THE STOCHASTIC FUBINI THEOREM REVISITED

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ABSTRACT. In this note we present an elementary and self-contained proof of the stochastic Fubini theorem, which states that one can interchange a Lebesgue integral and a stochastic integral. The integrability conditions we use are weaker and more natural than the usual conditions in the literature. In particular, we do not need integrability in Ω , and we use L^1 -integrability instead of L^2 -integrability in the additional parameter.

1. INTRODUCTION

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space with a complete and right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$. Let $S = M + A$ be a real-valued continuous semimartingale. Moreover, (X, Σ, μ) is a σ -finite measure space. Finally, it is convenient to fix a final time $T \in (0, \infty)$. With minor changes one can also take $T = \infty$.

In this paper we find conditions under which we can interchange a Lebesgue and a stochastic integral for a stochastic process depending on an additional parameter $x \in X$, i.e. $\phi : X \times [0, T] \times \Omega \rightarrow \mathbb{R}$ and ϕ satisfies certain measurability properties. There is already an extensive literature on such Fubini theorems (see [1, 2, 4, 5, 11, 12]). In this paper we prove the Fubini theorem under much weaker integrability conditions. To be more precise, we give sufficient conditions on ψ under which the following holds: almost surely one has

$$(1.1) \quad \int_X \int_0^t \psi(s, x) dS(s) d\mu(x) = \int_0^t \int_X \psi(s, x) d\mu(x) dS(r), \quad t \in [0, T].$$

In order to compare our results to the existing results it is convenient to replace S by standard real-valued Brownian motion w in this introduction. After this comparison we return to the general framework of arbitrary continuous semimartingales. Let $\psi : X \times [0, T] \times \Omega \rightarrow \mathbb{R}$ be jointly measurable and adapted, in the sense that for all $x \in X$, the process $\psi(x, \cdot, \cdot)$ is adapted. Replacing S by w , the identity in (1.1) takes the form

$$(1.2) \quad \int_X \int_0^t \psi(s, x) dw(s) d\mu(x) = \int_0^t \int_X \psi(s, x) d\mu(x) dw(r), \quad t \in [0, T].$$

Here $\psi : X \times [0, T] \times \Omega \rightarrow \mathbb{R}$ is jointly measurable and adapted (see Section 2 for details). It seems natural that for the integral with respect to μ an $L^1(X)$ -condition is needed, and for the stochastic integral with respect to w an $L^2(0, T)$ condition is needed. Most versions of the stochastic Fubini theorem in the literature use more

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integrability than this. Two popular integrability conditions are (cf. [2, Theorem 4.18] and [11, Theorem IV.65]):

$$(1.3) \quad \left(\int_0^T \int_X |\psi(x, t)|^2 d\mu(x) dt \right)^{1/2} < \infty \text{ almost surely and } \mu(X) < \infty.$$

$$(1.4) \quad \int_X \left(\mathbb{E} \int_0^T |\psi(x, t)|^2 dt \right)^{1/2} d\mu(x) < \infty,$$

where $\mathbb{E}$ stands for expectation. Condition 1.3 is unnatural because $L^2(X)$ -integrability is required. Condition (1.4) uses $L^1(X)$ -integrability, but is also unnatural since integrability in Ω is assumed. It is unclear how to localize (1.4), because the $L^1(X)$ norm is in the wrong order.

In [10] (1.2) has been proved under the assumption that

$$(1.5) \quad \int_X \left(\int_0^T |\psi(x, t)|^2 dt \right)^{1/2} d\mu(x) < \infty \text{ almost surely.}$$

The condition (1.5) is weaker than (1.3) and (1.4). The proof in [10] is based vector-valued stochastic integration for processes with values in $L^1(X)$. Recently, in [8] (1.2) has been proved under the condition (1.5) in a different way.

In this paper we give a self-contained and elementary proof of the stochastic Fubini theorem for continuous semimartingales S under a condition which reduces to (1.5) if $w = S$. Some ideas in the proof are inspired by the methods in [10], but we do not use vector-valued stochastic integration theory. We believe our approach is easier than that of [8]. Moreover, it covers the more general situation of continuous local martingales. The method of proof is robust in the sense that the main ideas could also be extended to other situations: vector-valued stochastic integrals, operator-valued integrands, etc. Under more measurability assumptions we expect that the results can also be extended to noncontinuous semimartingales. To keep the proofs transparent, we only consider integration with respect to continuous semimartingales.

It is quite a technical issue what measurability of the function ψ is required. There are many possibilities since ψ depends on three variables. For integration with respect to continuous semimartingales it seems most natural to assume progressive measurability (see Section 2 for details). However, if the semimartingale is just standard Brownian motion, then adaptedness suffices.

Recall that for $p \in [1, \infty)$ and a Banach space E , the space $L^p(\Omega; E)$ is the usual space of E -valued L^p -integrable random variables (see [3, Chapter 2]) equipped with the norm $\|\xi\|_{L^p(\Omega; E)} = (\mathbb{E}(\|\xi\|_E^p))^{1/p}$. The space $L^0(\Omega; E)$ is the complete metric space of random variables with the translation-invariant metric $\|\xi\|_{L^0(\Omega; E)} = \mathbb{E}(\|\xi\|_E \wedge 1)$. Moreover, $C([0, T]; E)$ is the space of continuous E -valued functions equipped with the norm $\|f\|_{C([0, T]; E)} = \sup_{t \in [0, T]} \|f(t)\|_E$.

2. MAIN RESULT

Let M be a continuous local martingale and let $[M] : [0, T] \times \Omega \rightarrow \mathbb{R}_+$ be its quadratic variation. Recall that $[M]$ is pathwise continuous and nondecreasing and $[M](0, \omega) = 0$ for almost all $\omega \in \Omega$. If M is a standard Brownian motion, then for almost all $\omega \in \Omega$, $[M](t, \omega) = t$. Recall that a process $\phi : [0, T] \times \Omega \rightarrow \mathbb{R}$ is progressively measurable if for all $t \in [0, T]$, $\phi|_{[0, t]}$ is $\mathcal{B}_{[0, t]} \otimes \mathcal{F}_t$ -measurable. Let $\mathcal{P}$

be the σ -algebra on $[0, T] \times \Omega$ generated by all progressively measurable processes. The σ -algebra $\mathcal{P}$ is called the progressive σ -algebra and one can show

$$\mathcal{P} = \{A \in \mathcal{B}_{[0, T]} \otimes \mathcal{A} : \mathbf{1}_A \text{ is progressively measurable}\}.$$

For $p \in [0, \infty]$, let $\mathcal{L}^p(M)$ be the closure of all progressively measurable processes $\phi : [0, T] \times \Omega \rightarrow \mathbb{R}$, with respect to

$$\|\phi\|_{\mathcal{L}^p(M)} := \left\| \left(\int_0^T |\phi(t)|^2 d[M](t) \right)^{1/2} \right\|_{L^p(\Omega)}.$$

Moreover, let $\mathcal{L}(M) := \mathcal{L}^0(M)$. Recall that every $\phi \in \mathcal{L}(M)$ is stochastically integrable with respect to M and $t \mapsto \int_0^t \phi(r, \cdot) dM(r)$ defines a continuous local martingale (see [6, Chapter 17] or [7, Chapter 3]). Let $A : [0, T] \times \Omega \rightarrow \mathbb{R}$ be a continuous and adapted process which is of finite variation and satisfies $A(0, \omega) = 0$ for all $\omega \in \Omega$. Let $L(A)$ be the class of progressively measurable processes for which $\int_0^T |\phi(t, \omega)| |dA(t, \omega)| < \infty$ for almost all $\omega \in \Omega$, where $|dA(t, \omega)|$ is the absolute value of the Lebesgue–Stieltjes measure associated to $A(\cdot, \omega)$. For a continuous semimartingale $S = M + A$ we write

$$\mathbb{L}(S) = \mathcal{L}(M) \cap L(A).$$

For $\phi \in \mathbb{L}(S)$ one can define

$$\int_0^t \phi(s) dS(s) = \int_0^t \phi(s) dM(s) + \int_0^t \phi(s) dA(s), \quad t \in [0, \infty),$$

and as a function of $t \in [0, T]$ and Ω , this is a continuous semimartingale again.

Let (X, Σ, μ) σ -finite measure space. We say that a function $\psi : X \times [0, T] \times \Omega \rightarrow \mathbb{R}$ is *progressively measurable* if for all $t \in [0, T]$, $\psi|_{[0, t]}$ is $\Sigma \otimes \mathcal{B}_{[0, t]} \otimes \mathcal{F}_t$.

Remark 2.1. In particular, every $\Sigma \otimes \mathcal{P}$ -measurable function ψ is progressively measurable. The reverse implication might not hold in general, i.e. it is unclear whether every progressively measurable function ψ is also $\Sigma \otimes \mathcal{P}$ -measurable.

The following stochastic Fubini theorem for integration with respect to continuous semimartingales is the main result of this paper:

Theorem 2.2. *Let $S = M + A$ be a continuous semimartingale. Let $\psi : X \times [0, T] \times \Omega \rightarrow \mathbb{R}$ be progressively measurable and such that for almost all $\omega \in \Omega$,*

$$(2.1) \quad \int_X \left(\int_0^T |\psi(x, t, \omega)|^2 d[M](t, \omega) \right)^{\frac{1}{2}} d\mu(x) < \infty,$$

$$(2.2) \quad \int_X \int_0^T |\psi(x, t, \omega)| |dA|(t, \omega) d\mu(x) < \infty.$$

Then the following assertions hold:

- (1) *For almost all $x \in X$, $\psi(x, \cdot, \cdot)$ is in $\mathbb{L}(S)$ and the process $\zeta : X \times [0, T] \times \Omega \rightarrow \mathbb{R}$ defined by*

$$\zeta(x, t, \omega) := \int_0^t \psi(x, r, \cdot) dS(r)$$

is measurable and for almost all $\omega \in \Omega$,

$$\sup_{t \in [0, T]} \int_X |\zeta(x, t, \omega)| d\mu(x) < \infty.$$

- (2) For almost all $(t, \omega) \times [0, T] \times \Omega$, the function $x \mapsto \psi(x, t, \omega)$ is integrable and the process $\eta : [0, T] \times \Omega \rightarrow \mathbb{R}$ defined by

$$\eta(t, \omega) = \int_X \psi(x, t, \omega) d\mu(x)$$

is in $\mathbb{L}(S)$.

- (3) For almost all $\omega \in \Omega$ for all $t \in [0, T]$, one has

$$(2.3) \quad \int_X \zeta(x, t, \omega) d\mu(x) = \left(\int_0^t \eta(r, \cdot) dS(r) \right)(\omega).$$

The Fubini theorem for integration with respect to A under the condition (2.2) is well-known. The novelty of the above result lies in the Fubini theorem for integration with respect to M under the assumption (2.1).

For each $\omega \in \Omega$, let $F(\omega)$ denote the space of progressively measurable $f : X \times [0, T] \times \Omega \rightarrow \mathbb{R}$ for which

$$\|f\|_{F(\omega)} := \int_X \left(\int_0^T |f(x, t)|^2 d[M](t, \omega) \right)^{\frac{1}{2}} d\mu(x) < \infty.$$

Clearly, $\|f\|_{F(\omega)}$ is measurable and we can write $\mathbb{E}\|f\|_F := \int_\Omega \|f\|_{F(\omega)} d\mathbb{P}(\omega)$.

In the proof of the theorem we need the following lemma which is well-known to experts. For convenience its proof will be given at the end of the paper.

Lemma 2.3. *Let $\psi : X \times [0, T] \times \Omega \rightarrow \mathbb{R}$ be progressively measurable and such that for almost all $\omega \in \Omega$, $\|\psi(\cdot, \cdot, \omega)\|_{F(\omega)} < \infty$. Then there exist functions $\psi_1, \psi_2, \dots : X \times [0, T] \times \Omega \rightarrow \mathbb{R}$ of the form*

$$(2.4) \quad \psi_n(x, t, \omega) = \sum_{k=1}^{K_n} f_k^n(x) \phi_k^n(t, \omega),$$

with $f_k^n \in L^1(X)$ a simple function, and each $\phi_k^n \in \mathcal{L}(M)$ a uniformly bounded step process such that for almost all $\omega \in \Omega$,

$$\lim_{n \rightarrow \infty} \|\psi(\cdot, \cdot, \omega) - \psi_n(\cdot, \cdot, \omega)\|_{F(\omega)} = 0.$$

In the case M is a Brownian motion, the proof of the lemma only requires adaptedness and joint measurability of ψ .

Proof of Theorem 2.2. By the Fubini theorem for σ -finite measures one can interchange the integrals with respect A and μ . Therefore, in the sequel it suffices to consider the case $S = M$.

Step 1: A preliminary maximal inequality

Let ψ be a function of the form

$$(2.5) \quad \psi(x, t, \omega) = \sum_{n=1}^N f_n(x) \phi_n(t, \omega),$$

where $f_n \in L^1(X)$, $n = 1, \dots, N$, and ϕ_n in $\mathcal{L}^1(M)$, $n = 1, \dots, N$. Let

$$(2.6) \quad \zeta(x, t, \omega) = \left(\int_0^t \psi(x, r, \cdot) dM(r) \right)(\omega) = \sum_{n=1}^N f_n(x) \left(\int_0^t \phi_n(r, \cdot) dM(r) \right)(\omega).$$

Then ζ is measurable and for all $x \in X$ and $\omega \in \Omega$, $\zeta(x, \cdot, \omega)$ is continuous. Moreover, for every $x \in X$, $\zeta(x, \cdot, \cdot)$ is a continuous martingale with quadratic variation $[\zeta(x, \cdot, \cdot)]$ given by

$$[\zeta(x, \cdot, \cdot)](s, \omega) = \int_0^s |\psi(x, t, \omega)|^2 d[M](t, \omega).$$

The Burkholder-Davis-Gundy inequalities (see [6, Theorem 17.7]) applied pointwise in X imply that there is a constant $C \geq 0$ such that

$$(2.7) \quad \begin{aligned} \int_{\Omega} \sup_{t \in [0, T]} \int_X |\zeta(x, t, \omega)| d\mu(x) d\mathbb{P}(\omega) &\leq \int_{\Omega} \int_X \sup_{t \in [0, T]} |\zeta(x, t, \omega)| d\mathbb{P}(\omega) d\mu(x) \\ &\leq C \int_{\Omega} \int_X \left(\int_0^T |\psi(x, t, \omega)|^2 d[M](t, \omega) \right)^{\frac{1}{2}} d\mathbb{P}(\omega) d\mu(x) = C\mathbb{E}\|\psi\|_F. \end{aligned}$$

Let $A, B : [0, T] \times \Omega \rightarrow \mathbb{R}$ be defined by

$$A(t, \omega) = C\|\psi(\cdot, \cdot, \omega)\mathbf{1}_{[0, t]}\|_{F(\omega)}, \quad B(t, \omega) = \int_X |\zeta(x, t, \omega)| d\mu(x).$$

Then both A and B are adapted and have continuous paths almost surely. Indeed, this follows from the dominated convergence theorem and (2.6). For any stopping-time τ with values in $[0, T]$ one has $\mathbb{E}B(\tau) \leq \mathbb{E}A(\tau)$. Indeed, this follows from (see [6, Proposition 17.15])

$$\zeta(x, \tau) = \int_0^T \mathbf{1}_{[0, \tau]}(r) \psi(x, r, \cdot) dM(r)$$

and (2.7) applied to $\mathbf{1}_{[0, \tau]}\psi$. Now a standard stopping-time argument (see [9]) implies that for all $\epsilon, \delta > 0$,

$$(2.8) \quad \mathbb{P}\left(\omega \in \Omega : \sup_{t \in [0, T]} \int_X |\zeta(x, t, \omega)| d\mu(x) \geq \epsilon\right) \leq C \frac{\mathbb{E}(\|\psi\|_F \wedge \delta)}{\epsilon} + \mathbb{P}(\|\psi\|_F \geq \delta).$$

Step 2: Approximation

Assume ψ is progressively measurable and satisfies (2.1). By Fubini's theorem we can find a set $X_0 \in \Sigma$ with $\mu(X \setminus X_0) = 0$ and such that for all $x \in X_0$ for almost all $\omega \in \Omega$,

$$\int_0^T |\psi(x, t, \omega)|^2 d[M](t, \omega) < \infty.$$

Since by Fubini's theorem $\psi(x, \cdot)$ is progressively measurable, it follows that for each $x \in X_0$ one can define

$$\zeta(x, t, \omega) = \int_0^t \psi(x, s, \cdot) dM(s)$$

and for $x \in X \setminus X_0$, we set $\zeta(x, t, \omega) = 0$. By Lemma 2.3 we can find a sequence of functions $(\psi_n)_{n \geq 1}$ of the form (2.5) such that for almost all $\omega \in \Omega$,

$$(2.9) \quad \lim_{n \rightarrow \infty} \|\psi(\cdot, \cdot, \omega) - \psi_n(\cdot, \cdot, \omega)\|_{F(\omega)} = 0.$$

Step 2a: The left-hand side of (2.3).

For each $n \geq 1$ let $\zeta_n \in L(\Omega; C([0, T]; L^1(X)))$ be defined by

$$\zeta_n(x, t, \omega) = \left(\int_0^t \psi_n(x, s, \cdot) dM(s) \right)(\omega).$$

It follows from (2.8) (applied with $\zeta_n - \zeta_m$) that $(\zeta_n)_{n \geq 1}$ is a Cauchy sequence in $L^0(\Omega; C([0, T]; L^1(X)))$. Therefore, $(\zeta_n)_{n \geq 1}$ converges to some $\tilde{\zeta}$ in $L^0(\Omega; C([0, T]; L^1(X)))$.

By taking subsequences one can identify this limit with ζ , in the sense that for almost all $\omega \in \Omega$, for all $t \in [0, T]$, for almost all $x \in X$, $\tilde{\zeta}(x, t, \omega) = \zeta(x, t, \omega)$. In the sequel we just replace ζ by $\tilde{\zeta}$. In particular, it follows that for all $\delta > 0$

$$(2.10) \quad \begin{aligned} & \mathbb{P}\left(\sup_{t \in [0, T]} \left| \int_X \zeta(x, t, \cdot) d\mu(x) - \int_X \zeta_n(x, t, \cdot) d\mu(x) \right| > \delta\right) \\ & \leq \mathbb{P}\left(\sup_{t \in [0, T]} \int_X |\zeta(x, t, \cdot) - \zeta_n(x, t, \cdot)| d\mu(x) > \delta\right) \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$.

Step 2b: the right-hand side of (2.3)

By Fubini's theorem and Hölder's inequality one has

$$\int_0^T \int_X |\psi(x, t, \omega)| d\mu(x) d[M](t, \omega) \leq \sqrt{T} \|\psi(\cdot, \cdot, \omega)\|_{F(\omega)} < \infty \quad \text{a.a. } \omega \in \Omega.$$

Therefore, for almost all $(t, \omega) \in [0, T] \times \Omega$, we can define $\eta : [0, T] \times \Omega \rightarrow \mathbb{R}$ as in (2). By Fubini's theorem η is progressively measurable. Let $\eta_n(t, \omega) = \int_X \psi_n(t, \omega) d\mu(x)$ for $n \geq 1$, where ψ_n is as in (2.9). By Fubini's theorem, η_n is progressively measurable. By Minkowski's inequality we obtain for almost all $\omega \in \Omega$,

$$\left(\int_0^T |\eta(t, \omega) - \eta_n(t, \omega)|^2 d[M](t, \omega) \right)^{1/2} \leq \|\psi(\cdot, \cdot, \omega) - \psi_n(\cdot, \cdot, \omega)\|_{F(\omega)},$$

and the latter goes to zero for almost all $\omega \in \Omega$. This shows that $\eta \in \mathcal{L}(M)$ and by [6, Lemma 17.12] we obtain that

$$(2.11) \quad \sup_{t \in [0, T]} \left| \int_0^t \eta(r, \cdot) dM(r) - \int_0^t \eta_n(r, \cdot) dM(r) \right| \rightarrow 0$$

in probability as $n \rightarrow \infty$.

Step 3: Conclusion

Clearly, for almost all $\omega \in \Omega$, for all $t \in [0, T]$,

$$\int_X \zeta_n(x, t, \omega) d\mu(x) = \left(\int_0^t \eta_n(r, \cdot) dM(r) \right)(\omega).$$

Now the result follows from (2.10) and (2.11). Indeed, for all $\epsilon > 0$, one has

$$\begin{aligned} & \mathbb{P}\left(\sup_{t \in [0, T]} \left| \int_X \zeta(x, t, \omega) d\mu(x) - \left(\int_0^t \eta(r, \cdot) dM(r) \right)(\omega) \right| > \epsilon\right) \\ & = \lim_{n \rightarrow \infty} \mathbb{P}\left(\sup_{t \in [0, T]} \left| \int_X \zeta_n(x, t, \omega) d\mu(x) - \left(\int_0^t \eta_n(r, \cdot) dM(r) \right)(\omega) \right| > \epsilon\right) = 0, \end{aligned}$$

and this implies (3). $\square$

Proof of Lemma 2.3. Let $(X_n)_{n \geq 1}$ in Σ be increasing with $\mu(X_n) < \infty$ for all $n \geq 1$, and such that $X = \bigcup_{n \geq 1} X_n$. Let

$$\tau_n = \inf\{t \in [0, T] : [M](t) \leq n\}.$$

For almost all $\omega \in \Omega$, the dominated convergence theorem implies that

$$\lim_{n \rightarrow \infty} \|\psi(\cdot, \cdot, \omega) - \psi(\cdot, \cdot, \omega) \mathbf{1}_{|\psi(\cdot, \cdot, \omega)| \leq n} \mathbf{1}_{X_n} \mathbf{1}_{[0, \tau_n(\omega)]}\|_{F(\omega)} = 0.$$

Therefore, we can assume that ψ is uniformly bounded, X is a finite measure space and M is a continuous martingale with $[M]_T$ uniformly bounded.

Let $\epsilon > 0$ be arbitrary. We show that there is a $\tilde{\psi} : X \times [0, T] \times \Omega \rightarrow \mathbb{R}$ of the form (2.4) such that

$$(2.12) \quad R := \int_X \int_\Omega \int_0^T |\tilde{\psi}(x, s, \omega) - \psi(x, s, \omega)|^2 d[M](s, \omega) d\mathbb{P}(\omega) d\mu(x) < \epsilon^2 / \mu(X).$$

As soon as we have shown this, it follows from Fubini's theorem and Hölder's inequality that $\mathbb{E} \|\tilde{\psi} - \psi\|_F^2 \leq \mu(X)R < \epsilon^2$, and therefore completes the proof.

Fix $\omega \in \Omega$ and $x \in X$. We now proceed as in [6, Lemma 17.3]. However, some measurability issues need to be checked since ψ depends on an additional parameter. For convenience of the reader we present the details. Define the strictly increasing process $A : [0, T] \times \Omega \rightarrow \mathbb{R}$ by $A(t, \omega) = [M](t, \omega) + t$. Let $\tau(s, \omega) = \sup\{t \in [0, T] : A(t, \omega) \leq s\}$. For $h > 0$, let

$$\begin{aligned} \psi^h(x, t, \omega) &= h^{-1} \int_{\tau((A(t, \omega) - h)_+)}^t \psi(x, s, \omega) dA(s, \omega) \\ &= h^{-1} \int_{(A(t, \omega) - h)_+}^{A(t, \omega)} \psi(x, \tau(s, \omega), \omega) ds. \end{aligned}$$

Note that

$$\psi^h(x, \tau(r, \omega), \omega) = \int_{(r-h)_+}^r \psi(x, \tau(s, \omega), \omega) ds$$

It follows from the Lebesgue differentiation theorem that for almost all $r \in [0, A(T, \omega)]$

$$\lim_{h \downarrow 0} \psi^h(x, \tau(r, \omega), \omega) = \psi(x, \tau(r, \omega), \omega).$$

Therefore, by the dominated convergence theorem we have

$$\begin{aligned} &\int_0^T |\psi^h(x, s, \omega) - \psi(x, s, \omega)|^2 dA(s, \omega) \\ &= \int_0^{A(T, \omega)} |\psi^h(x, \tau(s, \omega), \omega) - \psi(x, \tau(s, \omega), \omega)|^2 ds \rightarrow 0 \end{aligned}$$

as $h \downarrow 0$. Another application of the dominated convergence theorem shows that

$$(2.13) \quad \int_X \int_\Omega \int_0^T |\psi^h(x, s, \omega) - \psi(x, s, \omega)|^2 dA(s, \omega) d\mathbb{P}(\omega) d\mu(x) \rightarrow 0$$

as $h \downarrow 0$.

For all (x, ω) one has $s \mapsto \psi^h(x, s, \omega)$ is continuous. Moreover, for each $s \in [0, T]$, $\psi^h(\cdot, s, \cdot)$ is $\Sigma \otimes \mathcal{F}_t$ measurable. Indeed, let $f : \Sigma \times [0, T] \times \Omega \rightarrow \mathbb{R}$ be defined by

$$f(x, t, \omega) = \int_0^t \psi(x, s, \omega) dA(s, \omega).$$

Then for all $x \in X$ and $\omega \in \Omega$ the function f is continuous. Moreover, for all $t \in [0, T]$, $(x, \omega) \mapsto f(x, t, \omega)$ is $\Sigma \otimes \mathcal{F}_t$ -measurable. Therefore, f is progressively measurable. Note that $\tau((A(\cdot) - h)_+)$ is adapted. Fix $t \in [0, T]$. The mapping $v : X \times \Omega \rightarrow X \times [0, t] \times \Omega$ defined by

$$v(x, \omega) = (x, \tau((A(\omega) - h)_+), \omega)$$

is $(\Sigma \otimes \mathcal{F}_t) \rightarrow (\Sigma \otimes \mathcal{B}_{[0,t]} \otimes \mathcal{F}_t)$ -measurable. Therefore, $(x, \omega) \mapsto f(x, \tau((A(t, \omega) - h)_+), \omega)$ is $\Sigma \otimes \mathcal{F}_t$ -measurable as it is the composition $f|_{[0,t]} \circ v$, where $f|_{[0,t]}$ is $\Sigma \otimes \mathcal{B}_{[0,t]} \otimes \mathcal{F}_t$ -measurable.

Let $\epsilon > 0$ be arbitrary, and choose $h > 0$ such that the left-hand side of (2.13) is smaller than $\epsilon/3$. For all integers $n \geq 1$ let $\psi^{h,n} : X \times [0, T] \times \Omega \rightarrow \mathbb{R}$ be defined by

$$\psi^{h,n}(x, t, \omega) = \sum_{i=1}^n \mathbf{1}_{(t_{i-1}^n, t_i^n]}(t) \psi^h(x, t_{i-1}^n, \omega).$$

Here $t_i^n = Ti/n$. Then $\psi_n^h \rightarrow \psi^h$ pointwise as $n \rightarrow \infty$. Moreover, by the dominated convergence theorem

$$(2.14) \quad \int_X \int_\Omega \int_0^T |\psi^{h,n}(x, s, \omega) - \psi^h(x, s, \omega)|^2 dA(s, \omega) d\mathbb{P}(\omega) d\mu(x) \rightarrow 0$$

as $n \rightarrow \infty$. Choose n such that the left-hand side is smaller than $\epsilon/3$. For each $0 \leq i \leq n$, we can find a simple function $\xi_i : X \times \Omega \rightarrow \mathbb{R}$ which is a finite linear combination of functions $\mathbf{1}_{X_i} \mathbf{1}_{\Omega_i}$ with $X_i \in \Sigma$ and $\Omega_i \in \mathcal{F}_{t_{i-1}^n}$ such that

$$\|\xi - \psi^h(\cdot, t_{i-1}^n, \cdot)\|_{L^2(X \times \Omega)} < \epsilon/(3n).$$

Now let $\psi = \sum_{i=1}^n \mathbf{1}_{(t_{i-1}^n, t_i^n]} \xi_i$. Then by the triangle inequality we have

$$(2.15) \quad \int_X \int_\Omega \int_0^T |\psi(x, s, \omega) - \psi^h(x, s, \omega)|^2 dA(s, \omega) d\mathbb{P}(\omega) d\mu(x) < \epsilon.$$

□

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