorgenfrey line is well-ordered by the natural order of $\mathbb{R}$. *Hint*: Let X be such a compact set and $A \subseteq X$. Note that X is also compact as a subset of $\mathbb{R}$ and hence bounded. Let $a = \inf A$ and consider a finite subcover of the open cover $\bigl\{(-\infty, a]\bigr\} \cup \bigl\{(b, \infty) : b > a\bigr\}$ of X; deduce that $a \in A$.

An *initial segment* of a well-ordered set $(X, \prec)$ is a subset I with the property that $x \in I$ and $y \prec x$ imply $y \in I$. Note that either $I = X$ or $I = \{x : x \prec p\}$, where $p = \min X \setminus I$. It follows that initial segments are comparable with respect to $\subset$. We let $\mathcal{I}_X$ denote the family of all proper initial segments of X and $\mathcal{I}_X^+$ denotes $\mathcal{I}_X \cup \{X\}$, the family of all initial segments of X. We shall often use $\hat{p}$ as a convenient shorthand for $\{x : x \prec p\}$.

Induction and recursion

Well-orderings enable us to do proofs by induction and perform constructions by recursion.

1.1. Theorem (Principle of Induction). *Let $(X, \prec)$ be a well-ordered set and let A be a subset of X such that $\{y : y \prec x\} \subseteq A$ implies $x \in A$ for all $x \in X$. Then $A = X$.*

This is essentially a reformulation of the definition of well-ordering. The straightforward proof is by contraposition: if $A \neq X$ then $x = \min X \setminus A$ satisfies $\hat{x} \subseteq A$ yet $x \notin A$.

There is an alternative formulation that bears closer resemblance to the familiar principle of mathematical induction.

► **2**. Let $(X, \prec)$ be a well-ordered set and let A be a subset of X that satisfies 1) $\min X \in A$, 2) if $x \in A$ then $x + 1 \in A$, and 3) if x is a limit element and $\hat{x} \subseteq A$ then $x \in A$. Then $A = X$.

The main use of this principle is in showing that all elements of a well-ordered set have a certain property. By way of example we consider isomorphisms of well-ordered sets.

► **3**. Isomorphisms between well-ordered sets are unique: if f and g are order-preserving bijections between $(X, \prec)$ and $(Y, \sqsubset)$ then $f = g$. *Hint*: Let $I = \bigl\{x : f(x) = g(x)\bigr\}$ and apply the principle of induction.

► **4**. The well-ordered sets $(X, \prec)$ and $(\mathcal{I}_X, \subset)$ are isomorphic.

1.2. THEOREM (Principle of Recursion). *Let $(X, \prec)$ be a well-ordered set, Y any set and $\mathcal{F}$ the family of all maps f whose domain is an initial segment of X and whose range is in Y. For every map $F : \mathcal{F} \to Y$ there is a unique map $f : X \to Y$ such that $f(x) = F\bigl(f \restriction \hat{x}\bigr)$ for every x.*

The proof of this principle offers a good exercise in working with well-orders.

► **5**. Let $\mathcal{G}$ be the subfamily of $\mathcal{F}$ consisting of all approximations of f: these are functions g that satisfy $g(x) = F\bigl(g \restriction \{y : y \prec x\}\bigr)$ for all x in their domains.

a. If $g, h \in \mathcal{G}$ and $\operatorname{dom} g \subseteq \operatorname{dom} h$ then $g = h \restriction \operatorname{dom} g$. *Hint*: Use the principle of induction.

b. The union $f = \bigcup \mathcal{G}$ is a function that belongs to $\mathcal{G}$.

c. The domain of f is equal to X.

d. If f and g are two functions that satisfy the conclusion of the principle of recursion then $f = g$.

The Principle of Recursion formalises the idea that a function can be constructed by specifying its initial segments. By way of example we show how it can be used to show that any two well-ordered sets are comparable.

We compare well-ordered sets by 'being an initial segment of', more precisely we say that $(X, \prec)$ is shorter than $(Y, \sqsubset)$ if there is $y \in Y$ such that $(X, \prec)$ is isomorphic to the initial segment $\{z : z \sqsubset y\}$ of Y.

► **6**. The well-ordered set $(X, \prec)$ is shorter than $(\mathcal{I}_X^+, \subset)$.

► **7**. Let $(X, \prec)$ and $(Y, \sqsubset)$ be well-ordered sets. Let $\mathcal{F}$ be the set of maps whose domain is an initial segment of X and whose range is contained in Y. Define

$F : \mathcal{F} \to Y \cup \{Y\}$ by $F(f) = \min(Y \setminus \operatorname{ran} f)$ if the right-hand side is nonempty and $F(f) = Y$ otherwise. Apply the principle of recursion to F to obtain a map $f : X \to Y \cup \{Y\}$.

a. If $x \prec y$ in X and $f(y) \neq Y$ then $f(x) \sqsubset f(y)$.
b. If there is an x in X such that $f(x) = Y$ then Y is shorter than X.
c. If $Y \notin \operatorname{ran} f$ then X and Y are isomorphic if $\operatorname{ran} f = Y$, and X is shorter than Y if $\operatorname{ran} f \subset Y$.

► **8**. Let $(X, \prec)$ and $(Y, \sqsubset)$ be well-ordered sets. Then either Y is shorter than or isomorphic to X or $\mathcal{I}_X^+$ is shorter than or isomorphic to Y.

In practice a construction by recursion proceeds less formally. One could, for example, describe the construction of the map f from the Exercise 1.7 as follows: "define $f(\min X) = \min Y$ and, assuming $f(y)$ has been found for all $y \prec x$, put $f(x) = \min Y \setminus \{f(y) : y \prec x\}$ if this set is nonempty, and $f(x) = Y$ otherwise". One would then go on to check that f had the required properties.

► **9**. The natural well-order of $\mathbb{N}$ is shortest among all well-orders of $\mathbb{N}$.
Hint: Apply the procedure from Exercise 1.7 to $(\mathbb{N}, \in)$ and $(\mathbb{N}, \prec)$, where $\prec$ is any other well-ordering of $\mathbb{N}$.

► **10**. Construct well-orders of $\mathbb{N}$ with the following properties (*Hint*: try to find compact subsets of $\mathbb{S}$):
a. with two limit elements;
b. with infinitely many limit elements;
c. where the set of limit elements is isomorphic to the set from a.

2. Ordinals

Let $(X, \prec)$ be a well-ordered set; we have already encountered a copy of X that is more set-like than X itself: $(X, \prec)$ is isomorphic to its family $\mathcal{I}_X$ of proper initial segments, which is well-ordered by $\subset$. We say that $(X, \prec)$ is an *ordinal* if $(X, \prec) = (\mathcal{I}_X, \subset)$; by this we mean that $X = \mathcal{I}_X$ and that $x \prec y$ iff $\hat{x} \subset \hat{y}$.

► **1**. The set $\mathbb{N}$, as described in Appendix B, is an ordinal.

► **2**. Let $(X, \prec)$ be an ordinal (with at least ten elements).
a. The minimum element of X is $\varnothing$.
b. The next element of X is $\{\varnothing\}$.
c. The one after that is $\{\varnothing, \{\varnothing\}\}$.
d. Write down the next few elements of X.

► **3**. Let $(X, \prec)$ be an ordinal.
a. For every $x \in X$ we have $x = \hat{x}$.
b. If $x \in X$ then its direct successor (if it exists) is $x \cup \{x\}$.

c. Every element of X is an ordinal.
d. For $x, y \in X$ the following are equivalent: $x \prec y$, $x \subset y$ and $x \in y$.
e. The set X is transitive, i.e., if $y \in X$ and $x \in y$ then $x \in X$.
f. The set X is well-ordered by $\in$.

The conjunction of the last parts of Exercise 2.3 actually characterises ordinals.

► **4**. Let X be a transitive set that is well-ordered by $\in$; then $(X, \in)$ is an ordinal.

This characterisation of ordinals can be simplified with the aid of the Axiom of Foundation (see Appendix A).

► **5**. A set is an ordinal iff it is transitive and linearly ordered by $\in$.

This last characterisation is now taken to be the definition of ordinals. This elegant way of singling out prototypical well-ordered sets is due to Von Neumann and has the advantage of using nothing but sets and the $\in$-relation. The Cantorian definition of an ordinal was 'order type of well-ordered set', which essentially meant that every ordinal was a proper class of sets.

We shall use (lower case) Greek letters to denote ordinals and we reserve the letter ω to denote the ordinal $\mathbb{N}$. The class of ordinals is well-ordered by $\in$. We extend the meaning of the word 'sequence' to include maps whose domein is some ordinal and we use similar notation. Thus $\langle x_\alpha : \alpha < \beta \rangle$ abbreviates the map f whose domain is β and whose value at α is x_α. We will call this a sequence of length β, or a β-sequence.

► **6**. Let α and β be ordinals.
a. $\alpha \cap \beta$ is an ordinal, call it γ;
b. $\gamma = \alpha$ or $\gamma = \beta$;
c. $\alpha \in \beta$ or $\alpha = \beta$ or $\beta \in \alpha$.

► **7**. a. If α is an ordinal then so is $\alpha \cup \{\alpha\}$.
b. If x is a set of ordinals then $\bigcup x$ is an ordinal.

► **8**. Let x be a set of ordinals.
a. $\bigcup x = \sup x$;
b. $\bigcap x = \min x$.

Every well-ordered set is isomorphic to exactly one ordinal, which we refer to as its *type*. We write $\operatorname{tp} X = \alpha$ to express that α is the type of X.

► **9**. Isomorphic ordinals are identical. *Hint*: Let $f : X \to Y$ be an isomorphism between ordinals and $A = \{x \in X : x = f(x)\}$, show that $A = X$ (see Exercise 2.2 for inspiration).

Thus every well-ordered set is isomorphic to at most one ordinal, to show that every well-ordered set is isomorphic to some ordinal we shall need the Axiom of Replacement.

▶ **10**. Every well-ordered set is isomorphic to an ordinal. Let $(X, \prec)$ be a well-ordered set and $A = \{x \in X : \hat{x} \text{ is isomorphic to an ordinal}\}$.

a. $\min X \in A$ because $\widehat{\min X} = \varnothing$.

b. If $x \in A$ then $x + 1 \in A$: if $\operatorname{tp} \hat{x} = \alpha$ then $\operatorname{tp} \widehat{x+1} = \alpha \cup \{\alpha\}$.

c. If x is a limit and $\hat{x} \subseteq A$ then $x \in A$ because $\operatorname{tp} \hat{x} = \{\operatorname{tp} \hat{y} : y \prec x\}$.

d. $\operatorname{tp} X = \{\operatorname{tp} \hat{x} : x \in X\}$.

The Axiom of Replacement was used twice: in the limit case and when assigning a type to X itself. In both cases we had the assignment $y \mapsto \operatorname{tp} \hat{y}$ and a set S ($\hat{x}$ and X respectively); the Axiom of Replacement guarantees that the $\operatorname{tp} \hat{y}$ with $y \in S$ can be collected in a set. It still remains to prove of course that the set $\{\operatorname{tp} \hat{y} : y \in S\}$ is an ordinal that is isomorphic to S.

The countable ordinals

The well-orderings of the subsets of $\mathbb{N}$ all belong to $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ and hence form a set, which we denote by WO. By the Axiom of Replacement their types form a set as well. Thus the countable ordinals can be seen to form a set, we denote it by ω_1.

▶ **11**. The set ω_1 is an ordinal and hence uncountable. *Hint*: Apply Exercise 2.5 and the Axiom of Foundation.

▶ **12**. Let $\{\prec_k : k \in \mathbb{N}\}$ be a family of well-orderings of $\mathbb{N}$. Define $\langle k, l\rangle \sqsubset \langle m, n\rangle$ iff $k < m$ or $k = m$ and $l \prec_k n$.

a. The relation $\sqsubset$ is a well-ordering of $\mathbb{N}^2$.

b. For every k the well-ordering $\prec_k$ is shorter than $\sqsubset$. *Hint*: The procedure in Exercise 1.7 applied to $(\mathbb{N}, \prec_k)$ and $(\mathbb{N}^2, \sqsubset)$ yields a map of $\mathbb{N}$ into $k \times \mathbb{N}$.

c. If A is a countable subset of ω_1 then there is $\beta \in \omega_1$ such that $\alpha < \beta$ for all $\alpha \in A$. *Hint*: Use the Axiom of Choice.

▶ **13**. If $\alpha \in \omega_1$ is a limit then there is a strictly increasing sequence $\langle \alpha_n\rangle_n$ such that $\alpha = \sup_n \alpha_n$. *Hint*: $C = \{\beta : \beta < \alpha\}$ is countable; let $f : \mathbb{N} \to C$ be a bijection and recursively find $k_n \in \mathbb{N}$ such that $f(k_{n+1}) > f(n), f(k_n)$; put $\alpha_n = f(k_n)$.

We have two uncountable objects associated with $\mathbb{N}$: its power set $\mathcal{P}(\mathbb{N})$ and the set of countable ordinals ω_1. There is a natural map from $\mathcal{P}(\mathbb{N})$ (or rather $\mathcal{P}(\mathbb{N} \times \mathbb{N})$) onto ω_1: map $A \subseteq \mathbb{N}^2$ to the type of $(\mathbb{N}, A)$ if A is a well-ordering and to 0 otherwise. On the other hand, a choice of one well-order (of the right type) for each ordinal in ω_1 produces an injection of ω_1 into $\mathcal{P}(\mathbb{N})$. This was a blatant application of the Axiom of Choice and there is no easy description of such an injection from ω_1 into $\mathcal{P}(\mathbb{N})$ as it can be used to construct a nonmeasurable subset of $\mathbb{R}$. This should give some indication of the essential difference between the entities $\mathcal{P}(\mathbb{N})$ and ω_1.

On an elementary level we have the following nonexistence results.

▶ **14**.a. There is no map f from ω_1 into $\mathcal{P}(\mathbb{N})$ such that $f(\alpha)$ is a proper subset of $f(\beta)$ whenever $\alpha < \beta$. *Hint*: If f were such a map consider the set of α for which $f(\alpha+1) \setminus f(\alpha)$ is nonempty.

b. There is no map $f : \omega_1 \to \mathbb{R}$ such that $\alpha < \beta$ implies $f(\alpha) < f(\beta)$.

However, we can get a sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{P}(\mathbb{N})$ that is strictly increasing with respect to almost containment. When working with subsets of $\mathbb{N}$ 'almost' means 'with finitely many exceptions'. We attach an asterisk to a relation to indicate that it holds almost. Thus, $a \subseteq^* b$ means that $a \subseteq b$ with possibly finitely many points of a not belonging to b, in other words that $a \setminus b$ is finite. Similarly, $a \subset^* b$ means $a \subseteq^* b$ but $b \setminus a$ is infinite, and $a \cap b =^* \varnothing$ is expressed by saying that that a and b are almost disjoint.

▶ **15**. Let $\langle a_n \rangle_n$ be a sequence in $\mathcal{P}(\mathbb{N})$ such that $a_n \subset^* a_{n+1}$ for all n. There is a set $a \in \mathcal{P}(\mathbb{N})$ such that $a_n \subset^* a$ for all n and $a \subset^* \mathbb{N}$. *Hint*: Note that $a_n \setminus \bigcup_{m<n} a_m$ is always infinite; pick k_n in this difference and let $a = \mathbb{N} \setminus \{k_n : n \in \mathbb{N}\}$.

▶ **16**. There is a sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ in $\mathcal{P}(\mathbb{N})$ such that $x_\alpha \subset^* x_\beta$ whenever $\alpha < \beta$. *Hint*: Construct the x_α by recursion, applying Exercises 2.13 and 2.15 in the limit case.

Every ordinal (indeed every linearly ordered set) carries a natural topology, the *order topology*, which has the sets of the form $(\leftarrow, x)$ and $(x, \to)$ as a subbase. Unless explicitly specified otherwise we always assume, in topological situations, that ordinals carry their order topologies. The space ω_1 features in many counterexamples.

▶ **17**. The space ω_1 has the following properties.

a. It is first-countable: $\bigl\{(\beta, \alpha] : \beta < \alpha\bigr\}$ is a countable local base at α.

b. It is sequentially compact, i.e., every sequence has a converging subsequence. *Hint*: Given a sequence $\langle \alpha_n \rangle_n$ consider the minimal α for which $\{n : \alpha_n \leqslant \alpha\}$ is infinite.

c. The space ω_1 is not compact.

▶ **18**. If $f : \omega_1 \to \mathbb{R}$ is continuous then there is an α in ω_1 such that f is constant on the final segment $[\alpha, \omega_1)$.

a. For every α the set $F_\alpha = f\bigl[[\alpha, \omega_1)\bigr]$ is compact.

b. The set of α for which $F_{\alpha+1} \subset F_\alpha$ is countable. *Hint*: Choose an interval I_α with rational endpoints that meets $F_\alpha \setminus F_{\alpha+1}$; observe that $\alpha \mapsto I_\alpha$ is one-to-one.

c. Fix α such that $F_\beta = F_{\beta+1}$ for all $\beta \geqslant \alpha$. Then $F_\beta = F_\alpha$ for all $\beta \geqslant \alpha$.

d. There is only one point in F_α. *Hint*: If $x, y \in F_\alpha$ then there is a sequence $\langle \alpha_n \rangle_n$ such that $\alpha \leqslant \alpha_0 < \alpha_1 < \alpha_2 < \cdots$ and $f(\alpha_i) = x$ if i is even and $f(\alpha_i) = y$ if i is odd. Let $\beta = \sup_i \alpha_i$ and note that $x = f(\beta) = y$.

Cardinal numbers

We single out one important class of ordinals: the cardinal numbers or cardinals for short. Cardinal numbers will be used to measure the sizes of sets, rather than their order types. Accordingly, a *cardinal number* is an ordinal that is an *initial ordinal*, a 'smallest among equals': α is an initial ordinal if whenever β is an ordinal and $f : \alpha \to \beta$ a bijection one has $\beta \geqslant \alpha$. The contrapositive of this formulation reads: if $\beta < \alpha$ then there is no bijection between α and β. We generally reserve the letters κ, λ and μ for cardinal numbers.

▶ **19**.a. Every natural number is a cardinal number. *Hint*: Consider $I = \{n :$ there is no injection from $n+1$ into $n\}$.
b. The ordinal ω is a cardinal number. *Hint*: A bijection between ω and n induces an injection from $n+1$ into n.
c. The ordinal ω_1 is a cardinal number.
d. Every infinite cardinal is a limit ordinal.

The construction of ω_1 can be generalised to show that there is no largest cardinal number.

▶ **20**. Let X be a set and let $\mathrm{WO}(X)$ be the set of all well-orders of subsets of X.
a. The set of types of the elements of $\mathrm{WO}(X)$ is an ordinal, which we denote by X^+.
b. The ordinal X^+ is a cardinal. *Hint*: Every $\alpha < X^+$ admits an injection into X, but X^+ itself does not.

An important property of infinite cardinal numbers is that they are equal to their own squares. To see define a relation $\prec$ between pairs of ordinals as follows: $\langle\alpha,\beta\rangle \prec \langle\gamma,\delta\rangle$ iff 1) $\max\{\alpha,\beta\} < \max\{\gamma,\delta\}$, or 2) $\beta < \delta$, or 3) $\beta = \delta$ and $\alpha < \gamma$.

▶ **21**.a. If ξ is an ordinal then $\prec$ is well-ordering of $\xi \times \xi$.
b. If κ is an infinite cardinal and $\alpha,\beta \in \kappa$ then the order type of $\bigl\{\langle\gamma,\delta\rangle : \langle\gamma,\delta\rangle \prec \langle\alpha,\beta\rangle\bigr\}$ is smaller than κ. *Hint*: Determine the type of $\bigl\{\langle\gamma,\delta\rangle : \max\{\gamma,\delta\} = \alpha\bigr\}$ and apply induction with respect to κ.
c. If κ is an infinite cardinal then the order type of $\kappa \times \kappa$, with respect to $\prec$, is κ.

▶ **22**. If κ is a cardinal then κ^+ is the smallest cardinal that is larger than κ. In addition, if κ is infinite and $f : \kappa \to \kappa^+$ is any map then there is an $\alpha < \kappa^+$ such that $f(\beta) < \alpha$ for all $\beta < \kappa$.

The cardinals are, as a subclass of the ordinals, well-ordered. The finite cardinals correspond to the natural numbers; we index the infinite cardinals by the ordinals. Thus $\omega_0 = \omega$, $\omega_1 = \omega_0^+$, $\omega_2 = \omega_1^+$ and generally $\omega_{\alpha+1} = \omega_\alpha^+$ for every α. If α is a limit then $\sup\{\omega_\beta : \beta < \alpha\}$ is a cardinal, denoted ω_α.

It is customary to distinguish between the two identities of the cardinal numbers ω_α: when we think of it as an ordinal we keep writing ω_α, we write $\aleph_\alpha$ when treating it as a cardinal. Every cardinal of the form $\aleph_{\alpha+1}$ is called a *successor cardinal*; if α is a limit ordinal then we call $\aleph_\alpha$ a *limit cardinal*. Finally then, the *cardinality* of a set X is the unique cardinal number κ such that there is a bijection between X and κ.

Cofinality

We have seen that for every countable limit ordinal α there is a strictly increasing sequence $\langle\alpha_n\rangle_n$ such that $\alpha = \sup_n \alpha_n$. The set $A = \{\alpha_n : n \in \omega\}$ is *cofinal* in α in that for every $\beta \in \alpha$ there is a $\gamma \in A$ such that $\beta < \gamma$. Since every ordinal has a cofinal subset, to wit itself, we can define the *cofinality* of α, denoted $\operatorname{cf}\alpha$, to be the minimal type of a cofinal subset of α.

▶ **23**.a. $\operatorname{cf}\operatorname{cf}\alpha = \operatorname{cf}\alpha$;
b. $\operatorname{cf}\alpha$ is a cardinal number.

There are two types of cardinal numbers, those that equal their cofinalities, like $\aleph_0$ and $\aleph_1$, and those that do not, e.g., $\aleph_\omega$ has cofinality ω_0. We call κ *regular* if the former applies, i.e., if $\kappa = \operatorname{cf}\kappa$, and *singular* if $\operatorname{cf}\kappa < \kappa$.

▶ **24**. Every infinite successor cardinal is regular.

As we shall see regular cardinals are very often easier to handle than singular ones. As will become apparent, many recursive constructions use up small portions of a set per step; if the cardinality, κ, of the set is regular and each step uses up fewer than κ elements then the set will not be exhausted until the end of the construction.

3. Some combinatorics

The Pressing-Down Lemma

Let κ be a cardinal; a function $f : \kappa \to \kappa$ is said to be *regressive* if $f(\alpha) < \alpha$ for all $\alpha > 0$. For example the function $n \mapsto \max\{0, n-1\}$, is regressive on ω; note that this function is one-to-one on $\omega \setminus \{0\}$. On regular uncountable cardinals this is not possible.

3.1. THEOREM. *Let κ be regular and uncountable and $f : \kappa \to \kappa$ a regressive function. Then f is constant on an unbounded subset of κ.*

▶ **1**. Assume that for all α the preimage $f^{\leftarrow}(\alpha)$ is bounded, say by β_α.
a. If $\gamma < \kappa$ then $\sup\{\beta_\alpha : \alpha \leqslant \gamma\} < \kappa$.
Define $\gamma_0 = 0$ and, recursively, $\gamma_{n+1} = \sup\{\beta_\alpha : \alpha \leqslant \gamma_n\}$.
b. $\gamma_0 < \gamma_1 < \cdots$. *Hint*: $\gamma_{n+1} \geqslant \beta_{\gamma_n}$.
c. $\gamma = \sup_n \gamma_n < \kappa$.
d. $f(\gamma) \geqslant \gamma$, so f is not regressive.

This can be used, for example, to redo Exercise 2.18.

▶ **2**. If κ is regular and uncountable and $f : \kappa \to \mathbb{R}$ is continuous then f is constant on $[\alpha, \kappa)$ for some α.
a. For every n there is an α_n such that $\left|f(\beta) - f(\alpha_n)\right| < 2^{-n}$ whenever $\beta \geqslant \alpha_n$.
Hint: Define a regressive function f_n such that $\left|f(\beta) - f(\alpha)\right| < 2^{-n}$ whenever $f_n(\alpha) < \beta \leqslant \alpha$ and apply the pressing-down lemma.

b. The ordinal $\alpha = \sup_n \alpha_n$ is as required.

► **3**. Let α be an ordinal of countable cofinality; define a regressive $f : \alpha \to \alpha$ with all preimages $f^{\leftarrow}(\beta)$ bounded.

The Δ-system lemma

A family $\mathcal{D}$ of sets is a Δ-*system* if there is a single set R, the *root*, such that $D_1 \cap D_2 = R$ whenever $D_1, D_2 \in \mathcal{D}$ are distinct. So certainly a pairwise disjoint family is a Δ-system. The Δ-system lemma says that a large enough family of small enough sets can be thinned out to a large Δ-system.

► **4**. Let κ be a regular uncountable cardinal and $\{F_\alpha : \alpha < \kappa\}$ a family of finite sets. There is an unbounded subset A of κ such that $\{F_\alpha : \alpha \in A\}$ is a Δ-system.

a. Without loss of generality $F_\alpha \subseteq \kappa$ for all α.

b. The function $f : \alpha \mapsto \max\bigl(\{0\} \cup (F_\alpha \cap \alpha)\bigr)$ is regressive.

c. There are $\beta < \kappa$ and an unbounded set C in κ such that $F_\alpha \cap \alpha \subseteq \beta$ whenever $\alpha \in C$.

d. There are a finite subset R of β and an unbounded subset B of C such that $F_\alpha \cap \alpha = R$ whenever $\alpha \in B$.

e. There is an unbounded subset A of B such that $\max F_\alpha < \gamma$ whenever $\alpha < \gamma$ in A.

f. The set A is as required.

► **5**. a. The family $\{n : n \in \omega\}$ is a countable family of finite sets without a three-element Δ-sytem in it.

b. Let κ be singular of cofinality λ and let $\langle \alpha_\eta : \eta < \lambda \rangle$ be increasing and cofinal in κ. Let $\mathcal{F} = \bigl\{\{\alpha_\eta, \beta\} : \alpha_\eta < \beta < \alpha_{\eta+1}\bigr\}$. Every Δ-system in $\mathcal{F}$ is of cardinality less than κ.

Closed unbounded sets

We consider the order-topology on ordinals and in particular on regular cardinal numbers. Let κ be regular and uncountable. A *closed and unbounded set* in κ is one that is closed in the order topology and cofinal in κ. We write $\mathcal{C} = \{C : C \text{ is closed and unbounded in } \kappa\}$.

For the moment we concentrate on $\kappa = \omega_1$.

► **6**. $\mathcal{C}$ is closed under countable intersections.

a. If $C_0, C_1 \in \mathcal{C}$, then $C_0 \cap C_1 \in \mathcal{C}$. *Hint*: To show unboundedness let α be arbitrary and choose $\langle \alpha_n : n \in \omega \rangle$ strictly increasing with $\alpha_0 > \alpha$ and $\alpha_{2n+i} \in C_i$; consider $\sup_n \alpha_n$.

b. Let $\{C_n : n \in \omega\} \subseteq \mathcal{C}$, then $\bigcap_n C_n \in \mathcal{C}$. *Hint*: As above but now choose α_n in $\bigcap_{i \leqslant n} C_i$.

We can improve this exercise by using another kind of intersection. Let $\{A_\alpha : \alpha \in \omega_1\}$ be a family of subsets of ω_1. The *diagonal intersection* of this family is defined as

$$\mathop{\triangle}_{\alpha} A_\alpha = \bigl\{\delta : (\forall \gamma < \delta)(\delta \in A_\gamma)\bigr\}.$$

For families of closed unbounded sets this intersection is never empty.

▶ **7**. Let $\{C_\alpha : \alpha \in \omega_1\} \subseteq \mathcal{C}$, then $C = \triangle_\alpha C_\alpha \in \mathcal{C}$.
a. C is unbounded. *Hint*: Given α choose α_n recursively above α such that $\alpha_{n+1} \in \bigcap_{\beta \leqslant \alpha_n} C_\beta$ and consider $\sup_n \alpha_n$.
b. C is closed. *Hint*: If $\langle \alpha_n : n \in \omega \rangle$ is strictly increasing in C then $\alpha = \sup_n \alpha_n$ belongs to $\bigcap_{\beta < \alpha_n} C_\beta$ for all n.

▶ **8**. Generalize the exercises above to larger κ.

Stationary sets

A subset of a regular cardinal κ is said to be *stationary* if it meets every closed and unbounded subset of κ. Stationary sets play a large role in many set-theoretic and topological arguments, a we shall see later. As an example we show how the Pressing-Down Lemma can be strengthened.

▶ **9**. Let S be a stationary subset of a regular cardinal κ and $f : S \to \kappa$ a regressive function. Then f is constant on a stationary set. *Hint*: Assume that for every α there is a $C_\alpha \in \mathcal{C}$ that is disjoint from $f^{\leftarrow}(\alpha)$; consider a point δ in $\triangle_\alpha C_\alpha$.

Stationary subsets share a topological property with regular uncountable cardinals.

▶ **10**. Let S be an unbounded subset of some regular uncountable cardinal κ. Then S is stationary iff every continuous function $f : S \to \mathbb{R}$ is constant on a tail.
a. For $C \in \mathcal{C}$ (with unbounded complement) define a continuous function $f : \kappa \setminus C \to \mathbb{R}$ that is not constant on any tail.
b. If S is stationary and $f : S \to \mathbb{R}$ is continuous then there is an α such that f is constant on $S \setminus \alpha$. *Hint*: Apply the strong form of the Pressing-Down Lemma.

We can find large disjoint families of stationary sets on ω_1.

▶ **11**. For each $\alpha < \omega_1$ let $f_\alpha : \omega \to \alpha$ be a surjection. Define $A_{\beta,n} = \{\alpha : f_\alpha(n) = \beta\}$.
a. For all n: if $\beta \neq \gamma$ then $A_{\beta,n} \cap A_{\gamma,n} = \varnothing$.
b. For all β we have $(\beta, \omega_1) \subseteq \bigcup_n A_{\beta,n}$.
c. For every β there is an n_β such that A_{β,n_β} is stationary. *Hint*: If not we get $\{C_n : n \in \omega\} \subseteq \mathcal{C}$ with $\bigcap_n C_n \subseteq \beta + 1$.
d. There is $n \in \omega$ for which $S = \{\beta : n = n_\beta\}$ is stationary.
e. $\{A_{\beta,n} : \beta \in S\}$ is a disjoint family of stationary subsets.

4. Trees

A *tree* is a partially ordered set in which every set of predecessors is well-ordered. More formally, consider a partially ordered set $(P, \preccurlyeq)$ and, for $x \in P$ put $\hat{x} = \{y \in P : y \prec x\}$. We say that $(P, \preccurlyeq)$ is a tree if every set $\hat{x}$ is well-ordered.

▶ **1**. Let $\mathcal{Z}(\mathbb{S})$ denote the set of compact subsets of the Sorgenfrey line. Order $\mathcal{Z}(\mathbb{S})$ by 'being an initial segment of', i.e., $C \preccurlyeq D$ iff $C = D \cap (-\infty, \max C]$.
 a. The relation $\preccurlyeq$ is a partial order.
 b. $\hat{D} = \{C : C \prec D\}$ is well-ordered and isomorphic to $D \setminus \{\max D\}$.

▶ **2**. Let α be an ordinal and X a set. Let ${}^{<\alpha}X$ denote the set of functions with domain some β less than α and range contained in X; in short ${}^{<\alpha}X = \bigcup_{\beta<\alpha} {}^{\beta}X$. Then ${}^{<\alpha}X$ is a tree when ordered by inclusion. For every $s \in {}^{<\alpha}X$ the order type of $\hat{s}$ is its domain.

A special example is ${}^{<\omega}2$, the tree of finite sequences of zeros and ones, ordered by extension.

A tree is divided into levels: if $(T, \preccurlyeq)$ is a tree and α is an ordinal then T_α denotes the set of $t \in T$ for which $\hat{t}$ has type α. We write $\operatorname{ht} t = \operatorname{tp} \hat{t}$; thus $T_\alpha = \{t : \operatorname{ht} t = \alpha\}$. The minimal ordinal α for which $T_\alpha = \varnothing$ is called the *height* of T. The height of T exists because of the Axiom of Replacement.

▶ **3**. The height of a tree T is equal to $\sup\{\operatorname{ht} t + 1 : t \in T\}$.

▶ **4**. The height of the tree $\mathcal{Z}(\mathbb{S})$ is ω_1.

A *branch* of a tree (or a *path*) is a maximal linearly ordered subset.

▶ **5**. A subset B of a tree is a branch iff it is linearly ordered, contains $\hat{t}$ whenever $t \in B$, and there is no t such that $B \subseteq \hat{t}$.

▶ **6**. The tree $\mathcal{Z}(\mathbb{S})$ has no branches of type ω_1.

König's Lemma

A very useful result about infinite trees is the following.

4.1. Theorem (König's Lemma). *Let T be an infinite tree in which for every $n \in \omega$ the level T_n is finite. Then there is a sequence $\langle t_n \rangle_n$ in T such that $t_n \in T_n$ and $t_n < t_{n+1}$ for all n.*

▶ **7**. Prove König's Lemma. *Hint*: Construct $\langle t_n \rangle_n$ by recursion: choose $t_0 \in T_0$ with $\{s : t_0 < s\}$ infinite, then $t_1 \in T_1$ with $t_1 > t_0$ and $\{s : t_1 < s\}$ infinite, ...

König's Lemma has many applications.

▶ **8**. The topological product ${}^{\omega}2$ is compact. Let $\mathcal{U}$ be a family of open sets, no finite subfamily of which covers ${}^{\omega}2$ and let T be the set of s for which $[s]$ is not covered by a finite subfamily of $\mathcal{U}$, where for $s \in {}^{<\omega}2$ we put $[s] = \{x \in {}^{\omega}2 : s \subset x\}$.

a. The family $\{[s] : s \in {}^{<\omega}2\}$ is a base for the topology of ${}^{\omega}2$.
b. T is an infinite subtree of ${}^{<\omega}2$. *Hint*: If $s \in T$ then $s^\frown 0 \in T$ or $s^\frown 1 \in T$.
c. König's Lemma implies that $\mathcal{U}$ does not cover ${}^{\omega}2$.

This exercise has a converse.

▶ **9**. König's Lemma can be deduced from the compactness of ${}^{\omega}2$. Let T be an infinite tree with finite levels and consider the topological product $X = \prod_{n\in\omega} T_n$, where each T_n is given the discrete topology. For $n \in \omega$ put $F_n = \{x \in X : (\forall i < n)\bigl(x(i) < x(i+1)\bigr)\}$.
a. X is compact. *Hint*: X can be embedded into ${}^{\omega}2$ as a closed subset.
b. If $x \in \bigcap_{n\in\omega} F_n$ then $\langle x(n) : n \in \omega\rangle$ satisfies the conclusion of König's Lemma.
c. For each n the set F_n is clopen and nonempty; in addition $F_{n+1} \subseteq F_n$.
d. $\bigcap_{n\in\omega} F_n \neq \varnothing$.

As a further example we prove the simplest version of Ramsey's theorem. For this we establish some notation: $[\omega]^2$ denotes the family of 2-element subsets of ω. A map $c : [\omega]^2 \to 2$ is said to be a colouring of $[\omega]^2$ and a subset A of ω is said to be *c-homogeneous* or just *homogeneous* if c is constant on $[A]^2$. For ease of notation we identify $[\omega]^2$ with $\{\langle i, j\rangle : i < j < \omega\}$.

4.2. THEOREM (Ramsey's theorem). *For every colouring of $[\omega]^2$ there is an infinite homogeneous set.*

▶ **10**. Prove Ramsey's theorem. Given a colouring $c : [\omega]^2 \to 2$ define a subtree $T = \{t_n : n \in \omega\}$ of ${}^{<\omega}2$ as follows: $t_0 = \varnothing$; if $n > 0$ and the t_i for $i \in n$ have been found define $t_n \restriction m$ by recursion: if $t_n \restriction m = t_i$ for some $i \in n$ then put $t_n(m) = c(i, n)$, if $t_n \restriction m \neq t_i$ for all i then stop: $t_n = t_n \restriction m$.
a. The map $n \mapsto t_n$ is one-to-one.
b. If $s < t_n$ then $s = t_i$ for some $i \in n$.
c. If $t_i < t_m < t_n$ then $c(i, m) = c(i, n)$.
d. If $\{t_n : n \in A\}$ is a branch through T then A is *prehomogeneous*, i.e., if $i \in j \in k$ in A then $c(i, j) = c(i, k)$.
e. An infinite prehomogeneous set contains an infinite homogeneous set.

▶ **11**. Every sequence in $\mathbb{R}$ (or any linearly ordered set) has a monotone subsequence. *Hint*: Given such a sequence $\langle x_n\rangle_n$ define $f : [\omega]^2 \to 2$ by $f(i, j) = 1$ if $x_i < x_j$ and $f(i, j) = 0$ if $x_i \geqslant x_j$ (where $i \in j$ is tacitly assumed).

Aronszajn trees

The proof of König's Lemma was a fairly easy recursion and it may seem that a straightforward adaptation will show that every tree of height ω_1 with countable levels has a branch of type ω_1.

▶ **12**. Investigate where such an adaptation of the proof of König's Lemma is liable to break down.

An *Aronszajn tree* is a tree of height ω_1 with all levels countable, but without a branch of type ω_1.

▶ **13**. There is an Aronszajn tree T contained in $\{t \in \mathcal{Z}(\mathbb{S}) : t \subseteq \mathbb{Q}\}$. Construct T by recursion, one level at a time and maintaining the following property $\dagger_\alpha$: if $\gamma < \beta \leqslant \alpha$, $s \in T_\gamma$ and $q > \max s$, where $q \in \mathbb{Q}$, then there is $t \in T_\beta$ such that $s \preccurlyeq t$ and $\max t = q$. Set $T_0 = \{\varnothing\}$.

a. Given T_α put $T_{\alpha+1} = \{t \cup \{q\} : t \in T_\alpha, q \in \mathbb{Q}, q > \max t\}$. If T_α is countable then so is $T_{\alpha+1}$. If $\dagger_\alpha$ holds then so does $\dagger_{\alpha+1}$.

b. If α is a limit and T_β has been found for $\beta < \alpha$ such that $\dagger_\beta$ holds for all $\beta < \alpha$ choose an increasing sequence $\langle \alpha_n \rangle_n$ with $\alpha = \sup_n \alpha_n$. For every pair $\langle t, q \rangle$, where $t \in \bigcup_{\beta<\alpha} T_\beta$ and $q \in \mathbb{Q}$ with $q > \max t$, let n_0 be minimal so that $t \in \bigcup_{\beta<\alpha_{n_0}} T_\beta$ and choose a sequence $\langle t_n \rangle_{n \geqslant n_0}$ with $t_n \in T_{\alpha_n}$, $t \preccurlyeq t_n \preccurlyeq t_{n+1}$ and $q - 2^{-n} \leqslant \max t_n < q$ for all n. Put $s_{t,q} = \bigcup_n t_n \cup \{q\}$ and let T_α be the set of all the $s_{t,q}$ thus obtained. Then T_α is countable and $\dagger_\alpha$ holds.

c. If $n \in \omega$ and $t \in T_n$ then $\operatorname{tp} t = n$.

d. If $\alpha \geqslant \omega$ and $t \in T_\alpha$ then $\operatorname{tp} t = \alpha + 1$.

e. The tree $T = \bigcup_{\alpha \in \omega_1} T_\alpha$ is an Aronszajn tree.

▶ **14**. An alternative construction. There is a sequence $\langle r_\alpha : \alpha < \omega_1 \rangle$ such that r_α is an injective map from α into $\mathbb{N}$ and such that $r_\alpha =^* r_\beta \restriction \alpha$ whenever $\alpha < \beta$. The tree $T = \{r_\beta \restriction \alpha : \alpha \leqslant \beta < \omega_1\}$ is an Aronszajn tree. *Hint*: Refer to Exercise 2.16. Choose the r_α by recursion, making sure that $r_\alpha[\alpha] \subseteq x_\alpha$.

Chapter 2

Elementarity

This chapter introduces and studies elementary substructures 'of the universe'.

1. Definability

Our first task is to define what *definable* subsets of a set are. Intuitively these are sets determined by some formula, and this is how we shall work with them, but the formal definition is more algebraic in nature. The problem is that we cannot quite formalize a quantifier like "there exists a formula".

1.1. DEFINITION. For $n \in \omega$ and $i, j < n$ set

1. $\mathrm{Proj}(A, R, n) = \{s \in A^n : (\exists t \in R)(t \restriction n = s)\}$;
2. $\mathrm{Diag}_=(A, n, i, j) = \{s \in A^n : s(i) = s(j)\}$; and
3. $\mathrm{Diag}_\in(A, n, i, j) = \{s \in A^n : s(i) \in s(j)\}$.

Using these operations we define the *definable* relations on A, as follows. First by recursion on $k \in \omega$ and all n simultaneously define $\mathrm{Df}'(k, A, n)$ by

$$\begin{aligned}
\mathrm{Df}'(0, A, n) &= \{\mathrm{Diag}_=(A, n, i, j) : i, j < n\} \cup \{\mathrm{Diag}_\in(A, n, i, j) : i, j < n\} \\
\mathrm{Df}'(k+1, A, n) &= \mathrm{Df}'(k, A, n) \cup \{A^n \setminus R : R \in \mathrm{Df}'(k, A, n)\} \\
&\quad \cup \{R \cap S : R, S \in \mathrm{Df}'(k, A, n)\} \\
&\quad \cup \{\mathrm{Proj}(A, R, n) : R \in \mathrm{Df}'(k, A, n+1)\}
\end{aligned}$$

Once this is done we set $\mathrm{Df}(A, n) = \bigcup_{k\in\omega} \mathrm{Df}'(k, A, n)$. These are the *definable* n-ary relations on A.

The family of definable relations is closed under taking complements, intersections and projections.

1.2. LEMMA. *If* $R, S \in \mathrm{Df}(A, n)$ *then* $A^n \setminus R, R \cap S \in \mathrm{Df}(A, n)$ *and if* $R \in \mathrm{Df}(A, n+1)$ *then* $\mathrm{Proj}(A, R, n) \in \mathrm{Df}(A, n)$.

Taking complements, intersections and projections, correspond to applying $\neg$, $\wedge$ and $\exists v_i$ to formulas. The following lemma makes this connection more explicit.

1.3. LEMMA. *Let* $\varphi(x_0, \ldots, x_{n-1})$ *be a formula whose free variables are among* $x_0, \ldots, x_{n-1}$. *Then for every set* A

$$\{s \in A^n : \varphi^A(s(0), \ldots, s(n-1))\} \in \mathrm{Df}(A, n).$$

In order to make sense of this lemma we must delve into the notion of a formula, explain what free variables are, and define $\varphi^A(s(0), \ldots, s(n-1))$.

Formulas

We all have a good idea what a formula is and we usually know how to recognise one when we see it. However, when we want to treat formulas mathematically we have to formalise our 'good idea'. We begin by listing the basic symbols in the language of set theory. These are: $\wedge$, $\neg$, $\exists$, (,), $\in$, $=$ and infinitely many variables: v_i (one for each natural number i). Our formulas will be finite sequences of basic symbols.

1.4. DEFINITION. The formulas of set theory are built up as follows:

1. for all natural numbers i and j the expressions $v_i \in v_j$ and $v_i = v_j$ are formulas; and
2. if φ and ψ are formulas then so are $(\varphi) \wedge (\psi)$, $\neg(\varphi)$ and $\exists v_i(\varphi)$ for any i.

Note the parentheses, these help to keep everything tidy. In practice we would consider $v_0 \in v_2 \wedge \neg(v_3 = v_4)$ to be a good formula (and we shall often do so) but when we want to prove something about formulas we shall replace it with its correct form $(v_0 \in v_2) \wedge (\neg(v_3 = v_4))$. From elementary logic we know that the formulas allowed by Definition 1.4 express everything we want to express.

► **1**. The following are abbreviations for certain more complicated formulas: $\forall v_i(\varphi)$, $(\varphi) \vee (\psi)$, $(\varphi) \rightarrow (\psi)$, $(\varphi) \leftrightarrow (\psi)$, $v_i \notin v_j$ and $v_i \neq v_j$. Write down these complicated forms.

We shall explain most of the notions related to formulas by way of the following one

$$(1) \qquad (\exists v_0(v_0 \in v_1)) \wedge (\exists v_1(v_2 \in v_1)).$$

► **2**. A *subformula* of a formula φ is a consecutive sequence of symbols from φ that is itself a formula. Identify the subformulas of formula 1.

The *scope* of the occurrence of a quantifier $\exists v_i$ in a formula is the (unique) subformula beginning with that $\exists v_i$.

► **3**. a. Identify the scopes of $\exists v_0$ and $\exists v_1$ in formula 1.
b. Prove that the scope of an occurrence is well-defined.

An occurrence of a variable v_i in a formula is *bound* if it lies in the scope of an occurrence of $\exists v_i$ in that formula, otherwise it is *free*.

▶ **4**. Identify which occurrence of v_1 in formula 1 is free and which is bound.

The truth or falsity of a formula depends on its free-occurring variables, not on the bound variables. Therefore we would write formula 1 as $\varphi(v_1, v_2)$, to indicate that it is about the free variables v_1 and v_2. However, common usage is a bit more flexible: if necessary we will write our formula as, for example, $\varphi(v_0, v_1, v_2, v_3)$ to indicate that its free variables are *among* v_0, v_1, v_2 and v_3.

Now, if a, b, c and d are constants or other variables then $\varphi(a, b, c, d)$ is the result of replacing every *free* occurence of v_0, v_1, v_2 and v_3 by a, b, c and d respectively. Thus, $\varphi(4, 3, 2, 1)$ is

$$\bigl(\exists v_0(v_0 \in 3)\bigr) \wedge \bigl(\exists v_1(2 \in v_1)\bigr),$$

and $\varphi(4, v_0, v_5, 1)$ is

$$\bigl(\exists v_0(v_0 \in v_0)\bigr) \wedge \bigl(\exists v_1(v_5 \in v_1)\bigr).$$

The second substitution is unfortunate because it has changed the meaning of the first part of φ from "v_1 has an element" to "something is an element of itself". Such substitutions will not be allowed; we only consider *free* substitutions: a substitution $\varphi(y_1, y_2, y_3, y_4)$ is free if no free occurence of an original v_i is in the scope of a quantifier $\exists y_i$ (this only matters if y_i is a variable of course).

In Lemma 1.3 we substitute elements of A for the free occurences of the variables in φ^A; in that case there is no problem with bad substitutions: the elements of A are not variables.

Finally we define what φ^A (the *relativation* of φ to A) means:

1. $(v_i = v_j)^A$ is $v_i = v_j$ and $(v_i \in v_j)^A$ is $v_i \in v_j$;
2. $(\varphi \wedge \psi)^A$ is $\varphi^A \wedge \psi^A$ and $(\neg\varphi)^A$ is $\neg(\varphi^A)$; and
3. $\bigl(\exists v_i(\varphi)\bigr)^A$ is $\exists v_i\bigl((v_i \in A) \wedge (\varphi)^A\bigr)$.

Thus, informally, φ^A is φ with every $\exists v_i$ replaced by $\exists v_i \in A$.

▶ **5**. Give φ^A, where φ is formula 1.

Now we are ready to prove Lemma 1.3; we know what a formula is, we know what φ^A is and we know how substitutions work. We abbreviate the set $\bigl\{s \in A^n : \varphi^A\bigl(s(0), \dots, s(n-1)\bigr)\bigr\}$ as $G(\varphi, A)$ and prove the lemma by induction on the length of φ.

▶ **6**. a. If φ is $x_i \in x_j$ then $G(\varphi, A) = \mathrm{Diag}_{\in}(A, n, i, j)$.
b. If φ is $x_i = x_j$ then $G(\varphi, A) = \mathrm{Diag}_{=}(A, n, i, j)$.
c. $G(\varphi \wedge \psi, A) = G(\varphi, A) \cap G(\varphi, A)$.
d. $G(\neg\varphi, A) = A^n \setminus G(\varphi, A)$.

If $\varphi = \exists y(\psi)$ then there are two cases: y is not one of the variables $x_0, \dots, x_{n-1}$ or $y = x_j$ for some $j < n$.

e. If y is not one of $x_0, \ldots, x_{n-1}$ then $G(\varphi, A) = \mathrm{Proj}(A, G(\psi, A), n)$, where we write ψ as $\psi(x_0, \ldots, x_{n-1}, y)$.

In case, for example, $y = x_0$ take a variable z not occuring in φ, write the formula $\psi(z, x_1, \ldots, x_{n-1})$ as $\psi'(x_0, \ldots, x_{n-1}, z)$ and let φ' be $\exists z(\psi')$.

f. The substitution $x_0 \to z$ is free.

g. ψ and ψ' are logically equivalent, hence so are φ and φ'.
$G(\varphi, A) = G(\varphi', A) = \mathrm{Proj}(A, G(\psi', A), n)$.

2. Elementary substructures

In order to define what elementary substructures are we must count the definable relations.

2.1. Definition. By recursion on m, we define $\mathrm{En}(m, A, n)$, for all n simultaneously, as follows

1. If $m = 2^i \cdot 3^j$ and $i, j < n$ then $\mathrm{En}(m, A, n) = \mathrm{Diag}_{\in}(A, n, i, j)$.
2. If $m = 2^i \cdot 3^j \cdot 5$ and $i, j < n$ then $\mathrm{En}(m, A, n) = \mathrm{Diag}_{=}(A, n, i, j)$.
3. If $m = 2^i \cdot 3^j \cdot 5^2$ then $\mathrm{En}(m, A, n) = A^n \setminus \mathrm{En}(i, A, n)$.
4. If $m = 2^i \cdot 3^j \cdot 5^3$ then $\mathrm{En}(m, A, n) = \mathrm{En}(i, A, n) \cap \mathrm{En}(j, A, n)$.
5. If $m = 2^i \cdot 3^j \cdot 5^4$ then $\mathrm{En}(m, A, n) = \mathrm{Proj}(A, \mathrm{En}(i, A, n+1), n)$.
6. In all other cases $\mathrm{En}(m, A, n) = \varnothing$.

► **1**. For any A and n we have $\mathrm{Df}(A, n) = \{\mathrm{En}(m, A, n) : m \in \omega\}$.
a. $\forall n\bigl(\mathrm{En}(m, A, n) \in \mathrm{Df}(A, n)\bigr)$ for all m. *Hint*: by induction on m.
b. $\forall n\bigl(\mathrm{Df}'(k, A, n) \subseteq \{\mathrm{En}(m, A, n) : m \in \omega\}\bigr)$ for all k. *Hint*: by induction on k.
c. The set $\mathrm{Df}(A, n)$ is countable.

The proof of Lemma 1.3 yields the following improvement.

2.2. Lemma. *Let $\varphi(x_0, \ldots, x_{n-1})$ be a formula whose free variables are among $x_0, \ldots, x_{n-1}$. Then there is an m such that for every set A*

$$\bigl\{s \in A^n : \varphi^A\bigl(s(0), \ldots, s(n-1)\bigr)\bigr\} = \mathrm{En}(m, A, n).$$

Using the enumeration we define the relation $M \prec N$ between sets.

2.3. Definition. We say that M is an *elementary substructure* of N — notation $M \prec N$ — if $M \subseteq N$ and

$$\forall n, m\bigl(\mathrm{En}(m, M, n) = \mathrm{En}(m, N, n) \cap M^n\bigr).$$

The following Lemma connects this notion to formulas; Lemma 2.2 facilitates the proof.

2.4. Lemma. *Let $\varphi(x_0, \ldots, x_{n-1})$ be a formula whose free variables are among $x_0, \ldots, x_{n-1}$. Then $M \prec N$ implies*

$$\bigl\{s \in M^n : \varphi^M\bigl(s(0), \ldots, s(n-1)\bigr)\bigr\} = \bigl\{s \in N^n : \varphi^N\bigl(s(0), \ldots, s(n-1)\bigr)\bigr\} \cap M^n.$$

To get a feeling for what the definition and this lemma say we look at an important special case.

► **2**. If $a \in M \prec N$ and $a \cap N \neq \varnothing$ then $a \cap M \neq \varnothing$.

a. Prove this from the definition. *Hint*: Note that the assumption says $a \in \mathrm{Proj}(N, \mathrm{Diag}_{\in}(N, 2, 1, 0), 1) \cap M$.

b. Prove this using Lemma 2.4. *Hint*: Let $\varphi(v_0)$ be $\exists v_1(v_1 \in v_0)$ and note that the assumption says $a \in \{s \in N : \varphi^N(s)\} \cap M$.

A lot of arguments involving elementarity boil down to a clever application of this exercise: to see that something of the right kind is in M show that the set of things of the right kind belongs to M and that its intersection with N is nonempty.

Another special case is when $n = 0$. This is because $A^0 = \{\varnothing\}$ (only the empty function has domain 0). Therefore $\mathrm{En}(m, A, 0)$ is either 0 or 1. In Lemma 2.2 the case $n = 0$ corresponds to formulas without free variables, so-called *sentences*, for which φ^A is either false (if $\mathrm{En}(m, A, 0) = 0$) or true (if $\mathrm{En}(m, A, 0) = 1$). This leads to the following notion from Model Theory: A and B are *elementarily equivalent* if $\mathrm{En}(m, A, 0) = \mathrm{En}(m, B, 0)$ for all m; we write this as $A \equiv B$.

► **3**. If $A \prec B$ then $A \equiv B$.

► **4**. If m is of the form $2^i \cdot 3^j \cdot 5^4$ then $\mathrm{En}(m, A, 0) = 1$ iff $\mathrm{En}(i, A, 1) \neq \varnothing$. Therefore an elementary substructure of a nonempty set is nonempty.

► **5**. ω is the only elementary substructure of itself. Let $M \prec \omega$.

a. $\varnothing \in M$. *Hint*: use the sentence $\exists x\bigl(\forall y(y \notin x)\bigr)$.

b. If $n \in M$ then $n + 1 \in M$. *Hint*: use the formula $\exists y\bigl((x \in y) \wedge \forall z((x \in z) \rightarrow ((z = y) \vee (y \in z)))\bigr)$

The following fundamental result shows that a given structure has many elementary substructures. It is a special case of the Löwenheim-Skolem theorem from Model Theory.

2.5. Theorem. *Given N and $X \subseteq N$ there is an M such that $X \subseteq M$, $M \prec N$ and $|M| \leqslant \max(\aleph_0, |X|)$.*

► **6**. Prove Theorem 2.5. *Hint*: Let $\lhd$ be a well-ordering of N. For $m, n \in \omega$ define $H_{mn} : N^n \to N$ as follows. If m is of the form $2^i \cdot 3^j \cdot 5^4$ and $s \in \mathrm{En}(m, N, n) = \mathrm{Proj}(N, \mathrm{En}(i, N, n+1), n)$ then $H_{mn}(s)$ is the $\lhd$-first element of N such that $s^\frown x \in \mathrm{En}(i, N, n+1)$; in all other cases let $H_{mn}(s)$ be the $\lhd$-minimum of N. Let $X_0 = X$ and, recursively, let $X_{k+1} = X_k \cup \bigcup_{m,n\in\omega} H_{mn}[X_k^n]$. In the end let $M = \bigcup_{k\in\omega} X_k$.

a. For all k we have $|X_k| \leqslant \max(\aleph_0, |X|)$ and also $|M| \leqslant \max(\aleph_0, |X|)$.

b. For all m and n we have $\mathrm{En}(m, M, n) = \mathrm{En}(m, N, n) \cap M^n$. *Hint*: Induction on m, for all n simultaneously.

3. Elementary substructures of the Universe

In applications we work — intuitively — with elementary substructures of the set-theoretic universe but, because of things like 'the set of all sets', this can only be done on an intuitive level.

However, nothing prevents us from sharpening our intuition a bit before we put our method on a firm foundation. So, for the moment we treat V, the universe of all sets, as a set and fix an elementary substructure, M, of it. Observe that, because all sets are in V, for any formula φ the relativization φ^V is just φ itself.

What must be in M?

Certain things *must* be in M, simply because they are definable individuals. For instance, $\varnothing \in M$ because it is the unique set without elements. To see this note that the empty set is the only x that satisfies $\forall z(z \in x \longrightarrow z \neq z)$. Therefore if we write this formula as $\varphi(x)$ then the sentence $\exists x\varphi(x)$ is true in V, i.e., after applying Lemma 2.2 to get a number i for φ and setting $m = 2^i \cdot 3^0 \cdot 5^4$ we see that $\mathrm{En}(m, V, 0) = 1$ and hence $\mathrm{En}(m, M, 0) = 1$. Now apply Exercise 2.4 to see that both $\mathrm{En}(i, V, 1)$ and $\mathrm{En}(i, M, 1)$ are nonempty. But because $M \prec V$ we know that $\mathrm{En}(i, M, 1) = \mathrm{En}(i, V, 1) \cap M$. Now use uniqueness of $\varnothing$ to see that $\mathrm{En}(i, V, 1) = \{\varnothing\}$; therefore the only possibility is that $\mathrm{En}(i, M, 1) = \{\varnothing\}$ as well, and so $\varnothing \in M$.

▶ **1.** a. $\omega \subseteq M$. *Hint*: Apply Exercise 2.5 or do it now.
b. $\omega \in M$. *Hint*: ω is the (unique) minimal inductive set.
c. $\omega_1 \in M$. *Hint*: Find a formula that *defines* ω_1.

We can apply uniqueness to show that M is closed under various set-theoretic operations, the following exercise contains small sample.

▶ **2.** a. If $a \in M$ then $\bigcup a \in M$.
b. If $a, b \in M$ then $\{a, b\} \in M$ and so $a \cup b \in M$.
c. If $a \in M$ then $\mathcal{P}(a) \in M$.

The last part of this exercise gives rise to Skolem's paradox. In case M is countable the uncountable set $\mathcal{P}(\omega)$ belongs to M. Now, as M is an elementary substructure of the universe, all axioms of set theory are true in M, so M must somehow contain information that $\mathcal{P}(\omega)$ is uncountable. But $\mathcal{P}(\omega) \cap M$ is countable, so how can this be? The answer is that if f is a map from ω to $\mathcal{P}(\omega)$ that belongs to M then it is still subject to Cantor's diagonal argument, which yields a subset A of ω that is not in the range of f. So f belongs to $\{g : \exists x(x \in \mathcal{P}(\omega) \wedge x \notin \operatorname{ran} g)\} \cap M$, by elementarity f must therefore also belong to $\{g : \exists x(x \in M \wedge x \in \mathcal{P}(\omega) \wedge x \notin \operatorname{ran} g)\}$. This shows that no surjective map from ω onto $\mathcal{P}(\omega) \cap M$ can be a member of M.

► **3**. If F is a finite subset of M then $F \in M$. *Hint*: Fix $n \in \omega$ and a bijection $f : n \to F$. Apply Exercise 3.2 to show by induction that $\{f(j) : j < i\} \in M$ for every i.

► **4**. If $a \in M$ is countable then $a \subseteq M$. *Hint*: a belongs to $\bigl\{x : \exists f\bigl((\operatorname{dom} f = \omega) \wedge (\operatorname{ran} f = x)\bigr)\bigr\} \cap M$, so there is an $f \in M$ with $\operatorname{dom} f = \omega$ and $\operatorname{ran} f = x$. Use uniqueness to show that $f(i) \in M$ for all $i \in \omega$.

► **5**. If M is countable then $M \cap \omega_1$ is a countable ordinal.

Closed and unbounded sets and stationary sets

Countable M enable us to give fast proofs of facts about closed and unbounded sets and about stationary sets. As in the previous chapter we let $\mathcal{C}$ be the family of closed and unbounded subsets of ω_1. Also, for countable M we put $\delta_M = M \cap \omega_1$.

► **6**. a. $\mathcal{C} \in M$. *Hint*: Write down a formula that defines $\mathcal{C}$.
b. If M is countable and $C \in \mathcal{C} \cap M$ then $\delta_M \in C$. *Hint*: Every $\alpha < \delta_M$ belongs to M and to $\{\beta : \exists\gamma(\gamma \in C \wedge \gamma > \beta)\} \cap M$, deduce that $C \cap \delta_M$ is cofinal in δ_M.
c. If $\{C_n : n \in \omega\} \subseteq \mathcal{C}$ then $\bigcap_n C_n \in \mathcal{C}$. *Hint*: Let $\alpha \in \omega_1$ be arbitrary. Let M be countable with $\{C_n : n \in \omega\} \cup \alpha \subseteq M$. Consider δ_M.
d. If M is countable and $\langle C_\alpha : \alpha < \omega_1\rangle$ is a sequence in $\mathcal{C}$ that belongs to M then $\delta_M \in \triangle_\alpha C_\alpha$. *Hint*: If $\alpha \in M$ then $C_\alpha \in M$.

► **7**. Assume M is countable. If $S \in M \cap \mathcal{P}(\omega_1)$ and $\delta_M \in S$ then S is stationary. *Hint*: $S \in \bigl\{A \in M : \forall C\bigl((C \in M \wedge C \in \mathcal{C}) \to C \cap A \neq \varnothing\bigr)\bigr\}$.

4. Proofs using elementarity

We reprove some of the results from Chapter 1 using elementarity.

First the pressing-down lemma.

► **1**. Let $f : \omega_1 \to \omega_1$ be regressive and let M be countable with $f \in M$.
a. $\alpha = f(\delta_M) \in M$.
b. $S = \{\beta : f(\beta) = \alpha\}$ belongs to M and it is stationary.

Next the Δ-system lemma.

► **2**. Let $F = \langle F_\alpha : \alpha < \omega_1\rangle$ be a sequence of finite subsets of ω_1. Let M be countable with $F \in M$.
a. Let $R = F_{\delta_M} \cap \delta_M$, then $R \in M$.
b. The set $S = \{\alpha : R = F_\alpha \cap \alpha\}$ belongs to M.
c. The set $C = \{\alpha : (\forall\beta < \alpha)(\max F_\beta < \alpha)\}$ is closed and belongs to M; also $\delta_M \in C$, so C is unbounded as well.
d. The set $T = C \cap S$ is stationary and if $\alpha < \beta$ in T then $F_\alpha \cap F_\beta = R$.

And finally:

► **3**. Let $f : \omega_1 \to \mathbb{R}$ be continuous and let M be countable with $f \in M$.
a. Let $\varepsilon > 0$ and take $\alpha < \delta_M$ such that $\bigl|f(\beta) - f(\delta_M)\bigr| < \varepsilon$ whenever $\alpha < \beta \leqslant \delta_M$. Then $\bigl|f(\beta) - f(\gamma)\bigr| < 2\varepsilon$ whenever $\beta, \gamma > \alpha$.
b. f is constant on $[\delta_M, \omega_1)$.

5. Justification

Taking elementary substructures of the universe of all sets is not something that can be formalized in Set Theory. One can formalize the applications in Set Theory, however. To see this we must realize that the arguments use a limited supply of sets and simply take a large enough set that contains these sets and rework the argument inside that big set. The most popular of these large sets are called $H(\theta)$. We shall describe these and show how to work with them.

To define the $H(\theta)$ we must first define the *transitive closure* of sets. First, a set x is said to be *transitive* if it satisfies $(\forall y \in x)(y \subseteq x)$. Around every set we can find a smallest transitive set, as follows. Given x put $x_0 = x$ and, recursively, $x_{n+1} = x_n \cup \bigcup x_n$; in the end set $\operatorname{trcl} x = \bigcup_n x_n$.

► **1**. a. $\operatorname{trcl} x$ is transitive.
b. If y is transitive and $x \subseteq y$ then $\operatorname{trcl} x \subseteq y$.

Now we can define $H(\theta)$, for cardinal numbers θ:

$$H(\theta) = \{x : |\operatorname{trcl} x| < \theta\}.$$

Thus, e.g., $H(\aleph_1)$ is the set of all hereditarily countable sets.

► **2**. a. $\omega \in H(\aleph_1)$, $\omega_1 \in H(\aleph_2)$ and, generally, $\kappa \in H(\kappa^+)$ for all κ.
b. $\omega \subseteq H(\aleph_0)$, $\omega_1 \subseteq H(\aleph_1)$, and, generally, $\kappa \subseteq H(\kappa)$ for all κ.
c. $\mathcal{P}(\omega) \in H(\mathfrak{c}^+)$, $\mathcal{P}(\omega_1) \in H\bigl((2^{\aleph_1})^+\bigr)$, and, generally, $\mathcal{P}(\kappa) \in H\bigl((2^\kappa)^+\bigr)$ for all κ.
d. $\mathcal{P}(\omega) \subseteq H(\aleph_1)$, $\mathcal{P}(\omega_1) \subseteq H(\aleph_2)$, and, generally, $\mathcal{P}(\kappa) \subseteq H(\kappa^+)$ for all κ.

We check that the proofs from Section 4 can be done within relativelt small $H(\theta)$.

► **3**. a. Exercise 3.6 can be done for $M \prec H(\aleph_2)$. *Hint*: $\mathcal{C} \subseteq H(\aleph_2)$.
b. Exercise 3.7 can be done for $M \prec H(\aleph_2)$. *Hint*: Replace $C \in \mathcal{C}$ by a formula that expresses 'C is cub'.
c. The proofs of the pressing-down lemma and Δ-system lemma can be done with $M \prec H(\aleph_2)$.
d. Exercise 4.3 requires $M \prec H(\mathfrak{c}^+)$.

Theorem 2.5 admits refinements. The following will be needed in the proof of Arkhangel′skiĭ's theorem.

► **4**. Let $X \subseteq H(\theta)$ be of cardinality $\mathfrak{c}$ (or less). There is an M such that $X \subseteq M$, $M \prec H(\theta)$, $|M| \leqslant \mathfrak{c}$ and ${}^\omega M \subseteq M$. *Hint*: In the original proof redefine the X_k so that ${}^\omega X_k \subseteq X_{k+1}$.

Chapter 3

Arkhangel′skiĭ's theorem

A special case of the theorem of the title says that first-countable compact Hausdorff spaces have cardinality at most $\mathfrak{c}$. In the literature one can find three approaches to this result; we shall present each of these, in an attempt to show how better tools do make for lighter work. For expository purposes we confine ourselves to the basic case of first-countable compact Hausdorff spaces; at the end of this section we indicate possible generalizations.

1. First proof

This is essentially Arkhangel′skiĭ's original proof. We shall require a few preliminary topological results.

▶ **1**. Let X be first-countable Hausdorff space with a dense set of cardinality $\mathfrak{c}$ (or less); then $|X| \leqslant \mathfrak{c}$. *Hint*: Every point in the space is the limit of a sequence from the dense set.

▶ **2**. Let X be a first-countable compact Hausdorff space and A a closed subset of cardinality $\mathfrak{c}$ (or less); then $X \setminus A$ can be written as the union of no more than $\mathfrak{c}$ closed sets. *Hint*: Choose a countable local base $\mathcal{B}_x$ at each point x of A and consider the family of all finite covers of A whose members belong to $\bigcup_{x\in A} \mathcal{B}_x$.

1.1. Theorem. *Let X be a first-countable compact Hausdorff space; then $|X| \leqslant \mathfrak{c}$.*

▶ **3**. Prove Theorem 1.1. Let T denote the tree ${}^{<\omega_1}\mathfrak{c}$ of countable sequences of elements of $\mathfrak{c}$.

a. $|T| = \mathfrak{c}$.

Choose closed sets F_t, for all $t \in T$, and points x_t, for $t \in T$ of successor height, as follows. First, $F_\varnothing = X$ and $x_\varnothing$ is any point of X. Second, if $\operatorname{ht} t$ is a limit ordinal we let $F_t = \bigcap_{s<t} F_s$. Third, we define $F_{t,\alpha}$ and $x_{t,\alpha}$ for every $\alpha < \mathfrak{c}$: Let $A_t = \operatorname{cl}\{x_s : s \leqslant t\}$ and write $X \setminus A_t = \bigcup_{\alpha<\mathfrak{c}} G_{t,\alpha}$, where each $G_{t,\alpha}$ is closed. Now put $F_{t,\alpha} = F_t \cap G_{t,\alpha}$ and let $x_{t,\alpha}$ be any point of $F_{t,\alpha}$ unless this set is empty, in which case we let $x_{t,\alpha} = x_\varnothing$.

b. $F_t \subseteq A_t \cup \bigcup_{\alpha<\mathfrak{c}} F_{t,\alpha}$.

c. For every α we have $X = \bigcup\{A_t : \operatorname{ht} t = \alpha\} \cup \bigcup\{F_t : \operatorname{ht} t = \alpha\}$.

Let $T' = \{t : |F_t| \leqslant \mathfrak{c}\}$.

d. $\bigcup_{t\in T} A_t \cup \bigcup_{t\in T'} F_t$ has cardinality $\mathfrak{c}$ (or less).

Assume $X \neq \bigcup_{t\in T} A_t \cup \bigcup_{t\in T'} F_t$ and choose $x \in X$ outside the union.

e. There is a path P through T such that $x \in F_t$ for all $t \in P$.

f. $\mathrm{cl}\{x_s : s < t\} \cap \mathrm{cl}\{x_s : t \leqslant s, s \in P\} = \varnothing$, whenever $t \in P$.

g. If $y \in \mathrm{cl}\{x_s : s \in P\}$ then $y \in \mathrm{cl}\{x_s : s < t\}$ for some $t \in P$; therefore $\bigcap_{t \in P} \mathrm{cl}\{x_s : t \leqslant s, s \in P\} = \varnothing$.

h. X is compact, hence $\bigcap_{t \in P} \mathrm{cl}\{x_s : t \leqslant s, s \in P\} \neq \varnothing$.

2. Second proof

The first proof is tree-like; the second proof proceeds in a linear recursion.

► **1**. Prove Theorem 1.1. Fix for every $x \in X$ a countable local base $\mathcal{B}_x$. Recursively define closed sets F_α, for $\alpha \in \omega_1$, as follows. $F_0 = \{x_0\}$ for some x_0. If α is a limit ordinal let $F_\alpha = \mathrm{cl} \bigcup_{\beta < \alpha} F_\beta$. If F_α is given let $\mathcal{B}_\alpha = \bigcup_{x \in F_\alpha} \mathcal{B}_x$ and choose for every finite subfamily $\mathcal{U}$ of $\mathcal{B}_\alpha$ that covers F_α but not X one point $x_{\mathcal{U}} \in X \setminus \bigcup \mathcal{U}$ and let $F_{\alpha+1}$ be the closure of the union of F_α and the set of all points $x_{\mathcal{U}}$.

a. For every α we have $|F_\alpha| \leqslant \mathfrak{c}$ and $|\mathcal{B}_\alpha| \leqslant \mathfrak{c}$.

b. The set $F = \bigcup_\alpha F_\alpha$ is closed, hence compact.

Let $\mathcal{U}$ be a finite subfamily of $\bigcup_{x \in F} \mathcal{B}_x$ that covers F.

c. $\mathcal{U} \subseteq \mathcal{B}_\alpha$ for some α.

d. $\mathcal{U}$ covers X. *Hint*: $\mathcal{U}$ covers $F_{\alpha+1}$.

e. Deduce that $X = F$, hence $|X| \leqslant \mathfrak{c}$.

3. Third proof

The third proof is the second proof in disguise.

► **1**. Prove Theorem 1.1. Fix for every $x \in X$ a countable local base $\mathcal{B}_x$. Let θ be large enough so that X and the assignment $x \mapsto \mathcal{B}_x$ belong to $H(\theta)$. Take an elementary substructure M of $H(\theta)$, of cardinality $\mathfrak{c}$, and such that X and $x \mapsto \mathcal{B}_x$ belong to M and ${}^\omega M \subseteq M$.

a. $F = X \cap M$ is closed in X. *Hint*: If $x \in \mathrm{cl}(X \cap M)$ then some sequence in $X \cap M$ converges to x; the sequence belongs to M.

b. Every finite subfamily $\mathcal{U}$ of $\bigcup_{x \in F} \mathcal{B}_x$ belongs to M; if it covers F then it also covers X. *Hint*: $M \vDash (\forall x \in X)(\exists U \in \mathcal{U})(x \in U)$.

4. Extensions and generalizations

One can relax the assumptions of Theorem 1.1 considerably.

► **1**. Theorem 1.1 also holds for Lindelöf spaces. *Hint*: All the proofs go through with finite collections replaced by countable ones.

We can replace the assumption of first-countability by the conjunction of two weaker properties: *countable pseudocharacter*, i.e., points are G_δ-sets, and *countable tightness*, which means that whenever $x \in \operatorname{cl} A$ there is a countable subset B of A such that $x \in \operatorname{cl} B$.

First we rework Exercise 1.1.

▶ **2**. Let X be a Lindelöf space with countable pseudocharacter and countable tightness. If A is a subset of X of cardinality $\mathfrak{c}$ or less then also $|\operatorname{cl} A| \leqslant \mathfrak{c}$.

a. It suffices to show that $|\operatorname{cl} A| \leqslant \mathfrak{c}$ whenever A is countable. *Hint*: $\operatorname{cl} A = \bigcup\{\operatorname{cl} B : B \in [A]^{\leqslant\aleph_0}\}$.

Assume X itself is separable and let D be a countable dense subset.

b. For every x we have $\{x\} = \bigcap\{O : x \in O \text{ and } O \text{ is regular open}\}$.

c. X has at most $\mathfrak{c}$ regular open sets. *Hint*: If O is regular open then $O = \operatorname{int}\operatorname{cl}(O \cap D)$.

For every countable family $\mathcal{U}$ of regular open sets put $N_{\mathcal{U}} = X \setminus \bigcup \mathcal{U}$ and let $\mathcal{N}$ be the family of these $N_{\mathcal{U}}$'s.

d. If O is open and $x \in O$ then there is a $\mathcal{U}$ such that $x \in N_{\mathcal{U}} \subseteq O$. *Hint*: $X \setminus O$ is Lindelöf.

e. For every point x there is a countable subfamily $\mathcal{N}_y$ of $\mathcal{N}$ such that $\{x\} = \bigcap \mathcal{N}_y$.

f. The map $x \mapsto \mathcal{N}_y$ from X into $[\mathcal{N}]^{\leqslant\aleph_0}$ is one-to-one.

Exercise 1.2 needs less extra work.

▶ **3**. Let X be a Lindelöf space of countable pseudocharacter and A a closed subset of cardinality $\mathfrak{c}$ (or less); then $X \setminus A$ can be written as the union of no more than $\mathfrak{c}$ closed sets. *Hint*: Choose a countable family $\mathcal{B}_x$ of open sets at each point x of A with $\bigcap \mathcal{B}_x = \{x\}$ and consider the family of all countable covers of A whose members belong to $\bigcup_{x \in A} \mathcal{B}_x$.

▶ **4**. Use any of the three proofs to show that a Lindelöf Hausdorff space of countable pseudocharacter and countable tightness has cardinality at most $\mathfrak{c}$.

Chapter 4

Dowker spaces

Products of normal spaces need not be normal; the square of the Sorgenfrey line is the best known example of this phenomenon. Lots of effort has gone into investigating what normal spaces do have normal products. The simplest case has turned out to be one of the most interesting: when is $X \times [0,1]$ normal? The spaces whose product with the unit interval $I = [0,1]$ is normal were characterized by Dowker and normal spaces whose product with I is not normal are called *Dowker spaces*.

1. Normality in products

We exhibit two non-normal products.

We first consider the square of the Sorgenfrey line. Remember that a local base at a point a is given by $\bigl\{(b,a] : b < a\bigr\}$.

▶ **1**. The Sorgenfrey line is normal. *Hint*: Given F and G choose for every $a \in \mathbb{S}$ a point $x_a < a$ such that $(x_a, a] \cap F = \varnothing$ if $x \notin F$ and $(x_a, a] \cap G = \varnothing$ if $x \notin G$; now let $U = \bigcup_{a \in F}(x_a, a]$ and $V = \bigcup_{a \in G}(x_a, a]$.

▶ **2**. The Sorgenfrey plane $\mathbb{S}^2$ is not normal. Let $P = \bigl\{(p,-p) : p \in \mathbb{P}\bigr\}$ and $Q = \bigl\{(q,-q) : q \in \mathbb{Q}\bigr\}$, where $\mathbb{P}$ and $\mathbb{Q}$ are the sets of irrational and rational numbers respectively.

a. P and Q are closed in $\mathbb{S}^2$.

Let U be an open set around P and for $n \in \mathbb{N}$ put $P_n = \{p \in \mathbb{P} : (p - 2^{-n}, p] \times (-p - 2^{-n}, -p] \subseteq U\}$.

b. There is an n such that $\operatorname{int}\operatorname{cl} P_n \neq \varnothing$ in the usual topology of the real line.

c. If $q \in \mathbb{Q} \cap \operatorname{int}\operatorname{cl} P_n$ then $(q,-q) \in \operatorname{cl} U$.

The next example is slightly better because, as we shall see, it shows better how the ingredients in Dowker's characterization appear.

▶ **3**. Consider the ordinal spaces ω_1 and $\omega_1 + 1$.

a. ω_1 and $\omega_1 + 1$ are normal.

b. $\omega_1 \times \omega_1 + 1$ is not normal. *Hint*: Consider $F = \bigl\{(\alpha,\alpha) : \alpha \in \omega_1\bigr\}$ and $G = \bigl\{(\alpha,\omega_1) : \alpha < \omega_1\bigr\}$; apply the Pressing-Down Lemma to show that $G \cap \operatorname{cl} U \neq \varnothing$ whenever U is an open set around F.

2. Borsuk's theorem

One of the reasons for wanting to know when $X \times I$ is normal is the following theorem, due to Borsuk.

2.1. THEOREM (Borsuk's Homotopy Extension Theorem). *Let X be a space such that $X \times I$ is normal, let A be a closed subspace of X and let $f, g : A \to S^n$ be continuous and homotopic. If f admits a continuous extension to X then so does g and the extensions may be chosen homotopic, in fact by a homotopy that extends the given homotopy between f and g.*

Two maps $f, g : X \to Y$ are *homotopic* if there is a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all x. We call H a *homotopy* between f and g. Thus Borsuk's theorem asserts that homotopies between maps can be extended provided one of the maps can be extended. Note the codomain, this the n-sphere, i.e, the subspace $\{x : \|x\| = 1\}$ of $\mathbb{R}^{n+1}$. For other codomains the proof is quite easy, e.g., for I^n the proof below finishes after the first step.

► **1**. Prove Borsuk's Homotopy Extension Theorem. Let $h : A \times I \to S^n$ be a homotopy between f and g and let $F : X \to S^n$ be an extension of f. Let $B = (A \times I) \cup (X \times \{0\})$ and define $k : B \to S^n$ by $k(x, t) = h(x, t)$ if $t > 0$ and $k(x, 0) = F(x)$.

a. The map k can be extended to a neighbourhood U of B. *Hint*: Extend k to $K : X \times I \to D$, where D is the massive ball, and let $U = \{(x, t) : K(x, t) \neq 0\}$; compose $K \restriction U$ with the projection with 0 as its centre.
b. There is a neighbourhood V of A such that $V \times I \subseteq U$.
c. There is a continuous function $l : X \to I$ such that $l(x) = 1$ for $x \in A$ and $l(x) = 0$ for $x \notin U$.
d. The map $H : (x, t) \mapsto K(x, l(x) \cdot t)$ is the desired homotopy.

3. Countable paracompactness

The property that characterizes normality of $X \times [0, 1]$ is *countable paracompactness.* To define it we must first introduce the following notion.

3.1. DEFINITION. A collection $\mathcal{A}$ of sets in a space X is *locally finite* if every point of X has a neighbourhood that intersects only finitely many elements of $\mathcal{A}$.

► **1**. If $\mathcal{A}$ is locally finite then $\operatorname{cl} \bigcup \mathcal{A} = \bigcup \{\operatorname{cl} A : A \in \mathcal{A}\}$.

Given two covers $\mathcal{A}$ and $\mathcal{B}$ of a set we say that $\mathcal{A}$ is a refinement of $\mathcal{B}$ if for every $A \in \mathcal{A}$ there is a $B \in \mathcal{B}$ such that $A \subseteq B$.

3.2. DEFINITION. A space is *paracompact* is every open cover has a locally finite open refinement. It is *countably paracompact* is every countable open cover has a locally finite open refinement.

To get a feeling for what locally finite open refinements can do we have the following.

► **2**. a. A paracompact Hausdorff space is regular. *Hint*: Given a closed set F and $x \in X \setminus F$ choose, for every $y \in F$, an open set U_y with $y \in U_y$ and $x \notin \operatorname{cl} U_y$. Consider a locally finite open refinement of $\{X \setminus F\} \cup \{U_y : y \in F\}$.
b. A paracompact regular space is normal.

► **3**. A space is countably paracompact iff every countable open cover has a *countable* locally finite open refinement. *Hint*: If $\mathcal{V}$ is some locally finite open refinement of $\mathcal{U}$, choose $U_V \in \mathcal{U}$ with $V \subseteq U_V$ for every $V \in \mathcal{V}$. Put $W_U = \bigcup\{V : U_V = U\}$; then $\{W_U : U \in \mathcal{U}\}$ is locally finite and of cardinality not more than $\mathcal{U}$.

► **4**. Let $\mathcal{U}$ be a locally finite open cover of the normal space X. There is an open cover $\{V_U : U \in \mathcal{U}\}$ of X such that $\operatorname{cl} V_U \subseteq U$ for all U. *Hint*: Well-order $\mathcal{U}$ by $\prec$ and define V_U by recursion on U: first put $F_U = X \setminus \bigl(\bigcup_{W \prec U} V_W \cup \bigcup_{W \succ U} W\bigr)$ and then choose V_U with $F_U \subseteq V_U$ and $\operatorname{cl} V_U \subseteq U$.

The following theorem gives more characterizations of countable paracompactness.

3.3. THEOREM. *The following are equivalent for a space X.*

1. *X is countably paracompact;*
2. *if $\{U_n : n \in \omega\}$ is an increasing open cover of X then there is a sequence $\{F_n : n \in \omega\}$ of closed sets with $F_n \subseteq U_n$ for all n and $X = \bigcup_n \operatorname{int} F_n$; and*
3. *if $\{F_n : n \in \omega\}$ is a decreasing sequence of closed sets in X with empty intersection then there is a sequence $\{U_n : n \in \omega\}$ of open sets with $F_n \subseteq U_n$ for all n and $\bigcap_n \operatorname{cl} U_n = \varnothing$.*

► **5**. Prove Theorem 3.3.
a. Prove 1 implies 2. *Hint*: Apply Exercise 3.3 to get $\{V_n : n \in \omega\}$ and put $F_n = X \setminus \bigcup_{m>n} V_m$.
b. Prove 2 implies 1. *Hint*: Given $\{U_n : n \in \omega\}$ apply 2 to $\{\bigcup_{m \leqslant n} U_n : n \in \omega\}$ and put $V_n = U_n \setminus \bigcup_{m<n} F_m$.
c. Prove 2 and 3 are equivalent.

The following is the characterization of countable paracompactness that is used most often.

► **6**. A normal space X is countably paracompact iff whenever $\{F_n : n \in \omega\}$ is a decreasing sequence of closed sets in X with empty intersection there is a sequence $\{U_n : n \in \omega\}$ of open sets with $F_n \subseteq U_n$ for all n and $\bigcap_n U_n = \varnothing$.

The following theorem is the promised characterization of normality of $X \times [0,1]$.

3.4. THEOREM. *The product $X \times [0,1]$ is normal iff X is normal and countably paracompact.*

The proof is in the following two exercises.

► **7**. Assume $X \times [0,1]$ is normal.

a. X is normal.

b. X is countably paracompact. *Hint*: Let $\{F_n : n \in \omega\}$ be a decreasing sequence of closed sets with empty intersection. Let $F = \bigcup_n \bigl(F_n \times [2^{-n}, 1]\bigr)$ and $G = X \times \{0\}$.

► **8**. Assume X is normal and countably paracompact. Let F and G be closed and disjoint in $X \times [0,1]$. Let $\mathcal{B}$ be a countable base for the topology of $[0,1]$, closed under finite unions. For $x \in X$ let $F_x = \{t \in [0,1] : (x,t) \in F\}$ and define G_x similarly.

a. F_x and G_x are closed and disjoint.

b. For every x there is a $B \in \mathcal{B}$ with $F_x \subseteq B$ and $\operatorname{cl} B \cap G_x = \varnothing$.

c. If $B \in \mathcal{B}$ then $U_B = \{x : F_x \subseteq B \text{ and } \operatorname{cl} B \cap G_x = \varnothing\}$ is open in X.

Take a locally finite open cover $\{V_B : B \in \mathcal{B}\}$ of X with $\operatorname{cl} V_B \subseteq U_B$ for all B and let $V = \bigcup_{B \in \mathcal{B}} (V_B \times B)$.

d. $F \subseteq V$ and $\operatorname{cl} V \cap G = \varnothing$. *Hint*: $\{V_B \times B : B \in \mathcal{B}\}$ is locally finite.

Chapter 5

Balogh's Dowker space

Balogh's example is constructed using pairs of elementary substructures of the universe. To see how it works we look at an easier example first.

1. An example of Rudin's

We discuss an example of a normal space that is not collectionwise Hausdorff, it is an adaptation of an example due to Rudin.

The space will have $\mathfrak{c} \cup [\mathfrak{c}]^2$ as its underlying set, where $[\mathfrak{c}]^2$ denotes $\bigl\{\{\alpha,\beta\} : \alpha < \beta < \mathfrak{c}\bigr\}$. Each of the points $\{\alpha,\beta\}$ will be isolated. For each α we will find a filter $\mathcal{F}_\alpha$ of subsets of $\mathfrak{c}$ and define the neighbourhoods of α to be the sets of the form $U(\alpha,F) = \{\alpha\} \cup \bigl\{\{\alpha,\beta\} : \beta \in F\bigr\}$, with $F \in \mathcal{F}_\alpha$.

► **1.** a. $U(\alpha,F) \cap U(\beta,G) \subseteq \bigl\{\{\alpha,\beta\}\bigr\}$.
b. $U(\alpha,F) \cap U(\alpha,G) \neq \varnothing$ iff $\alpha \in G$ and $\beta \in F$.

We shall choose for every $\alpha \in \mathfrak{c}$ and every subset A of $\mathfrak{c}$ a subset $F(\alpha,A)$ and let $\mathcal{F}_\alpha$ be the filter generated by $\{F(\alpha,A) : A \subseteq \mathfrak{c}\}$. Note that $\mathcal{F}_\alpha$ may be an improper filter.

Normality will be achieve by ensuring that $\beta \notin F(\alpha,A)$ or $\alpha \notin F(\beta,A)$ whenever $\alpha \notin A$ and $\beta \in A$.

To this end we define $I(\alpha,A) = A$ if $\alpha \in A$ and $I(\alpha,A) = \mathfrak{c} \setminus A$ if $\alpha \notin A$. We shall also define sets $J(\alpha,A)$ for all α and A and put

$$F(\alpha,A) = I(\alpha,A) \cup \{\beta > \alpha : \beta \in J(\alpha,A)\} \cup \{\beta < \alpha : \alpha \notin J(\beta,A)\}.$$

This already gives us normality.

► **2**. If $A \subseteq \mathfrak{c}$, $\alpha \in A$ and $\beta \notin A$ then $\alpha \notin F(\beta,A)$ or $\beta \notin F(\alpha,A)$.

Notice that every element of $\mathcal{F}_\alpha$ is determined by a finite family of subsets of $\mathfrak{c}$, so, to get our space to be not collectionwise Hausdorff, we must consider all possible assignments $f : \alpha \mapsto \mathcal{A}_\alpha$ of finite families of subsets of $\mathfrak{c}$ and, somehow, ensure that there are disctinct α and β with $\alpha \in \bigcap_{A\in\mathcal{A}_\beta} F(\beta,A)$ and $\beta \in \bigcap_{A\in\mathcal{A}_\alpha} F(\alpha,A)$.

Our strategy for dealing with $2^{\mathfrak{c}}$ such assignments in only $\mathfrak{c}$ steps is based on the following idea. Take a countable elementary substructure M of the universe that has the assignment f in it and look at the restriction of f to M, i.e., the map $f^M : \mathfrak{c} \cap M \to \bigl[\mathcal{P}(M \cap \mathfrak{c})\bigr]^{<\omega}$ defined by $f^M(\beta) = \{A \cap M : A \in \mathcal{A}_\alpha\}$. There are only $\mathfrak{c}$ many such restrictions and they give us enough information to deal with all possible assignments using just $\mathfrak{c}$ many points.

So let $\bigl\{(A_\beta, f_\beta) : \beta \in P\bigr\}$ enumerate the family of all pairs of the form $(M \cap \mathfrak{c}, f^M)$, where M is an elementary substructure of $H(\theta)$ and where we assume that $P \subseteq \mathfrak{c}$ and the enumeration is such that always $A_\beta \subseteq \beta$.

Given (A_β, f_β) define a function $g_\beta : \mathbb{N} \to A_\beta$, as follows. Assume $g_\beta \restriction n_k$ is known and put $\mathcal{B}_k = \bigcup_{i<n_k} f_\beta\bigl(g_\beta(i)\bigr)$, so $\mathcal{B}_k$ is a finite family of subsets of A_β. For every function $\chi : \mathcal{B}_k \to \{0,1\}$ choose, if possible, a point α_χ not in $\{g_\beta(i) : i < n_k\}$ such that for all $B \in \mathcal{B}_k$ we have $\alpha_\chi \in B$ iff $\chi(B) = 1$. Extend g_β to some $n_{k+1} > n_k$ so that $\{g_\beta(i) : n_k \leqslant i < n_{k+1}\}$ counts the set of α_χ's.

► **3**. The map g_β is one-to-one and defined on all of $\mathbb{N}$.

Define $f'_\beta : \mathbb{N} \to \bigl[\mathcal{P}(A)\bigr]^{<\omega}$ by $f'_\beta(i) = f_\beta\bigl(g_\beta(i)\bigr) \setminus \mathcal{B}_k$ for $n_k \leqslant i < n_{k+1}$. Now we can define the sets $J(\alpha, A)$:

$$J(\alpha, A) = \bigl\{\beta > \alpha : (\exists i)\bigl(\alpha = g_\beta(i) \wedge A \cap A_\beta \in f'_\beta(i)\bigr)\bigr\}.$$

With this the definition of the $F(\alpha, A)$ is complete.

Let $f : \mathfrak{c} \to \bigl[\mathcal{P}(\mathfrak{c})\bigr]^{<\omega}$ be given and fix a countable elementary substructure M of the universe with $f \in M$. Fix β with $\mathfrak{c} \cap M = A_\beta$ and $f^M = f_\beta$.

► **4**. There is a k such that $A \in f(\beta)$ and $A \cap M \in f'_\beta(i)$ imply $i < n_k$.

Define $\chi : \mathcal{B}_k \to \{0,1\}$ by: if $i < n_k$ and $B \in f_\beta\bigl(g_\beta(i)\bigr)$ and then $\chi(B \cap M) = 1$ iff $\beta \in B$.

► **5**. a. χ is well-defined, i.e., there are no $B, C \in \mathcal{B}_k$ with $B \cap M = C \cap M$ and $\beta \in B \setminus C$. *Hint*: elementarity.
b. α_χ is defined. *Hint*: elementarity.

► **6**. $\beta \in F(\alpha_\chi, A)$, whenever $A \in f(\alpha_\chi)$. Fix $j \in [n_k, n_{k+1})$ with $\alpha_\chi = g_\beta(j)$.
a. If $A \cap M \in f'_\beta(i)$ for some $i < n_k$ then $\beta \in A$ iff $\alpha_\chi \in A$, hence $\beta \in I(\alpha_\chi, A)$.
b. If $A \cap M \in f'_\beta(j)$ then $\beta \in J(\beta, A)$.

► **7**. $\alpha_\chi \in F(\beta, A)$, whenever $A \in f(\beta)$.
a. If $A \cap M \in f'_\beta(i)$ for some $i < n_k$ then $\beta \in A$ iff $\alpha_\chi \in A$, hence $\alpha_\chi \in I(\beta, A)$.
b. If $A \cap M \notin f'_\beta(i)$ for any $i < n_k$ then $A \cap M \notin f'_\beta(j)$ and so $\beta \notin J(\alpha_\chi, A)$, whence $\alpha_\chi \in F(\beta, A)$.

2. Balogh's example

Balogh's example is, to some extent, similar in spirit to Rudin's example but much more complicated.

The underlying set of our space X will be $\mathfrak{c}\times\omega$. As above we will construct, for each α, a filter $\mathcal{F}_\alpha$ and use these filters to define the topology: U is open iff whenever $(\alpha, n+1) \in U$ there is an $F \in \mathcal{F}_\alpha$ such that $\{(\beta, n) : \beta \in F\} \subseteq U$.

► **1**. a. For every n the set $U_n = \mathfrak{c} \times [0, n]$ is open.
b. For every n the set $L_n = \mathfrak{c} \times \{n\}$ is relatively discrete.

The hard part will be to ensure that the space is normal and not countably paracompact. Normality is handled much like in Rudin's example: there will be $F(\alpha, A)$ in $\mathcal{F}_\alpha$ such that $F(\alpha, A) \cap F(\beta, \mathfrak{c} \setminus A) = \varnothing$ whenever $\alpha \in A$ and $\beta \notin A$. Countable paracompactness follows because, for every n, a closed set contained in $\mathfrak{c} \times n$ must be 'small', in fact so small that whenever we choose closed sets $F_n \subseteq \mathfrak{c} \times n$ for every n, their union will not even cover $\mathfrak{c} \times \{0\}$.

The following combinatorial lemma lies at the basis of the construction.

2.1. Lemma. *There is a map $c \mapsto d_c$ from ${}^{\mathfrak{c}}2$ to itself such that whenever $f : \mathfrak{c} \to \omega$, $g : \mathfrak{c} \to [{}^{\mathfrak{c}}2]^{<\omega}$ and $h : \mathfrak{c} \to [\mathfrak{c}]^{<\omega}$ are given we can find $\alpha < \beta$ in $\mathfrak{c}$ with $f(\alpha) = f(\beta)$, if $c \in g(\alpha)$ then $c(\alpha) = d_c(\beta)$, and $\beta \notin h(\alpha)$.*

The construction

Given the lemma, the construction proceeds as follows. For $\alpha \in \mathfrak{c}$, $s \in [{}^{\mathfrak{c}}2]^{<\omega}$ and $a \in [\mathfrak{c}]^{<\omega}$ put

$$F(\alpha, s, a) = \bigl\{\beta \in \mathfrak{c} : (\forall c \in s)\bigl(d_c(\beta) = c(\alpha)\bigr)\bigr\} \setminus a.$$

Furthermore, for each α, let $\mathcal{F}_\alpha$ be the family of all sets of the form $F(\alpha, s, a)$.

► **2**. $F(\alpha, s_1, a_1) \cap F(\alpha, s_2, a_2) = F(\alpha, s_1 \cup s_2, a_1 \cup a_2)$.

It is very well possible that $F(\alpha, s, a) = \varnothing$ for some α, s and a; for example when $c(\alpha) = 1$ and d_c is constantly 0: in that case $F(\alpha, \{c\}, \varnothing) = \varnothing$. We will see however that this does not happen too often.

Normality

The space is even hereditarily normal.

Let H and K be separated subsets of X, i.e., $H \cap \operatorname{cl} K = \operatorname{cl} H \cap K = \varnothing$. We have to find disjoint open sets around H and K.

► **3**. It suffices to find, for each n, open sets V_n and W_n with $H \cap L_n \subseteq V_n$ and $\operatorname{cl} V_n \cap K = \varnothing$, as well as $K \cap L_n \subseteq W_n$ and $\operatorname{cl} W_n \cap H = \varnothing$.
Hint: Let $V = \bigcup_n (V_m \setminus \bigcup_{m \leqslant n} \operatorname{cl} W_n)$ and $W = \bigcup_n (W_m \setminus \bigcup_{m \leqslant n} \operatorname{cl} V_n)$.

► **4**. Let $A \subseteq \mathfrak{c}$ and $n \in \omega$. Then $A \times \{n\}$ and $(\mathfrak{c} \setminus A) \times \{n\}$ have disjoint open neighbourhoods.
a. The statement holds for $n = 0$. *Hint*: See Exercise 2.1.
b. If the statement holds for n then it holds for $n + 1$. *Hint*: Let c be the characteristic function of A and show that $F(\alpha, \{c\}, \varnothing)$ and $F(\beta, \{c\}, \varnothing)$ are disjoint whenever $\alpha \in A$ and $\beta \notin A$. Look at $A' \times \{n\}$, where $A' = \bigcup_{\alpha \in A} F(\alpha, \{c\}, \varnothing)$.

▶ **5**. If $m < n$ then $K \cap L_m$ and $H \cap L_n$ have disjoint open neighbourhoods.
a. There are disjoint open sets O_K and O_H in U_m such that $L_n \cap \operatorname{cl} K \subseteq O_K$ and $L_n \setminus \operatorname{cl} K \subseteq O_H$.
b. The set $O_H^* = O_H \cup (U_n \setminus (U_m \cup \operatorname{cl} K))$ is open and contains $H \cap L_n$.
c. O_H^* and O_K are as required.

▶ **6**. There are disjoint open sets V_n and O around $H \cap L_n$ and K respectively.
a. There are disjoint open sets V_n and O' around $H \cap L_n$ and $K \cap U_n$ respectively. *Hint*: Apply the previous two exercises.
b. The set $O = O' \cup (X \setminus (U_n \cup \operatorname{cl} H))$ is open and as required.

Countable paracompactness

We call a subset A of $\mathfrak{c}$ *separated* if we can find for each $\alpha \in A$ a set $F_\alpha \in \mathcal{F}_\alpha$ such that $\alpha \notin F_\beta$ and $\beta \notin F_\alpha$ whenever $\alpha \neq \beta$ in A. A set is *σ-separated* if it is the union of countably many separated sets.

▶ **7**. $\mathfrak{c}$ is not σ-separated. *Hint*: Apply Lemma 2.1.

▶ **8**. Let $n \in \omega$ and $A \subseteq \mathfrak{c}$. Then $A \setminus \varphi(A)$ is separated, where $\varphi(A) = \bigl\{\alpha : (\alpha, n+1) \in \operatorname{cl}\bigl(A \times \{n\}\bigr)\bigr\}$.

▶ **9**. If $n \in \omega$ and F_n is closed and a subset of U_n then $A_n = \{\alpha : (\alpha, 0) \in F_n\}$ is the union of $n+1$ many separated sets. *Hint*: $\varphi^{n+1}(A_n) = \varnothing$.

▶ **10**. X is not countably paracompact.

Proof of Lemma 2.1

The proof of Lemma 2.1 is much like that in Section 1: we try to deal with $2^{\mathfrak{c}}$ many possibilities by looking at their restrictions to countable elementary substructures of $H(\theta)$, where θ is sufficiently big, larger than $2^{2^{\mathfrak{c}}}$ will work.

However, we need an extra twist to the construction. Assume we have f, g and h as in the lemma. We take two countable elementary substructures M and N of $H(\theta)$, with $f, g, h \in M$ and $M \in N$. We define $A = \mathfrak{c} \cap N$ and $B = \{c \restriction A : c \in {}^{\mathfrak{c}}2 \cap M\}$.

▶ **11**. If $\alpha \in N$ and $\beta \notin A$ then $\beta \notin h(\alpha)$.

▶ **12**.a. For every n the preimage $f^{\leftarrow}(n)$ belongs to M.
b. If $\beta \notin A$ and $f(\beta) = n$ then $f^{\leftarrow}(n)$ is uncountable.

These two exercises show that it is quite easy to find $\alpha < \beta$ with $f(\alpha) = f(\beta)$ and $\beta \notin h(\alpha)$: simply take β outside A and $\alpha \in A$ with $f(\alpha) = f(\beta)$.

To get, given β, an α such that $d_c(\beta) = c(\alpha)$ for all $c \in g(\alpha)$ we have to do more work.

▶ **13**. If $c \in {}^{\mathfrak{c}}2 \cap N \setminus M$ then $c \restriction A \notin B$. *Hint*: If $c' \in {}^{\mathfrak{c}}2 \cap M$ then $c' \neq c$, use elementarity.

For $\alpha \in \mathfrak{c}$ define $e_\alpha : g(\alpha) \to 2$ by $e_\alpha(c) = c(\alpha)$.

► **14**. The function e_α depends only on the restriction of g to N, defined by $g^N(\alpha) = \{c \restriction N : c \in g(\alpha)\}$.

► **15**. Let $E = g(\beta) \cap M$ and $n = f(\beta)$, and put $e = e_\beta \restriction E$.

a. The set $H = \{\gamma : f(\gamma) = n,\ E \subseteq g(\gamma)$ and $e = e_\gamma \restriction E\}$ belongs to M and is cofinal in $\mathfrak{c}$.

b. If F is a finite subset of M and $\alpha \in \mathfrak{c} \cap M$ then there is a $\gamma \in H \cap M$ with $\gamma > \alpha$ and $F \cap g(\gamma) \setminus E = \varnothing$.

c. Choose, in M, a maximal subset K of H such that $g(\gamma) \cap g(\delta) = E$ whenever $\gamma \neq \delta$ in K. Then K is uncountable. *Hint*: If K is countable then $K \subseteq M$; consider $K \cup \{\beta\}$.

d. If $\alpha \in K$ and $c \in E$ then $c(\alpha) = c(\beta)$.

This gives us a clue as to how to define the value $d_c(\beta)$ for certain c: if there is an $\alpha \in K$ with $c \in g(\alpha)$ then $d_c(\beta) = c(\beta) = c(\alpha)$ if $c \in E$ and $d_c(\beta) = c(\alpha)$ if $c \notin E$. If there is no such α then $d_c(\beta)$ is not important, so we set $d_c(\beta) = 0$. However, this assumes that we know M and N, whereas we need to define the d_c knowing only $\mathfrak{c} \cap M$, $\mathfrak{c} \cap N$ and the restrictions of f, g and h.

To give the true definition we let $\{(a_\beta, A_\beta, B_\beta, f_\beta, g_\beta, h_\beta) : \beta \in P\}$ enumerate the set of structures of the form

$$(\mathfrak{c} \cap M, \mathfrak{c} \cap N, \{c \restriction (\mathfrak{c} \cap N) : c \in M\}, f \restriction N, g^N, h \restriction N),$$

where $M, N \prec H(\theta)$, $M \in N$ and $f, g, h \in M$. Also, g^N is defined on A by $g^N(\alpha) = \{c \restriction (\mathfrak{c} \cap N) : c \in g(\alpha)\}$. We assume P and the enumeration are chosen so that always $A_\beta \subseteq \beta$.

Fix $\beta \in P$ and consider the βth structure $(a_\beta, A_\beta, B_\beta, f_\beta, g_\beta, h_\beta)$.

Inspired by Exercise 2.15 we consider triples (n, E, e), where $n \in \mathbb{N}$, $E \in [a_\beta]^{<\aleph_0}$ and $e : E \to \{0, 1\}$. For each such triple put

$$H(n, E, e) = \{\gamma \in A_\beta : f(\gamma) = n, g(\gamma) \cap B_\beta = E \text{ and } e = e_\gamma \restriction E\}.$$

Here we define e_γ as above: $e_\gamma(c) = c(\gamma)$ for $c \in g_\beta(\gamma)$.

Still using Exercise 2.15 as our guideline we consider the set I_β of those (n, E, e) for which $H(n, E, e)$ has an infinite subset $K(n, E, e)$ such that $g_\beta(\gamma) \cap g_\beta(\delta) = E$ whenever $\gamma \neq \delta$ in $K(n, E, e)$.

► **16**. There are an infinite set J_β in A_β and a function $u_\beta : J_\beta \to [{}^{A_\beta}2]^{<\aleph_0}$ with disjoint values such that for every $(n, E, e) \in I_\beta$ there are infinitely many $\gamma \in J_\beta \cap K(n, E, e)$ with $u_\beta(\gamma) = g_\beta(\gamma) \setminus E$.

Now we define the d_c:

1. if $\beta \in P$ and $c \restriction A_\beta \in B_\beta$ then set $d_c(\beta) = c(\beta)$;
2. if $\beta \in P$ and $c \restriction A_\beta \notin B_\beta$ but $c \restriction A_\beta \in u_\beta(\alpha)$ for a (unique) $\alpha \in J_\beta$ then set then $d_c(\beta) = c(\alpha)$;

3. in all other cases set $d_c(\beta)=0$.

   This definition works.

► **17**. Let f, g and h be given and take M and N with $f,g,h\in M$, $M\in N$ and $M,N\prec H(\theta)$. Fix β with $(a_\beta,A_\beta,B_\beta,f_\beta,g_\beta,h_\beta)=(\mathfrak{c}\cap M,\mathfrak{c}\cap N,\{c\restriction A:c\in M\},f\restriction N,g^N,h\restriction N)$. Let $n=f(\beta)$, $E=g(\beta\cap M)$ and $e=e_\beta\restriction E$. If $\alpha\in J_\beta\cap K(n,E,e)$ then $f(\alpha)=f(\beta)$, $\beta\notin h(\alpha)$ and $d_c(\beta)=c(\alpha)$ for all $c\in g_(\alpha)$.

Chapter 6

Rudin's Dowker space

Rudin's Dowker space is of a totally different nature than that of Balogh; it was based on an example of Misčenko's of a linearly Lindelöf space that is not Lindelöf.

1. Description of the space

We work in the product $P = \prod_{n=1}^{\infty}(\omega_n + 1)$ of the successors of the first $\aleph_0$ many uncountable ordinals. We give P the *box topology*, where each ordinal has its usual *order topology*. The box topology has the family of all open boxes as a base; an *open box* is simply a product $\prod_{n=1}^{\infty} O_n$, where O_n is open in $\omega_n + 1$.

We consider two subspaces of P:

$$X' = \{x \in P : (\forall n)(\operatorname{cf} x_n > \omega_0)\}$$

and its subset

$$X = \{x \in P : (\exists i)(\forall n)(\omega_i > \operatorname{cf} x_n > \omega_0)\}.$$

The space X is Rudin's Dowker space. The rest of this chapter will be devoted to verifying this.

A nice base

We need an easy-to-handle base for the topology of X' and X. To this end we introduce the following notation. For $x, y \in P$ we say $x < y$ if $x_n < y_n$ for all n and $x \leqslant y$ means $x_n \leqslant y_n$ for all n. For $x, y \in P$ with $x < y$ we use $(x, y]$ to denote the set $\{z \in X' : (\forall n)(x_n < z_n \leqslant y_n)\}$, i.e, $(x, y] = X' \cap \prod_{n=1}^{\infty}(x_n, y_n]$.

▶ **1**. If $x \in X'$ then $\bigl\{(y, x] : y < x\bigr\}$ is a local base at x.

We shall be using the family $\mathcal{B} = \bigl\{(x, y] : x, y \in P,\ x < y\bigr\}$ as a base for the open sets of X'. The following consequence of the choice of points in X' will be very useful.

▶ **2**. X' is a P-space, i.e., if $\mathcal{U}$ is a countable family of open sets then $\bigcap \mathcal{U}$ is open.

2. X is normal

To prove X is normal we prove two things:

1. every open cover of X' has a disjoint open refinement, and
2. if A and B are closed and disjoint *in* X then their closures *in* X' are disjoint too.

▶ **1**. The two statements above imply that X is indeed normal.

The property that every open cover has a disjoint open refinement is called *ultraparacompactness*; it is (much) stronger than ordinary paracompactness.

X' is ultraparacompact

Let $\mathcal{O}$ be an open cover of X'. We build a sequence $\langle \mathcal{U}_\alpha : \alpha < \omega_1 \rangle$ of open covers of X' such that

1. each $\mathcal{U}_\alpha$ is a disjoint open cover and a subfamily of $\mathcal{B}$,
2. if $\alpha < \beta$ then $\mathcal{U}_\beta$ is a refinement of $\mathcal{U}_\alpha$,
3. if $U \in \mathcal{U}_\alpha$ and $U \subseteq O$ for some $O \in \mathcal{O}$ then $U \in \mathcal{U}_{\alpha+1}$, and
4. if $U \in \mathcal{U}_\alpha$, say $U = (x, y]$, and $U \subseteq O$ for no $O \in \mathcal{O}$ then for every $V \in \mathcal{U}_{\alpha+1}$ with $V \subseteq U$ and $V = (u, v]$ there is some n such that $v_n < y_n$ or $V \subseteq O$ for some $O \in \mathcal{O}$.

▶ **2**. Let $y \in X'$ and denote for $\alpha < \omega_1$ the unique element of $\mathcal{U}_\alpha$ that contains y by $(u_\alpha, v_\alpha]$.

a. For every n there is an α_n such that $v_\alpha(n) = v_{\alpha_n}(n)$ whenever $\alpha \geqslant \alpha_n$.

Let $\alpha_y = \sup_n \alpha_n$ and $\beta = \alpha_y + 1$.

b. There is an $O \in \mathcal{O}$ with $(u_\beta, v_\beta] \subseteq O$.

c. If $\gamma \geqslant \beta$ then $(u_\gamma, v_\gamma] = (u_\beta, v_\beta]$.

▶ **3**. The family $\{(u_{\alpha_y}, v_{\alpha_y}] : y \in X'\}$ is a disjoint open refinement of $\mathcal{O}$.

To construct the sequence we start with $\mathcal{U}_0 = \{X'\}$. Note that $X' = (0, t]$, where $t_i = \omega_i$ for all i.

To make $\mathcal{U}_{\alpha+1}$ from $\mathcal{U}_\alpha$ let $U \in \mathcal{U}_\alpha$, say $U = (x, y]$. If there is an $O \in \mathcal{O}$ with $U \subseteq O$ put $\mathcal{I}_U = \{U\}$. If not then consider two cases.

$y \in X'$ Take $z < y$ so that $x < z$ and $(z, y] \subseteq O$ for some $O \in \mathcal{O}$. For every subset A of $\mathbb{N}$ put

$$V_A = \{u \in (x, y] : (\forall i \in A)(u_i \leqslant z_i) \wedge (\forall i \notin A)(u_i > z_i)\}.$$

Set $\mathcal{I}_U = \{V_A : A \subseteq \mathbb{N}\}$.

$y \notin X'$ Fix n with $\operatorname{cf} y_n = \omega_0$ and fix an increasing cofinal sequence $\langle \alpha_i \rangle_i$ of ordinals in y_n with $\alpha_0 = x_n$. For $i \in \omega$ put $V_i = \{u \in (x, y] : \alpha_i < u_n \leqslant \alpha_{i+1}\}$ and let $\mathcal{I}_U = \{V_i : i \in \omega\}$.

Disjoint closed sets in X have disjoint closures in X'

Let A and B be closed and disjoint in X. Define $A_n = \{x \in A : (\forall i)(\operatorname{cf} x_i \leqslant \aleph_n)\}$ and define B_n similarly.

► **4**. It suffices to show that for every n the sets A_n and B_n have disjoint closures in X'. *Hint*: $A = \bigcup_n A_n$ and X' is a P-space.

Fix n and take $x \in X' \setminus X$. Let θ be large enough and take a countable elementary substructure M_0 of $H(\theta)$ with $A, B, x, X', X, P \in M_0$. Use M_0 as the starting point of a sequence $\langle M_\alpha < \alpha < \omega_n \rangle$ of elementary substructures of $H(\theta)$ such that $M_\alpha \cup \{M_\alpha\} \subseteq M_{\alpha+1}$ (and $|M_{\alpha+1}| \leqslant \max\{|M_\alpha|, \aleph_0\}$) for all α and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ whenever α is a limit. In the end let $M = \bigcup_{\alpha<\omega_n} M_\alpha$.

► **5**. a. For every limit ordinal β the set M_β is an elementary substructure of $H(\theta)$.
b. For every α we have $\alpha \subseteq M_\alpha$.
c. For every α we have $|M_\alpha| = \max\{|\alpha|, \aleph_0\}$.

Define $\hat{x}$ by $\hat{x}_i = \sup M \cap x_i$ and for every $\alpha < \omega_n$ define u_α by $u_\alpha(i) = \sup M_\alpha \cap x_i$.

► **6**. a. If $\operatorname{cf} x_i \leqslant \aleph_n$ then $\hat{x}_i = x_i$. *Hint*: There is $C \in M_0$ with $|C| \leqslant \aleph_n$ that is cofinal in $x(i)$. Show that $C \subseteq M$.
b. If $\operatorname{cf} x(i) > \aleph_n$ then $\hat{x}_n(i) < x(i)$ and $\operatorname{cf} \hat{x}_n(i) = \aleph_n$.
c. $\hat{x}_n \in X$ for all n.

► **7**. a. There is an α such that $(u_\alpha, \hat{x}] \cap A = \varnothing$ or $(u_\alpha, \hat{x}] \cap B = \varnothing$.
b. For this α we have $(u_\alpha, x] \cap A_n = \varnothing$ or $(u_\alpha, x] \cap B_n = \varnothing$.

► **8**. X is collectionwise normal, i.e., if $\mathcal{F}$ is a discrete collection of closed sets then there is a disjoint family $\{U_F : F \in \mathcal{F}\}$ of open sets with $F \subseteq U_F$ for all F. *Hint*: $\{\operatorname{cl}_{X'} F : F \in \mathcal{F}\}$ is discrete.

3. X is not countably paracompact

We apply Exercise 3.6. For $n \geqslant 1$ let $F_n = \{x \in X : (\forall i \leqslant n)(x_i = \omega_i)\}$; we show that $\bigcap_{n=1}^\infty U_n \neq \varnothing$ whenever $\langle U_n \rangle_n$ is a sequence of open sets with $U_n \supseteq F_n$ for all n.

► **1**. The sets F_n are indeed closed and $\bigcap_{n=1}^\infty F_n = \varnothing$.

The key to the proof is the following lemma. We let t denote the top of P, i.e., $t_i = \omega_i$ for all i.

3.1. Lemma. *If U is an open set around F_n then there is an $x < t$ such that $X \cap (x, t] \subseteq U$.*

Given this lemma, the rest of the proof is easy.

► **2**. $\bigcap_{n=1}^\infty U_n \neq \varnothing$. *Hint*: Apply Lemma 3.1 infinitely often to get an $x < t$ with $(x, t] \cap X \subseteq \bigcap_n U_n$.

Proof of Lemma 3.1

Let n and U be given. Choose θ large enough, so that $P, X, U \in H(\theta)$.

► **3**. There is $M \prec H(\theta)$ with $\omega_n \cup \{P, X, U\} \subseteq M$, $|M| = \aleph_n$ and such that $M \cap \prod_{i>n} \omega_i$ is cofinal in $\prod_{i>n}(M \cap \omega_n)$. ***Hint***: Obtain M as the union of a sequence $\langle M_\alpha : \alpha < \omega_n \rangle$, where $M_\alpha \in M_{\alpha+1}$ for all α.

Define $x \in P$ by $x_i = \omega_i$ for $i \leqslant n$ and $x_i = \sup M \cap \omega_i$ for $i > n$.

► **4**. $x \in F_n$.

Choose $x' < x$ so that $(x', x] \cap X \subseteq U$.

► **5**. a. There is $z \in M \cap P$ with $x' \leqslant z < x$.
b. For every $u \in M \cap (z, t] \cap X$ we have $t \in U$.
c. $(z, t] \cap X \subseteq U$.

This completes the proof. In the next chapter we shall see that X has a (much smaller) subspace that is also a Dowker space.

Chapter 7

A Dowker space of size $\aleph_{\omega+1}$

In this chapter we show that Rudin's Dowker space has a closed subspace of cardinality $\aleph_{\omega+1}$ that is also a Dowker space.

1. A special sequence in $\prod_{n=1}^{\infty} \omega_n$

We shall not be working with the full product $\prod_{n=1}^{\infty} \omega_n$ but with a subproduct over a subset B of $\mathbb{N}$. For $B \subseteq \mathbb{N}$ we write P_B for $\prod_{n\in B}(\omega_n + 1)$ and $Q_B = \prod_{n\in B} \omega_n$. As before, for $x, y \in P_B$ we write

- $x < y$ if $(\forall n \in B)(x_n < y_n)$;
- $x \leqslant y$ if $(\forall n \in B)(x_n \leqslant y_n)$;
- $x <^* y$ if $\{n : x_n \geqslant y_n\}$ is finite;
- $x \leqslant^* y$ if $\{n : x_n > y_n\}$ is finite; and
- $x = *y$ if $\{n : x_n \neq y_n\}$ is finite.

For us the following result, which we quote without proof, will give us our Dowker subspace.

1.1. Theorem. *There are a subset B of $\mathbb{N}$ and a sequence $\langle x_\alpha : \alpha < \aleph_{\omega+1}\rangle$ in Q_B such that*

1. *the sequence is increasing with respect to $<^*$, i.e., $x_\alpha <^* x_\beta$ whenever $\alpha < \beta$; and*
2. *the sequence is cofinal, i.e., if $x \in Q_B$ then there is an α with $x <^* x_\alpha$.*

We call a sequence as in this theorem an $\aleph_{\omega+1}$-scale. We can improve the scale a bit.

▶ **1**. We can assume that for every δ of uncountable cofinality *if* $\{x_\alpha : \alpha < \delta\}$ has a least upper bound with respect to $\leqslant^*$ *then* x_δ is such a least upper bound.
Hint: Start with some $\aleph_{\omega+1}$-scale $\langle y_\alpha : \alpha < \aleph_{\omega+1}\rangle$. Given $\langle x_\alpha : \alpha < \delta\rangle$ let x_δ be a least upper bound in case $\operatorname{cf}\delta > \omega_0$ and a least upper bound exists, otherwise let $x_\delta = y_\beta$, where β is minimal with $x_\alpha <^* y_\beta$ for all α.

We shall need many least upper bounds. We write $B_k = B \cap (k, \omega)$ when $k \in \mathbb{N}$.

1.2. Lemma. *Let $k \geqslant m$ and let $\langle \alpha_\eta : \eta < \aleph_m\rangle$ be strictly increasing with $\delta = \sup_\eta \alpha_\eta < \aleph_{\omega+1}$. Let $\langle y_\eta : \eta < \aleph_m\rangle$ be a sequence in Q_{B_k} that is* *increasing with respect to* $<$ *and such that* $y_\eta =^* x_{\alpha_\eta}$ *for all* η*, and let* y *be the pointwise least upper bound of the* y_η*.*

1. $y \in Q_{B_k}$ *and it is a least upper bound of* $\langle x_{\alpha_\eta} : \eta < \aleph_m \rangle$*;*
2. $\operatorname{cf} y(n) = \aleph_m$ *for all* $n > k$*; and*
3. $y =^* x_\delta$*.*

▶ **2**. Prove Lemma 1.2.

a. $\operatorname{cf} y(n) = \aleph_m$ for $n > k$. *Hint*: The sequence is $<$-increasing.

b. $y \in Q_{B_k}$. *Hint*: Use $\operatorname{cf} y(n) = \aleph_m$.

Let y' be any upper bound of the x_{α_η} and $C = \{n \in B : n > k \text{ and } y'(n) < y(n)\}$.

c. For $n \in C$ there is η_n with $y_{\eta_n}(n) > y'(n)$.

d. $\sup\{\eta_n : n \in C\} = \eta < \aleph_m$.

e. $y_\eta(n) \geqslant y_{\eta_n}(n) > y'(n)$ for $n \in C$.

f. C is finite. *Hint*: $y_\eta =^* x_{\alpha_\eta} \leqslant^* y'$.

g. $y =^* x_\delta$.

2. The space

We use a subproduct of Rudin's space X, to wit the space X_B, where $X_B = \{x \restriction B : x \in X\}$.

▶ **1**. The space X_B is also a Dowker space.

The desired subspace Z of X_B is now easily defined:

$$Z = \{x \in X_B : (\exists \alpha < \aleph_{\omega+1})(x =^* x_\alpha)\}.$$

Some simple but useful observations about Z.

▶ **2**. The space Z has cardinality $\aleph_{\omega+1}$. *Hint*: If $\alpha < \aleph_{\omega+1}$ then $\{x \in X_B : x =^* x_\alpha\}$ has cardinality $\aleph_{\omega+1}$.

▶ **3**. If $x \in Z$ then there is a unique α with $x =^* x_\alpha$.

▶ **4**. If $x, y \in Z$ then $x <^* y$ or $y <^* x$ or $x =^* y$.

Z is collectionwise normal

To show that Z is (collectionwise) normal it suffices to show that Z is closed in X_B.

▶ **5**. Assume $m \leqslant k$ and let $\langle y_\eta : \eta < \aleph_m \rangle$ be a sequence in Z such that $\langle y_\eta \restriction B_k : \eta < \aleph_m \rangle$ is $<$-increasing and let y be its pointwise supremum. Then $y \in Z$. *Hint*: Apply Lemma 1.2; for every η there is one α_η with $y_\eta =^* x_{\alpha_\eta}$.

Now we show that Z is closed in X_B. Fix a point $t \in X_B$ that is in the closure of Z. For every $z \in Z$ put $E(z, t) = \{n \in B : z_n = t_n\}$. Of course we seek a z so that $E(z, t)$ is cofinite.

► **6**. If $z \in Z$ and $z \leqslant t$ then $E(z,t)$ is finite or cofinite. Define y by $y_n = 0$ if $n \in E(z,t)$ and $y_n = z_n$ if $n \notin E(z,t)$.

a. $y < t$ and there is $x \in Z \cap (y,t]$.

b. If $z <^* x$ then $E(z,t)$ is finite. *Hint*: $\{n : z_n < x_n\} \cap E(z,t) = \varnothing$.

c. If $x =^* z$ or $x <^* z$ then $E(z,t)$ is cofinite. *Hint*: $\{n : x_n < z_n\} \subseteq E(z,t)$.

For $w \subseteq B$ let $Z_w = \{z \in Z : z \leqslant t \text{ and } E(z,t) = w\}$. Note that Z_w is only nonempty if w is finite or cofinite.

► **7**. There are w such that $t \in \operatorname{cl} Z_w$. *Hint*: X_B is a P-space, show that $\operatorname{cl}\{z \in Z : z \leqslant t\} = \bigcup_w \operatorname{cl} Z_w$.

Fix some w such that $t \in \operatorname{cl} Z_w$; if this w is infinite we are done, so assume it is finite. Also put $M_m = \{n \in B : \operatorname{cf} t_n = \aleph_m\}$ and $M_{<m} = \bigcup_{i<m} M_i$.

► **8**. There is an m such that M_m is infinite.

Let m be minimal such that M_m is infinite. Let $k = \max(M_{<m} \cup w)$. For each $n \in M_n$ fix an increasing and cofinal sequence $\langle \gamma^n_\eta : \eta < \aleph_m \rangle$ in t_n.

► **9**. There is a sequence $\langle y_\eta : \eta < \aleph_m \rangle$ in Z_w such that

1. $y_\eta \leqslant t$ for all η;
2. if $\eta < \zeta$ then $y_\eta < y_\zeta < t$ on B_k; and
3. $y_\eta(n) \geqslant \gamma^n_\eta$ for $n \in B_k \cap M_m$.

► **10**. Let y be the pointwise supremum of the sequence $\langle y_\eta : \eta < \aleph_m \rangle$.

a. $y \in Z$.

b. $E(y,t)$ is cofinite. *Hint*: $M_n \subseteq E(y,t)$.

c. $t \in Z$.

Z is not countably paracompact

This is relatively easy. Let $F_n = \{z \in X_B : (\forall i \in B)(i \leqslant n \to z_i = \omega_i)\}$ for every n.

► **11**. If U is open in Z and $U \supseteq F_n \cap Z$ then there is $z \in Z$ with $(z,t] \cap Z \subseteq U$.

a. The set $V = U \cup (X_B \setminus Z)$ is open in X_B and $F_n \subseteq V$.

b. There is $x \in X_B$ with $(x,t] \cap X_B \subseteq V$.

c. There is $z \in Z$ with $z > x$.

► **12**. If U_n is open and $U_n \supseteq F_n$ for all n then $\bigcap_n U_n \neq \varnothing$.

Appendix A

The Axioms of Set Theory

In the parlance of Mathematical Logic, Set Theory is a first-order theory with equality and one binary predicate, denoted $\in$, with the following axioms.

The Axiom of Extensionality. Sets with the same elements are equal: $(\forall x)(x \in a \leftrightarrow x \in b) \rightarrow (a = b)$.

The Axiom of Pairing. For any two sets a and b there is a third set having only a and b as its elements: $(\forall a)(\forall b)(\exists c)(\forall x)\bigl(x \in c \leftrightarrow (x = a \vee x = b)\bigr)$.

The Axiom of Union. For any set a there is a set consisting of all the elements of the elements of a: $(\forall a)(\exists b)(\forall x)\bigl(x \in b \leftrightarrow (\exists y)(y \in a \wedge x \in y)\bigr)$.

The Axiom of Power Set. For any set a there is a set consisting of all the subsets of a: $(\forall a)(\exists b)(\forall x)\bigl(x \in b \leftrightarrow (\forall y)(y \in x \rightarrow y \in a)\bigr)$.

The Axiom of Separation. If φ is a property, possibly with a parameter p, then for every a and p there is a set that consists of those elements of a that satisfy φ: $(\forall a)(\forall p)(\exists b)(\forall x)(x \in b \leftrightarrow (x \in a \wedge \varphi(x, p)))$.

The Axiom of Replacement. If F is a function then for every set a its image $F[a]$ under F is a set: $(\forall a)(\exists b)(\forall y)(y \in b \leftrightarrow (\exists x)(x \in a \wedge F(x) = y))$.

The Axiom of Infinity. There is an infinite set: $(\exists a)(\varnothing \in a \wedge (\forall x)(x \in a \rightarrow x \cup \{x\} \in a)$.

The Axiom of Foundation. Every nonempty set has a $\in$-minimal element: $(\forall a)\bigl(a \neq \varnothing \rightarrow (\exists b)(b \in a \wedge (\forall c)(c \in b \rightarrow c \notin a))\bigr)$.

The Axiom of Choice. Every set of nonempty sets has a choice function: $(\forall a)(\exists b)\bigl((\forall x \in a)(\exists y \in x)(\langle x, y\rangle \in b) \wedge (\forall x)(\forall y)(\forall z)((\langle x, y\rangle \in b \wedge \langle x, z\rangle \in b) \rightarrow y = z)\bigr)$.

These axioms form the starting point for Set Theory, just like Euclid's axioms were the starting point for Euclidean geometry.

The Axiom of Extensionality connects $=$ and $\in$; it mirrors the way in which we normally show that sets are equal.

The Axiom of Pairing, combined with the Axiom of Extensionality, lets us define a new 'function': $\{a, b\}$ is the *unique* c such that $(\forall x)(x \in c \leftrightarrow (x = a \vee x = b))$. We can then form $\{a\} = \{a, a\}$, the singleton set, and $\bigl\{\{a\}, \{a, b\}\bigr\}$, the *ordered pair*, usually denoted $\langle a, b\rangle$.

▶ **13**. Verify that $\langle a,b\rangle = \langle c,d\rangle$ iff $a=c$ and $b=d$.

The Axioms of Union and Power Set give additional operations: $\bigcup a = \bigl\{y : (\exists x \in a)(y \in x)\bigr\}$ and $\mathcal{P}(a) = \bigl\{x : (\forall y \in x)(y \in a)\bigr\}$. By combining Union and Pairing we can form $a \cup b = \bigcup\{a,b\}$.

The Axioms of Separation and Replacement deserve special consideration; both in fact represent an infinite list of axioms, one for each property or function. As such they should properly called axiom schemas. As an application let $\varphi(x,p)$ be $x \in p$; then for any sets a and p the set $\{x \in a : x \in p\}$ exists — it is of course nothing but $a \cap p$. Similarly, if $\varphi(x,p)$ is $(\forall y \in p)(x \in p)$ and $a \in p$ then $\{x \in a : \varphi(x,p)\}$ defines $\bigcap p$.

In the Replacement schema proper one considers formulas that define functions: if $\varphi(x,y,p)$ satisfies $(\forall x)(\forall y)(\forall z)(\varphi(x,y,p) \wedge \varphi(x,z,p) \rightarrow y = z)$ then $(\forall a)(\exists b)(\forall y)\bigl(y \in b \leftrightarrow (\exists x \in a)\varphi(x,y,p)\bigr)$.

The Axiom of Infinity may look strange at first but we must realize that none of the axioms so far can express the notions of 'finite' and 'infinite' in any way. As can be seen in Appendix B the present formulation leads to a satisfactory *set* that does all we may expect of the natural numbers. A set as in the Axiom of Infinity is called *inductive* and an inductive set deserves to be called infinite because it contains the chain $\varnothing \in \{\varnothing\} \in \bigl\{\varnothing, \{\varnothing\}\bigr\} \in \cdots$ that goes on forever. The Axiom of Foundation is also called the Axiom of Regularity because it proscribes infinite chains $\cdots \in x_2 \in x_1 \in x_0$ and thus ensures that the universe of sets can be built up by iterating the power set operation, thus: $V_0 = \varnothing$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ and $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ if α is a limit ordinal.

▶ **14**. Every set belongs to V_α for some α. *Hint*: Given a let $b = \bigl\{x \in \operatorname{trcl} a : (\exists\alpha)(x \in V_\alpha)\bigr\}$. If $b \neq \operatorname{trcl} a$ let x be $\in$-minimal in $\operatorname{trcl} a \setminus b$; by the Axiom of Replacement $x \subseteq V_\alpha$ for some α and so $x \in V_{\alpha+1}$ and hence $x \in b$.

The Axiom of Choice accounts for the C in ZFC. Because of its nonconstructive nature — the existence of the choice function is simply asserted, no description is given — it is treated with suspicion by some. We will use it freely in this book and at some places point out some of its stranger consequences.

Appendix B

Basics of Set Theory

In this chapter we collect some notions from Set Theory that are used throughout the book. We take the opportunity to illustrate how familiar set-theoretic operations can be justified on the basis of the axioms presented in Chapter A.

1. The Natural numbers

To see how $\mathbb{N}$ can be conceived as a set we apply the Axiom of Infinity to get an inductive set I. This means that $\varnothing \in I$ and that $x \cup \{x\} \in I$ whenever $x \in I$.

▶ **1**. There is a smallest inductive set.
  a. Apply the Power Set and Separation Axioms to construct $\mathbb{N} = \bigcap\{X : X \subseteq I$ and X is inductive$\}$.
  b. The set $\mathbb{N}$ is inductive and a subset of any other inductive set.

Thus, the official Set-Theoretic definition of $\mathbb{N}$ is that it is the smallest inductive set. We make some abbreviations: $0 = \varnothing$, $1 = \{0\}$, $2 = \{0, 1\}$, and so on.

▶ **2**. The set $\mathbb{N}$, together with the operation $n \mapsto n \cup \{n\}$, satisfies Peano's Axioms for the natural numbers.

This exercise allows us to define addition and multiplication as usual and $m < n$ by $(\exists k)(k \neq 0 \wedge n = m + k)$.

▶ **3**. The order $<$ is identical to $\in$.

2. Ordered pairs and sequences

We defined ordered pairs in Appendix A. In a similar fashion one can define ordered triples: $\langle x, y, z\rangle = \bigl\{\{x\}, \{x, y\}, \{x, y, z\}\bigr\}$

3. Products and relations

Relations abound in mathematics; they have a reasonably simple mathematical foundation.

Products

Given two sets X and Y the product $X \times Y$ is defined to be the set of ordered pairs $\langle x, y \rangle$ with $x \in X$ and $y \in Y$. The next exercise shows that this is a sound definition.

▶ **1**. The existence of $X \times Y$ can be deduced from the Axioms of Pairing, Union, Power Set and Separation.
a. If $x \in X$ and $y \in Y$ then $\langle x, y \rangle \in \mathcal{P}\big(\mathcal{P}(X \cup Y)\big)$.
b. $X \times Y = \big\{ z \in \mathcal{P}\big(\mathcal{P}(X \cup Y)\big) : (\exists x \in X)(\exists y \in Y)(z = \langle x, y \rangle) \big\}$.

There will be situations where the Power Set Axiom is not available; we can avoid it in building $X \times Y$.

▶ **2**. The existence of $X \times Y$ can be deduced from the Axioms of Pairing, Union, and Replacement.
a. Given $x \in X$ use the map $y \mapsto \langle x, y \rangle$ to deduce that $\{x\} \times Y$ is a set.
b. Use the map $x \mapsto \{x\} \times Y$ to deduce that $\mathcal{X} = \big\{ \{x\} \times Y : x \in X \big\}$ is a set.
c. $X \times Y = \bigcup \mathcal{X}$ is a set.

Relations

A relation is a set of ordered pairs. Its domain is the set of its first coordinates and its range the set of its second coordinates.

▶ **3**. a. z is an ordered pair iff $(\exists u \in z)(\exists v \in z)(\exists x \in u)(\exists y \in v)\big(z = \langle x, y \rangle\big)$.
b. x is the first coordinate of z iff $(\exists v \in z)(\exists y \in v)\big(z = \langle x, y \rangle\big)$.
c. y is the second coordinate of z iff $(\exists u \in z)(\exists x \in u)\big(z = \langle x, y \rangle\big)$.

Bibliography

[1] A. V. Arkhangel′skiĭ, *On the cardinality of bicompacta satisfying the first axiom of countability*, Soviet Mathematics Doklady **10** (1969), 951–955. MR 40#4922

[2] Zoltan T. Balogh, *A small Dowker space in* ZFC, Proceedings of the American Mathematical Society **124** (1996), 2555–2560. MR 96j:54026

[3] Maxim R. Burke and Menachem Magidor, *Shelah's* pcf *theory and its applications*, Annals of Pure and Applied Logic **50** (1990), no. 3, 207–254. MR 92f:03053

[4] Ryszard Engelking, *General topology. revised and completed edition*, Sigma Series in Pure Mathematics, no. 6, Heldermann Verlag, Berlin, 1989. MR 91c:54001

[5] Menachem Kojman and Saharon Shelah, *A ZFC Dowker space in* $\aleph_{\omega+1}$*: an application of PCF theory to topology*, Proceedings of the American Mathematical Society **126** (1998), no. 8, 2459–2465. MR 98m:54011

[6] Kenneth Kunen, *Set theory. an introduction to independence proofs*, Studies in Logic and the foundations of mathematics, no. 102, North-Holland, Amsterdam, 1980b. MR 82f:03001

[7] Mary Ellen Rudin, *A normal space X for which $X \times I$ is not normal*, Fundamenta Mathematicae **73** (1971/72), no. 2, 179–186. MR 45 #2660

[8] ———, *Two problems of Dowker*, Proceedings of the American Mathematical Society **91** (1984), 155–158. MR 85i:54022b