Functional Calculus Methods for Evolution Equations

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1 Semigroups

In this first chapter we shall introduce the notion of a strongly continuous semigroup and its generator. There are various ways to do this, and we have chosen to follow [2] starting from the Cauchy problem instead of [4], which begins with the semigroup itself. The presentation is self-contained, although we assume that the reader is familiar with basic operator theory on Banach spaces. Looking at such a distinguished reference as [4] it is obvious that the material included here is quite sparse. However, our focus is on functional calculus and not on semigroups.

An abstract Cauchy problem has the form

\[ (ACP)_x \quad \frac{d}{dt} u(t) = A u(t), \quad u(0) = x \in X \]

where \( X \) is a Banach space and \( A \) is a closed linear operator on \( X \). This means that \( \mathcal{D}(A) \subseteq X \) is a linear subspace called the domain of \( A \) and \( A : \mathcal{D}(A) \to X \) is linear. Moreover, the graph of \( A \)

\[ \text{graph}(A) := \{(x, Ax) \mid x \in \mathcal{D}(A)\} \]

is a closed subspace of \( X \oplus X \). A (mild) solution of \( (ACP)_x \) is a continuous mapping \( u : [0, \infty) \to X \) such that \( u(0) = x \) and

\[ \int_0^t u(s) \, ds \in \mathcal{D}(A) \quad \text{and} \quad A \int_0^t u(s) \, ds = u(t) - u(0) \quad (t > 0). \]

(Since \( A \) is closed, this implies that \( u(0) = x \).)

Exercise 1: A classical solution of \( (ACP)_x \) is a function \( u \in C^1(\mathbb{R}_+; X) \) such that \( u(0) = x \) and

\[ u(t) \in \mathcal{D}(A), \quad u'(t) = A u(t) \quad (t \geq 0). \]
Prove that every classical solution is a mild solution.

We say that the (ACP) for $A$ is **well-posed** if for each $x \in X$ there is a **unique mild solution** of $(ACP)_x$. In the following, we shall suppose that we are given a well-posed (ACP) associated with an operator $A$. Let us introduce the mapping

$$u : \mathbb{R}_+ \times X \to X$$

where $u(\cdot, x) : \mathbb{R}_+ \to X$ is the unique solution of $(ACP)_x$, for each $x \in X$. For each $t \geq 0$ we write

$$T(t) := u(t, \cdot) : X \to X.$$

Here are the main properties of these operators.

**1.1 Theorem** Let the (ACP) associated with a closed linear operator $A$ on a Banach space $X$ be well-posed, and let $T := (T(t))_{t \geq 0}$ be defined as above. Then the following assertions hold:

a) Each $T(t)$ is a bounded linear operator.

b) $T$ satisfies the semigroup law, i.e.

$$T(0) = I, \quad T(t + s) = T(t)T(s) \quad (t, s \geq 0).$$

c) $T$ is strongly continuous, i.e., $t \mapsto T(t)x : \mathbb{R}_+ \to X$ is continuous for all $x \in X$.

d) If $x \in \mathcal{D}(A)$, then $T(t)x \in \mathcal{D}(A)$ and $AT(t)x = T(t)Ax$ for all $t \geq 0$. Moreover, $T(\cdot)x$ is a classical solution of $(ACP)_x$.

**Proof.**

a) Let $x, y \in X$. Then $u(\cdot, x) + u(\cdot, y)$ is a mild solution of $(ACP)_{x+y}$, hence by uniqueness must be equal to $u(\cdot, x+y)$. Analogously for $\lambda x$, $\lambda \in \mathbb{C}$.

To prove boundedness (= continuity), consider the mapping

$$T := (x \mapsto T(\cdot)x) : X \to C(\mathbb{R}_+; X) =: \mathcal{X}.$$

Note that $\mathcal{X}$ is a Fréchet space. By the Closed Graph Theorem [9, Theorem 2.15] it suffices to prove that the graph of $T$ is closed. So suppose that $x_n \to x$ in $X$ and $Tx_n \to u$ within $\mathcal{X}$. This means that $T(t)x_n \to u(t)$ uniformly in $t$ from finite intervals. Using the closedness of $A$ one readily proves that $u$ is a mild solution, and since $u(0) = \lim_n T(0)x_n = x$, $u(t) = T(t)x$, i.e. $u = Tx$.

b) $T(0) = I$ is clear. Let $t, s \geq 0$. Let $v(t) := T(s + t)x$. Then

$$\int_0^t v(r) \, dr = \int_s^{s+t} T(r)x \, dr = \int_0^{t+s} T(r)x \, dr - \int_0^s T(r)x \, dr \in \mathcal{D}(A),$$
and
\[ A \int_0^t v(r) \, dr = A \int_0^{t+s} T(r) x \, dr - A \int_s^t T(r) x \, dr = T(t+s) x - x - [T(s)x - x] \]
\[ = T(t+s) x - T(s)x = v(t) - v(0). \]
Hence by uniqueness \( v(t) = u(t, T(s)x) = T(t)T(s)x. \)

c) The strong continuity is clear from the construction.
d) Let \( x \in D(A) \) and define \( v(t) = \int_0^t T(s)Axds + x. \) Then \( v(t) \in D(A) \) and \( Av(t) = T(t)Ax - Ax + Ax = T(t)Ax, \) for all \( t \geq 0. \) On the other hand, \( v \) is continuously differentiable with \( v'(t) = T(t)Ax, \) and so \( v \) is a classical solution of \( (ACP)_x. \) Hence by uniqueness, \( v(t) = T(t)x. \) The assertions follow. \( \Box \)

A family of \( (T(t))_{t \geq 0} \) of operators on a Banach space \( X \) is called a strongly continuous semigroup (or: \( C_0 \)-semigroup) if it has the properties a)–c) of Theorem 1.1. In the case that it is induced by an \( (ACP) \) associated with an operator \( A, \) then \( A \) is called the generator of the semigroup. We shall see in a moment that each \( C_0 \)-semigroup has a unique generator.

1.2 Fact Let \( T := (T(t))_{t \geq 0} \) be a \( C_0 \)-semigroup on the Banach space \( X. \) Then there are constants \( M \geq 1, \omega \in \mathbb{R} \) such that
\[ \|T(t)\| \leq Me^{\omega t} \quad (t \geq 0). \]
Moreover, defining
\[ R_\lambda x := \int_0^\infty e^{-\lambda t}T(t)x \, dt \quad (x \in X, \text{Re} \lambda > \omega) \]
one has
1) \( \|R_\lambda\| \leq \frac{M}{\text{Re} \lambda - \omega} \) \( (\text{Re} \lambda > \omega), \]
2) \( R_\lambda - R_\mu = (\mu - \lambda)R_\mu R_\lambda \) \( (\text{Re} \lambda, \text{Re} \mu > \omega), \]
3) \( \lim_{\lambda \to +\infty} \lambda R_\lambda x = x \) \( (x \in X). \)
The identity 2) is called the resolvent identity. The number
\[ \omega_0(T) := \inf\{\omega \in \mathbb{R} \mid \exists M \geq 1 : (1.1) \text{ holds}\} \in [-\infty, \infty) \]
is called the (exponential) growth bound of \( T. \)

Proof. Since \( T \) is strongly continuous, \( \sup_{t \in [0,1]} \|T(t)x\| < \infty \) for every \( x \in X. \) By the uniform boundedness principle, \( M := \sup_{t \in [0,1]} \|T(t)\| < \infty. \) Now one
uses the semigroup property. Write an arbitrary $t \geq 0$ as $t = n + r$ with $n \in \mathbb{N}_0$ and $r \in [0,1)$. Then $\|T(t)\| = \|T(r)T(1)^n\| \leq MM^n \leq MM^t = Me^{t \log M}$.

The estimate 1) is now straightforward, noting that $|e^{-\lambda t}| = e^{-t \Re \lambda}$.

To prove 2) take $\lambda, \mu \in \{\Re z > \omega\}$ and compute

$$R_\lambda R_\mu = \int_0^\infty e^{-\lambda t} T(t) \int_0^t e^{-\mu s} T(s) x \, ds = \int_0^\infty e^{-\lambda t} \int_0^t e^{-\mu s} T(t + s) x \, ds \, dt$$

$$= \int_0^\infty e^{-\lambda t} \int_0^t e^{-\mu (s + t)} T(s) x \, ds \, dt = \int_0^\infty e^{-\mu s} \int_0^s e^{-(\lambda - \mu) \tau} T(s) x \, ds \
= \int_0^\infty e^{-\mu s} \frac{e^{-(\lambda - \mu) s} - 1}{\mu - \lambda} T(s) x \, ds = \frac{R_\lambda x - R_\mu x}{\mu - \lambda}.$$ 

To prove 3) let $x \in X$, $\delta > 0$ and $\lambda > \omega, 0$. Then

$$\lambda R_\lambda x - x = \int_0^\delta \lambda e^{-\lambda t} T(t) x \, dt + \int_\delta^\infty \lambda e^{-\lambda t} T(t) x \, dt - e^{-\lambda \delta} x,$$

hence

$$\|\lambda R_\lambda x - x\| \leq \sup_{0 \leq t \leq \delta} \|T(t) x - x\| + \left( Me^{\omega \delta} \frac{\lambda}{\lambda - \omega} + 1 \right) e^{-\lambda \delta} \|x\|.$$

Given $\varepsilon > 0$ one first chooses $\delta > 0$ such that $\|T(t) x - x\| < \varepsilon$ for $t \in [0, \delta]$. Then as $\lambda \to \infty$ the second summand becomes arbitrarily small. □

By combining 2) and 3) one sees that each operator $R_\lambda$, $(\Re \lambda > \omega)$ is injective. The resolvent identity now implies that all the operators $R_\lambda$ are linked via one single (unbounded) operator. Before we can formulate the result we have to recall some basic spectral theory for unbounded operators.

For a general closed operator $A : \mathcal{D}(A) \to X$ on a Banach space $X$ we call

$$\varrho(A) := \{\lambda \in \mathbb{C} : \lambda I - A : \mathcal{D}(A) \to X \text{ bijective}\}$$

the resolvent set of $A$, and

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda I - A : \mathcal{D}(A) \to X \text{ bijective}\}$$

its resolvent. (Note that since $A$ is closed, also $(\lambda - A)^{-1}$ is closed, and by the Closed Graph Theorem must be bounded.) The set

$$\sigma(A) := \mathbb{C} \setminus \varrho(A)$$

is called the spectrum of $A$. 
1.3 Fact If $\lambda, \mu \in \varrho(A)$ then

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A).$$

If $\lambda \in \varrho(A)$ and $|\mu - \lambda| < \|R(\lambda, A)\|^{-1}$ then $\mu \in \varrho(A)$ and

$$R(\mu, A) = \sum_{n \geq 0} (\lambda - \mu)^n R(\lambda, A)^{n+1}.$$

In particular $\|R(\lambda, A)\| \text{dist}(\lambda, \sigma(A)) \geq 1$. Moreover, $R(\cdot, A) : \varrho(A) \to \mathcal{L}(X)$ is holomorphic, with

$$\frac{d^n}{d\lambda^n} R(\lambda, A) = n!(-1)^{n+1} R(\lambda, A)^{n+1}.$$ 

Proof. (Exercise) \qed

The resolvent identity is also a sufficient criterion to ensure that a given point is in the resolvent set, as the following exercise shows.

Exercise 2: Let $A$ be an operator on a Banach space $X$, and let $\mu \in \varrho(A)$. Prove that for $Q \in \mathcal{L}(X), \lambda \in \mathbb{C}$ the following assertions are equivalent:

(i) $\lambda \in \varrho(A)$ and $Q = R(\lambda, A)$.

(ii) $R(\mu, A) - Q = (\lambda - \mu)R(\mu, A)Q = (\lambda - \mu)QR(\mu, A)$.

1.4 Fact Let $\Omega \subset \mathbb{C}$ arbitrary and $(R_\lambda)_{\lambda \in \Omega} \subset \mathcal{L}(X)$ any family that satisfies the resolvent identity. If one/each of the operators $R_\lambda$ is injective, then there exists a unique closed operator $B$ such that $\Omega \subset \varrho(B)$ and $R_\lambda = R(\lambda, B)$ for all $\lambda \in \Omega$.

Proof. It is clear how the operator $B$ must look like, namely $B = \mu I - R_\mu^{-1}$ for all $\mu \in \Omega$. This proves uniqueness. For existence, fix one $\mu \in \Omega$ and define $B := \mu I - R_\mu^{-1}$. Then $\mu \in \varrho(B)$ and $R_\mu = R(\mu, B)$. The rest follows from Exercise 2 above. \qed

Remark: Fact 1.4 is still true if the operator $R_\lambda$ are not injective. However, this requires the notion of a multivalued operator, see [5, Appendix A].

The above results show that to a $C_0$-semigroup $T$ there exists an operator $B$ such that $\sigma(B) \subset \{\Re z \leq \omega\}$ and $R(\lambda, B) = R_\lambda$ for $\Re \lambda > \omega$. Moreover, by 3) of Theorem 1.1, $B$ is densely defined, i.e., $\overline{D(B)} = X$. We shall see in the following theorem that $B$ is in fact the generator of the semigroup.
1.5 Theorem Let $T$ be a $C_0$-semigroup on a Banach space $X$, and let $B$ be the operator whose resolvent is given by

$$R(\lambda, B) = \int_0^\infty e^{-\lambda t}T(t) \, dt \quad (\text{Re} \, \lambda > \omega).$$

Then the following statements hold.

a) The (ACP) associated to $A := B$ is well-posed and $T$ is its solution semigroup.

b) If $T$ arises as the solution semigroup of an (ACP) associated to a closed operator $A$, then $B = A$.

Hence $B$ is the uniquely determined generator of $T$.

Proof. a) Define $A := B$. Let $x \in X$ and define $u(t) := T(t)x$ and $v(t) := \int_0^t u(s) \, ds$. By definition $u$ is a mild solution of $(ACP)_x$ if $(v(t), T(t)x - x) \in \text{graph}(A)$ for all $t \geq 0$. This is equivalent to

$$v(t) = R_\lambda [x - T(t)x + \lambda v(t)], \quad \text{i.e.} \quad \lambda R_\lambda v(t) = v(t) + R_\lambda T(t)x + R_\lambda x.$$

To establish this, we integrate by parts:

$$\lambda R_\lambda v(t) = \int_0^\infty \lambda e^{-\lambda r} \int_0^r T(s + r)x \, ds \, dr = \int_0^\infty \lambda e^{-\lambda r} \int_r^{t+r} T(s)x \, ds \, dr$$

$$= -e^{-\lambda r}(v(t + r) - v(r)) \bigg|_0^\infty + \int_0^{\infty} e^{-\lambda r}(T(t + r)x - T(r)x) \, dr$$

$$= v(t) + R_\lambda T(t)x - R_\lambda x.$$

It remains to show the uniqueness of the mild solutions. By linearity one may reduce it to mild solutions with initial value 0. Suppose that $u$ is a mild solution with $u(0) = 0$ and define $v(t) := \int_0^t u(s) \, ds$. Then $v(t) \in \mathcal{D}(A)$ and

$$v'(t) = -Av(t).$$

Define $S(t)x := \int_0^t T(s)x \, ds$ for $x \in X, t \geq 0$, fix $\tau > 0$ and consider

$$f(t) := S(\tau - t)v(t), \quad t \in [0, \tau].$$

Then $f$ is differentiable (product rule) with

$$f'(t) = -T(\tau - t)v(t) + S(\tau - t)u(t) = -T(\tau - t)v(t) - S(\tau - t)Av(t) = v(t)$$

since for each $x \in \mathcal{D}(A)$ we have

$$S(\tau - t)A x = \int_0^{\tau - t} T(s)A x \, ds = A \int_0^{\tau - t} T(s)x \, ds = x - T(\tau - t)x.$$
This being true for all \( \tau > 0 \) we conclude that \( v \equiv 0 \), hence \( u \equiv 0 \).

b) We first show that \( \mathcal{D}(B) \subset \mathcal{D}(A) \) and \( I = (\lambda - A)R_\lambda \) for large \( \lambda \). This is equivalent to

\[
(R_\lambda, \lambda R_\lambda - I) : X \oplus X \longrightarrow \text{graph}(A).
\]

Now, integration by parts yields

\[
R_\lambda x = \int_0^\infty e^{-\lambda t} T(t)x \, dt = e^{-\lambda t} \int_0^t T(s)x \, ds \bigg|_0^\infty + \int_0^\infty \lambda e^{-\lambda t} \int_0^t T(s)x \, ds \, dt
\]

Since \( T \) solves the (ACP) for \( A \),

\[
\left( \int_0^t T(s)x \, ds, T(t)x - x \right) : \mathbb{R}_+ \longrightarrow \text{graph}(A) \subset X \oplus X
\]

is continuous. Multiplying by \( \lambda e^{-\lambda t} \) and integrating yields therefore

\[
(R_\lambda x, \lambda R_\lambda x - x) \in \text{graph}(A)
\]

since \( A \) is closed. It is left to show that \( \mathcal{D}(A) \subset \mathcal{D}(B) \). Take \( x \in \mathcal{D}(A) \) and define \( y := R_\lambda(\lambda - A)x \in \mathcal{D}(B) \). Then it is clear that \( (\lambda - A)(x - y) = 0 \). Then \( v(t) := e^{\lambda t}(x - y) \) is a classical solution of \( (ACP)_{x-y} \), hence by uniqueness, \( T(t)(x - y) = e^{\lambda t}(x - y) \). Taking norms yields

\[
\|x - y\| = e^{-\lambda t} \|T(t)(x - y)\| \leq Me^{-(\lambda - \omega)t} \|x - y\| \quad (t \geq 0).
\]

As \( \lambda > \omega \), this is only possible if \( x - y = 0 \). \( \square \)

**Exercise 3**: Let \( T \) be a \( C_0 \)-semigroup with generator \( A \). Show that for \( \alpha \in \mathbb{R} \)

\[
T_\alpha(t) := e^{\alpha t}T(t) \quad (t \geq 0)
\]

defines a \( C_0 \)-semigroup with generator \( A + \alpha I \).

**Exercise 4**: Let \( T \) be a \( C_0 \)-semigroup with generator \( A \). Show that

\[
\mathcal{D}(A) = \left\{ x \mid \lim_{t \searrow 0} \frac{1}{t} (T(t)x - x) \text{ exists} \right\}
\]

and \( Ax = \lim_{t \searrow 0} \frac{1}{t} (T(t)x - x) \) for \( x \in \mathcal{D}(A) \). (In [4], this is the definition of the generator of a \( C_0 \)-semigroup).
1.6 Fact Let \((T(t))_{t \geq 0}\) be a \(C_0\)-semigroup with generator \(A\). Then the following assertions are equivalent.

(i) There is a \(t_0 > 0\) such that \(T(t_0)\) is invertible.

(ii) There is a \(C_0\)-group \(U\) such that \(U(t) = T(t)\) for all \(t \geq 0\).

(iii) The operator \(-A\) generates a \(C_0\)-semigroup \(S\).

In this case \(S(t) = U(-t)\) for all \(t \geq 0\).

Proof. Suppose that (i) holds. Then \(T(t_0 - s)T(s) = T(t_0)\) is invertible, and so \(T(t)\) is invertible for all \(t \in [0, t_0]\). Hence \(T(t)\) is invertible for all \(t \geq 0\). Define \(U(t) = T(t)\) for \(t \geq 0\) and \(U(t) = T(t)^{-1}\) for \(t \geq 0\). Obviously, \(U\) is a group. To prove strong continuity, fix \(\tau > 0\). Then for \(x \in X\) and \(s \in [0, \tau]\) \(T(s)^{-1}x = T(\tau - s)T(\tau)^{-1}x\), and this is continuous.

Suppose that (ii) holds. Let \(B\) be the generator of the semigroup \((U(-t))_{t \geq 0}\). We show that \(B = -A\), using the result of Exercise 4 above. Let \(x \in \mathcal{D}(A)\). Then \((1/t)[U(-t)x - x] = -U(-t)(1/t)[U(t)x - x] \to -Ax\) as \(t \searrow 0\). Hence \(x \in \mathcal{D}(B)\) and \(Bx = -Ax\). If one interchanges the roles of \(A\) and \(B\) one obtains \(B \subset -A\).

Suppose that (iii) holds. One has to show that \(S(t) = T(t)^{-1}\). Fix \(\tau > 0\) and \(x \in \mathcal{D}(A) = \mathcal{D}(-A)\). Consider the function

\[v(t) := \begin{cases} S(t)T(\tau)x - T(\tau - t)x & (0 \leq t \leq \tau), \\ T(t - \tau)S(\tau)T(\tau)x - T(t - \tau)x & (t \geq \tau). \end{cases}\]

Then \(v' = Av\) and so \(v\) is a classical solution of the (ACP) for \(A\). Since \(v(0) = 0\), by uniqueness one must have \(v(t) = 0\) for all \(t \geq 0\). In particular for \(t = \tau\), hence \(S(\tau)T(\tau)x = x\). Since \(\mathcal{D}(A)\) is dense, \(I = S(\tau)T(\tau) = T(\tau)S(\tau)\), and this was to prove. \(\square\)

2 Functional Calculus

Consider a semigroup \(T\) with generator \(A\). Since \(T\) satisfies

\[T(t + s) = T(t)T(s)\] and \(\frac{d}{dt}T(t) = AT(t)\] \((t, s \geq 0)\)
at least formally, one often writes $T(t) = e^{tA}$. It seems that $A$ is “inserted” into the scalar function $e^{tz}$. Even more elementary, $R(\lambda, A)$ seems to be the result of inserting $A$ into the scalar function $(\lambda - z)^{-1}$. The scalar functions are related via the Laplace transform:

$$(\lambda - z)^{-1} = \int_0^\infty e^{-\lambda t} e^{t z} \, dt$$

and so are the operators!

Now, the idea of a functional calculus consists in describing a method of “insertion” of an operator $A$ into as many scalar functions as possible, in a reasonable way. “Reasonable” means here that formulae which link the functions transform after inserting the operator into formulae which link the operators. This concept is not restricted to semigroup generators. However, they form a simple case where we shall demonstrate the whole process in detail.

Of course one has to start somewhere. In the case of semigroup generators, one may begin with the semigroup, and this leads to the so-called Phillips calculus (see below), but this is far too special, since we want a generic approach. The easiest thing is to start with the generator, i.e. with its resolvent. That gives us for free the result of inserting $A$ into functions of the form

$$\frac{1}{\lambda - z} \quad (\lambda \in \varrho(A)).$$

Now the Cauchy integral formula tells us, that an arbitrary holomorphic function has the form

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) \frac{d\lambda}{\lambda - z}$$

where $\Gamma$ is a suitable contour, and $z$ is from the (suitably defined) “interior” of the contour. But how are we to choose $f$ and $\Gamma$?

To answer this, let us think of the easiest non-trivial situation. This is when $X = \mathbb{C}$ and $A$ is just multiplication with the number $a \in \mathbb{C}$. Clearly, inserting $A$ into $f$ should yield multiplication with $f(a)$, and so $a$ should be in the domain of definition of $f$. Now, in this situation we have $\{a\} = \sigma(A)$. A little less elementary, one could think of $X$ being a function space and $A$ is multiplication with a function $a$. Then $f(A)$ should be multiplication with the function $f \circ a$, and so the range of $a$ should be contained in the domain of $f$.

But usually the range (or the essential range) of $a$ coincides with the spectrum of the operator $A$ (see the next exercise). As a conclusion we obtain that in order to be in accordance with elementary situations we need $\sigma(A) \subset \text{dom}(f)$, and $\Gamma \subset \varrho(A)$ surrounding $\sigma(A)$ once.

**Exercise 5:** (See also [5, Section 1.4]) Let $(\Omega, \Sigma, \mu)$ be any measure space, and
$a : \Omega \to \mathbb{C}$ be a measurable function. Define $A$ on $X = L^2(\Omega, \Sigma, \mu)$ by

$$Af = af \quad \text{for} \quad f \in \mathcal{D}(A) = \{ f \in X \mid af \in X \}.$$ 

Show that $A$ is closed, and that $A$ is bounded if $a \in L^\infty$. Show further that

$$\sigma(A) = \text{essran}(a) = \{ \lambda \in \mathbb{C} \mid \forall \varepsilon > 0 : \mu\{|a - \lambda| < \varepsilon\} > 0 \}$$

and that for $\lambda \in \mathbb{C} \setminus \sigma(A)$ one has

$$R(\lambda, A)f = (\lambda - a)^{-1}f \quad (f \in X).$$

The simplest situation differing from the multiplicator examples is the one of a bounded operator $A \in L(X)$ and a holomorphic function $f$ defined on a neighbourhood $U$ of $\sigma(A)$. One can then find a (generalized) contour $\Gamma \subset U \setminus \sigma(A)$ that surrounds (in the sense of the index) each point in $\sigma(A)$ exactly once in positive sense. The (generalized) Cauchy integral formula\footnote{This, sometimes called the “global” or “homological”, version of the Cauchy formula often is not part of the standard curricula on elementary complex analysis. However, it is the real useful form, which also avoids complicated definitions like integration over piecewise smooth boundaries of compact sets. It can be found, e.g., in [8, Theorem 10.35].} reads

$$f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - a} \, dz \quad (a \in \sigma(A)).$$

Hence one defines

$$f(A) := \frac{1}{2\pi i} \int_\Gamma f(z)R(z, A) \, dz.$$ 

This approach, due to DUNFORD and RIESZ and therefore named the Dunford–Riesz calculus, is our guiding example. A detailed account in the context of Banach algebras is in [3, Section VII.4]. However, we deal with two additional difficulties: 1. Our operators $A$ are in general unbounded and as such their spectrum is unbounded. (Indeed, one may view unboundedness as a spectral condition, involving the improper spectral value $\infty$.) 2. The functions we are interested in are not always defined on a whole neighbourhood of the spectrum. (Consider the case of a semigroup generator: the exponential function has an essential singularity at $\infty$, which is a spectral value for each unbounded operator.)

These difficulties can be overcome by considering infinite contours and functions that vanish at a sufficiently high order in the singularities, in order to make the Cauchy integral converge. For this it is crucial to know the growth behaviour of the resolvent near the singular points. We shall now demonstrate the approach
in the case of semigroup generators. To be more flexible, we in fact shall not use the generator property but only the growth condition of the resolvent.

**First Step: The type of the operator**

The operators we have in mind are of “half-plane type” in the following sense. For $\omega \in [-\infty, \infty]$ define

$$L_\omega := \{ z \in \mathbb{C} \mid \text{Re} z < \omega \}, \quad R_\omega := \{ z \in \mathbb{C} \mid \text{Re} z > \omega \}$$

the open left and right half-planes defined by the abscissa $\text{Re} z = \omega$, where in the extremal cases one half-plane is understood to be empty, and the other is the whole complex plane. We say that an operator $A$ on a Banach space $X$ is of **half-plane type** $\omega \in \mathbb{R} \cup \{-\infty\}$, if $\sigma(A) \subset L_\omega$ and

$$M_\alpha := M_\alpha(A) := \sup \{ \| R(z, A) \| \mid \text{Re} z \geq \alpha \} < \infty$$

for every $\alpha > \omega$. We call

$$s_0(A) := \min \{ \omega \mid A \text{ is of half-plane type } \omega \} \in [-\infty, \infty]$$

the **abscissa of uniform boundedness** (in short: the $aub$) of the operator $A$. Then $A$ is of half-plane type $s_0(A)$.

**Second Step: The class of elementary functions**

An operator of half-plane type has its spectrum in a left half-plane $L_\omega$, and “surrounding” such a half-plane in the positive direction basically means to go from $\alpha - i\infty$ to $\alpha + i\infty$ for some $\alpha > \omega$. One now looks at the formula

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z) R(z, A) \, dz$$

with $\Gamma = \alpha + i\mathbb{R}$, and asks how $f$ should be in order that the integral converges. This leads to the class of elementary functions. For $\omega \in \mathbb{R}$ let

$$\mathcal{E}(L_\omega) := \{ f : L_\omega \rightarrow \mathbb{C} \mid f \text{ is holomorphic and}$$

$$\exists M, s > 0 : |f(z)| \leq M |z|^{-(1+s)} \text{ as } z \rightarrow \infty \}. $$

Clearly $\mathcal{E}(L_\omega)$ is an algebra such that

$$\frac{1}{(\lambda - z)(\mu - z)} \in \mathcal{E}(L_\omega) \quad (\text{Re} \lambda, \text{Re} \mu > \omega).$$

One has to make sure that Cauchy’s integral formula holds for these functions.
2.1 Fact Let $f \in E(L_{\alpha})$, and let $\delta < \alpha$. Then
\[
f(a) = \frac{1}{2\pi i} \int_{\text{Re } z = \delta} \frac{f(z)}{z - a} \, dz \quad (\text{Re } a > \delta)
\]
and
\[
0 = \frac{1}{2\pi i} \int_{\text{Re } z = \delta} f(z) \, dz.
\]
The direction of integration is bottom up, i.e., from $\delta - i\infty$ to $\delta + i\infty$.

Proof. Fix $\text{Re } a < \delta$. To establish the formula we employ the usual Cauchy theorem with contour being the boundary of the rectangle $\text{Im } z \in [-R, R]$, $\text{Re } z \in [\delta, \delta']$, for $\delta' < \text{Re } a$, and $R > 0$ large. If we let $R \to \infty$ we see that
\[
f(a) = \frac{1}{2\pi i} \int_{\text{Re } z = \delta} \frac{f(z)}{z - a} \, dz - \frac{1}{2\pi i} \int_{\text{Re } z = \delta'} \frac{f(z)}{z - a} \, dz.
\]
(Decay of $f$ for $|\text{Im } z| \to \infty$ makes the integrals converge and the integrals over the upper and lower rectangle sides vanish as $R$ becomes large.) As a consequence of this representation we see that the value of the second integral does not depend on $\delta'$. So we may let $\delta' \to -\infty$ without changing its value. But then this value has to be equal to 0, because of the decay of $f$. The arguments in the second case are similar. □

Third Step: The elementary functional calculus
Let $A$ be an operator of half-plane type $\omega$, and let $\alpha > \omega$, and let $f \in E(L_{\alpha})$. We have seen that
\[
f(a) = \frac{1}{2\pi i} \int_{\text{Re } z = \delta} f(z) \frac{dz}{z - a} \quad (\text{Re } a < \delta < \alpha)
\]
holds. Since the resolvent $R(\cdot, A)$ is bounded on the vertical lines $\{\text{Re } z = \delta\}$, $\delta > \omega$, the integral
\[
\Phi(f) := f(A) := \frac{1}{2\pi i} \int_{\text{Re } z = \delta} f(z)R(z, A) \, dz
\]
converges absolutely.

2.2 Fact The definition of $\Phi(f)$ does not depend on $\delta \in (\omega, \alpha)$.

Proof. We note that $[z \mapsto f(z)R(z, A)]$ is holomorphic on the strip $\omega < \text{Re } z < \alpha$. The Cauchy integral theorem is valid also for Banach space valued functions.\(^2\) We fix $\delta_1 < \delta_2$ and write $\gamma_{R, \pm}(t) = t \pm iR$, $t \in [\delta_1, \delta_2]$, $R > 0$. Then

\(^2\)This may be seen in two ways. Either one looks at the proof anew and realizes that all can be done for functions valid in Banach spaces, or one applies linear functionals and via the Hahn-Banach theorem reduces everything to the scalar case. See also the discussion in [9, Theorem 3.31]
by Cauchy’s theorem
\[ \int_{\text{Re}z=\delta_1} f(z)R(z,A) \, dz = \lim_{R \to \infty} \int_{\text{Re}z=\delta_1,|\text{Im}z| \leq R} f(z)R(z,A) \, dz \]
\[ = \lim_{R \to \infty} \int_{\gamma_{R,-}} \ldots - \int_{\gamma_{R,+}} \ldots - \int_{\text{Re}z=\delta_2,|\text{Im}z| \leq R} \ldots \]
\[ = \lim_{R \to \infty} \int_{\text{Re}z=\delta_2,|\text{Im}z| \leq R} f(z)R(z,A) \, dz = \int_{\text{Re}z=\delta_2} f(z)R(z,A) \, dz \]

since \( \left\| \int_{\gamma_{R,z}} f(z)R(z,A) \, dz \right\| \to 0 \) as \( R \to \infty \). (The resolvent is uniformly bounded and \( f \) is uniformly small as \( R \to \infty \).) \( \square \)

2.3 Theorem  The so defined mapping \( \Phi : \mathcal{E}(L_\alpha) \to \mathcal{L}(X) \) has the following properties:

a) \( \Phi \) is a homomorphism of algebras, i.e.,
\[ \Phi(f + g) = \Phi(f) + \Phi(g) \quad \text{and} \quad \Phi(fg) = \Phi(f)\Phi(g) \]
for all \( f, g \in \mathcal{E}(L_\alpha) \).
b) If \( T \in \mathcal{L}(X) \) commutes with \( A \) (see Exercise 6 below) then it commutes with every \( \Phi(f) \), \( f \in \mathcal{E}(L_\alpha) \).
c) \( \Phi(f(z)(\lambda - z)^{-1}) = \Phi(f)R(\lambda, A) \) for all \( \text{Re} \lambda > \alpha \).
d) \( \Phi((\lambda - z)^{-1}(\mu - z)^{-1}) = R(\lambda, A)R(\mu, A) \) for all \( \text{Re} \lambda, \text{Re} \mu > \alpha \).

Proof. a) Additivity is clear, so let’s prove the multiplicativity. Take \( f, g \in \mathcal{E}(L_\alpha) \). Choose \( \omega < \delta < \delta' < \alpha \). Then, by Fubini’s theorem and the resolvent identity,
\[ \Phi(f)\Phi(g) = \frac{1}{2\pi i} \int_{\text{Re}z=\delta} f(z)R(z,A) \, dz \cdot \frac{1}{2\pi i} \int_{\text{Re}w=\delta'} g(w)R(w,A) \, dw \]
\[ = \left( \frac{1}{2\pi i} \right)^2 \int_{\text{Re}z=\delta} \int_{\text{Re}w=\delta'} f(z)g(w)R(z,A)R(w,A) \, dz \, dw \]
\[ = \left( \frac{1}{2\pi i} \right)^2 \int_{\text{Re}z=\delta} \int_{\text{Re}w=\delta'} f(z)g(w) \frac{[R(z,A) - R(w,A)]}{w - z} \, dz \, dw \]
\[ = \frac{1}{2\pi i} \int_{\text{Re}z=\delta} \frac{1}{2\pi i} \int_{\text{Re}w=\delta'} \frac{g(w)}{w - z} \left[ R(z,A) - R(w,A) \right] \, dz \, dw \]
\[ = \frac{1}{2\pi i} \int_{\text{Re}z=\delta} \frac{1}{2\pi i} \int_{\text{Re}w=\delta'} \frac{f(z)g(w)}{w - z} \, dz \, dw \]
\[ = \frac{1}{2\pi i} \int_{\text{Re}z=\delta} \frac{1}{2\pi i} \int_{\text{Re}w=\delta'} \frac{g(w)}{w - z} \, dw \]
\[ = \frac{1}{2\pi i} \int_{\text{Re}z=\delta} \frac{1}{2\pi i} \int_{\text{Re}w=\delta'} \frac{f(z)g(w)}{w - z} \, dz \, dw = \Phi(fg). \]
In the last line we used Fact 2.1.

b) If $T$ commutes with $A$, $TR(z, A) = R(z, A)T$ for every $z \in \rho(A)$. Hence the assertion is trivial.

c) By the resolvent identity and Fact 2.1,

\[ \Phi(f)R(\lambda, A) = \frac{1}{2\pi i} \int_{|\text{Re } z| = \delta} f(z)R(z, A)R(\lambda, A) \, dz \]

we shift the path to the right, i.e., let $\delta \to \infty$. When passing the abscissas $\delta = \text{Re } \lambda$ and $\delta = \text{Re } \mu$ the residue theorem yields some additive contributions which sum up to $R(\lambda, A)R(\mu, A)$ by the resolvent identity; if $\delta > \text{Re } \lambda, \text{Re } \mu$, the integral does not change any more as $\delta \to \infty$ and hence it is equal to zero. A more concise form of reasoning goes like this. Choose $\omega < \delta < \text{Re } \lambda, \text{Re } \mu < \delta'$ and $R > \max|\text{Im } \lambda|, |\text{Im } \mu|$. Let $\Gamma_R$ be the (positively oriented) boundary of the rectangle $\text{Re } z \in [\delta, \delta'], \text{Im } z \in [-R, R]$. The function $g(z) := R(z, A)(\lambda - z)^{-1}(\mu - z)^{-1}$ has simple poles in $z = \lambda, \mu$, so the residue theorem yields

\[ \frac{1}{2\pi i} \int_{\Gamma_R} \frac{1}{(\lambda - z)(\mu - z)} R(z, A) \, dz = \frac{-R(\lambda, A)}{\mu - \lambda} + \frac{-R(\mu, A)}{\lambda - \mu} = -R(\lambda, A)R(\mu, A). \]

If we let $R \to \infty$ we obtain

\[ -f(A) + \frac{1}{2\pi i} \int_{|\text{Re } z| = \delta'} \frac{R(z, A) \, dz}{(\lambda - z)(\mu - z)} = -R(\lambda, A)R(\mu, A). \]

This shows that the integral over $\delta' + i\mathbb{R}$ is independent of $\delta'$, and so must be equal to 0, as we can make it small by choosing $\delta'$ large.

The mapping $\Phi = (f \mapsto f(A)) : \mathcal{E}(\mathcal{L}_0) \to \mathcal{L}(X)$ is called the elementary functional calculus for the half-plane type operator $A$.

**Exercise 6:** Let $A$ be an (unbounded) operator on a Banach space $X$. A bounded operator $T \in \mathcal{L}(X)$ is said to **commute** with $A$ if

$$(x, y) \in \text{graph}(A) \implies (Tx, Ty) \in \text{graph}(A)$$
for all $x,y \in X$. This is sometimes written as $TA \subset AT$. Show that if $A$ is bounded then $T$ commutes with $A$ if and only if $AT = TA$. Show that if $\lambda \in \sigma(A)$ then $T$ commutes with $A$ if and only if $T$ commutes with $R(\lambda, A)$.

### 3 Extended Functional Calculus

The elementary functional calculus is the major step. However, to recover the semigroup we need $f(z) = e^{tz}$ and these functions are not contained in the elementary classes $\mathcal{E}(L_0)$. This is not surprising, because not every operator of half-plane type is the generator of a $C_0$-semigroup\(^3\). Our aim is to define $e^{tA}$ for every half-plane type operator, but as a closed operator which is not necessarily bounded. The idea to do this is called regularization. By this we mean that although $e^{tz}$ does not belong to $\mathcal{E}(L_0)$, the “regularized” function $(\lambda - z)^{-2}e^{tz}$ in fact does, for arbitrary $\lambda > \alpha$. One may then define

$$e^{tz}(A) := (\lambda - A)^2 \left[ \frac{e^{tz}}{(\lambda - z)^2} \right] (A).$$

Here we use the general definition of the product of two unbounded operators:

$$\mathcal{D}(AB) = \{ x \in \mathcal{D}(A) \mid Ax \in \mathcal{D}(B) \}, \quad (AB)x := A(Bx) \quad (x \in \mathcal{D}(AB)).$$

(This makes multiplication associative, and the identity operator is neutral.)

**Exercise 7:** Let $A$ be a closed operator and $T \in \mathcal{L}(X)$. Show that $AT$ is closed.

The general principle of extension by regularization is formulated in the following setup.

**Fourth Step: Extension by Regularization**

An abstract functional calculus consists of a triple $(\mathcal{E}, \mathcal{M}, \Phi)$, where $\mathcal{M}$ is a commutative algebra with unit 1, $\mathcal{E} \subset \mathcal{M}$ is a subalgebra (usually not containing 1), and $\Phi : \mathcal{E} \longrightarrow \mathcal{L}(X)$ is an algebra homomorphism. The elements of the set

$$\text{Reg}(\mathcal{E}) := \{ e \in \mathcal{E} \mid \Phi(e) \text{ injective} \}$$

\(^3\)Examples are not so easy to find, at least the straightforward constructions fail. One may think of operators that are multiplication operators with respect to some not unconditional Schauder bases, see [5, Chapter 9].
are called **regularizers**, and the abstract functional calculus is **proper** if \( \text{Reg}(\mathcal{E}) \neq \emptyset \). An element \( f \in \mathcal{M} \) is said to be **regularized** by \( e \in \text{Reg}(\mathcal{E}) \) if \( ef \in \mathcal{E} \). In this case we call \( e \) a **regularizer for** \( f \) and define

\[
\Phi(f) := \Phi(e)^{-1} \Phi(ef).
\]

(This yields a closed operator by Exercise 7.) Using the commutativity of \( \mathcal{E} \) one shows easily that the definition of \( \Phi(f) \) does not depend on the particular regularizer of \( f \). Moreover, if \( f \in \mathcal{E} \), then the new definition of \( \Phi(f) \) is consistent with the original.

**Exercise 8:** Prove that the definition of \( \Phi(f) \) is independent of the regularizer \( e \), cf. [5, Section 1.2.1].

An element \( f \in \mathcal{F} \) is called **regularizable** if there exists a regularizer for \( f \).

The set of all regularizable elements is denoted by

\[
\mathcal{F}_r := \{ f \in \mathcal{F} \mid g \text{ regularizable} \}.
\]

**3.1 Theorem** Let \((\mathcal{E}, \mathcal{F}, \Phi)\) be a proper abstract functional calculus. Then the following assertions hold.

a) \( 1 \in \mathcal{M}_r \) and \( \Phi(1) = I \).

b) If \( f, g \in \mathcal{F}_r \) then \( f + g, fg \in \mathcal{F}_r \) and

\[
\Phi(f) + \Phi(g) \subset \Phi(f + g) \quad \text{and} \quad \Phi(f)\Phi(g) \subset \Phi(fg).
\]

Moreover, \( \mathcal{D}(\Phi(g)) \cap \mathcal{D}(\Phi(fg)) = \mathcal{D}(\Phi(f)\Phi(g)) \). In particular, if \( \Phi(g) \in \mathcal{L}(X) \)

\[
\Phi(f) + \Phi(g) = \Phi(f + g) \quad \text{and} \quad \Phi(f)\Phi(g) = \Phi(fg).
\]

c) Let \( f, g \in \mathcal{M} \) such that \( fg = 1 \) and \( f \in \mathcal{M}_r \). Then \( g \in \mathcal{M}_r \) iff \( \Phi(f) \) is injective, and in this case \( \Phi(g) = \Phi(f)^{-1} \).

d) If \( T \in \mathcal{L}(X) \) commutes with each \( \Phi(e), e \in \mathcal{E} \), then it commutes with each \( \Phi(f), f \in \mathcal{M}_r \).

**Proof.** a) Let \( e \in \text{Reg}(\mathcal{E}) \). Then \( e \) regularizes \( 1 \), and so

\[
\Phi(1) = \Phi(e)^{-1} \Phi(e1) = \Phi(e)^{-1} \Phi(e) = I.
\]

b) We prove the statement for the product leaving the sum case as an exercise. Let \( e_1 \) be a regularizer for \( f \) and \( e_2 \) be a regularizer for \( g \). Since \( \Phi(e_1e_2) = \]
\( \Phi(e_1)\Phi(e_2) \) is injective and \( (e_1e_2)(fg) = (e_1f)(e_2g) \in \mathcal{E} \), the element \( e := e_1e_2 \) is a regularizer for \( fg \). Furthermore

\[
\Phi(f)\Phi(g) = \Phi(e_1)^{-1}\Phi(e_1 f)\Phi(e_2)^{-1}\Phi(e_2 g) \subset \Phi(e_1)^{-1}\Phi(e_2)^{-1}\Phi(e_1 f)\Phi(e_2 g) = (\Phi(e_2)\Phi(e_1))^{-1}\Phi(e_1 f e_2 g) = \Phi(e)^{-1}\Phi(efg) = \Phi(fg).
\]

To prove the statement for the domains, note that the inclusion already yields that \( \mathcal{D}(\Phi(f)\Phi(g)) \subset \mathcal{D}(\Phi(fg)) \cap \mathcal{D}(\Phi(g)) \). To prove the converse inclusion, let \( x \in \mathcal{D}(\Phi(g)) \cap \mathcal{D}(\Phi(fg)) \). This means that there are elements \( y, z \) such that

\[
\Phi(e_2)y = \Phi(e_2)\Phi(e_1)z = \Phi(e_1)\Phi(e_2)x = \Phi(e_1 f)\Phi(e_2)y = \Phi(e_2)\Phi(e_1 f)y,
\]

and since \( \Phi(e_2) \) is injective it follows that \( \Phi(e_1)z = \Phi(e_1 f)y \). This means that \( \Phi(g)x = y \in \mathcal{D}(\Phi(f)) \).

c) Let \( e \) be a regularizer for \( f \). Then \( \Phi(e) \) is injective and \( e' := ef \in \mathcal{E} \). Hence \( e'g = ef g = e1 = e \in \mathcal{E} \). If \( \Phi(f) \) is injective, then \( \Phi(e') = \Phi(f e) = \Phi(f)\Phi(e) \) by part b), and this is injective. Hence \( e' \) is a regularizer for \( g \). If on the other hand \( g \) is regularizable, then \( \Phi(g)\Phi(f) \subset \Phi(fg) = \Phi(1) = I \), by a) and b). Hence \( \Phi(f) \) is injective. Moreover, by b), \( \mathcal{D}(\Phi(g)\Phi(f)) = \mathcal{D}(\Phi(f)) \), and so \( \Phi(f)^{-1} \subset \Phi(g) \). Symmetry in \( f \) and \( g \) yields also \( \Phi(g)^{-1} \subset \Phi(f) \), and so the assertion is proved.

d) This is straightforward. \( \square \)

**Exercise 9:** Provide the remaining parts of the proof of Theorem 3.1.

We may now apply this abstract setting to our case of operators of half-plane type. Let \( A \) be of half-plane type \( \omega \in \mathbb{R} \) and let \( \omega < \alpha \). We set \( \mathcal{E} := \mathcal{E}(L_\alpha) \) and

\[
\mathcal{F} := \mathcal{M}(L_\alpha)
\]

the space of all meromorphic functions on \( L_\alpha \). Fix \( \lambda > \alpha \). Then the function \( e := (\lambda - z)^{-2} \) is contained in \( \mathcal{E} \) and \( e(A) = R(\lambda, A)^2 \) is injective. Hence it is a regularizer for every bounded, holomorphic function on \( R_\alpha \). Therefore

\[
H^\infty(L_\alpha) \subset \mathcal{M}(L_\alpha)_r.
\]

In particular

\[
e^tA := (e^{tz})(A) \quad (t \geq 0)
\]
is defined, and \( \mathcal{D}(A^2) = \mathcal{R}(R(\lambda, A)^2) \subset \mathcal{D}(e^{tA}) \).

**Exercise 10:** Let \( A \) be an operator of half plane type \( \omega \), and let \( \alpha > \omega \). Let \( f \in \mathcal{M}(L_\alpha) \) such that there is \( s > 0 \) such that \( f = O(|z|^s) \) as \( z \to \infty \). Show that \( f(A) \) is defined. Show that for \( f(z) = z \) one obtains \( f(A) = A \).

**Exercise 11:** Let \( A \) be an operator of half plane type \( \omega \), and define \( B := A - \omega \). Show that \( B \) is of half-plane type 0. Moreover, show that \( \Psi = [f(z) \mapsto -\to f(z - \omega)] \) provides an isomorphism of \( \mathcal{E}(L_0), \mathcal{M}(L_0) \) to \( \mathcal{E}(L_\omega), \mathcal{M}(L_\omega) \). Prove the formula
\[
[f(z - \omega)](A) = f(B)
\]
first for elementary functions \( f \) and then for general \( f \). This is a simple example of a composition rule, see below for a more involved instance.

### 3.2 Lemma

Let \( A \) be an operator of half-plane type \( \omega \). Then for each \( x \in \mathcal{D}(A^2) \) and \( \alpha > \omega \), the function
\[
(t \mapsto e^{-\alpha t}e^{tA}x) : [0, \infty) \to X
\]
is continuous and bounded, and its Laplace transform is
\[
\int_0^\infty e^{-\lambda t}e^{tA}x \, dt = R(\lambda, A)x \quad (\text{Re} \, \lambda > \alpha).
\]

**Proof.** Fix \( \lambda > \alpha \) and write \( e^{tA}x = (e^{tz}/(\lambda - z)^2)(A)(\lambda - A)^2x \). Then the continuity in \( t \) is clear from Lebesgue’s theorem. The rest follows by Fubini’s theorem. \( \square \)

### 3.3 Lemma

Let \( A \) be a densely defined operator of half-plane type \( \omega < \infty \). Then \( \mathcal{D}(A^2) \) is dense.

**Proof.** For \( x \in \mathcal{D}(A) \) and \( n \in \mathbb{N}, n > \omega \), we write \( nR(n, A)x = x + R(n, A)Ax \). The right hand side is bounded in \( n \), hence \( R(n, A)x \to 0 \) as \( n \to \infty \). But \( \mathcal{D}(A) \) is dense and the operators \( (R(n, A))_{n \geq n_0} \) are uniformly bounded. Hence \( R(n, A)x \to 0 \) for all \( x \in X \). But this implies that \( nR(n, A)x = x + R(n, A)Ax \to x \) whenever \( x \in \mathcal{D}(A) \); so \( \mathcal{D}(A) \subset \mathcal{D}(A^2) \) which implies that \( \mathcal{D}(A^2) \) is dense in \( X \). \( \square \)

### 3.4 Theorem

Let \( A \) be an operator of half-plane type. Then \( A \) is the generator of a \( C_0 \)-semigroup \( T \) if and only if \( A \) is densely defined and \( e^{tA} \) is a bounded operator for all \( t \in [0, 1] \) satisfying \( \sup_{t \in [0, 1]} \|e^{tA}\| < \infty \). In this case, \( T(t) = e^{tA} \) for all \( t \geq 0 \).
Proof. Let $A$ generate a $C_0$-semigroup $(T(t))_{t\geq 0}$. Then $A$ is densely defined. Hence $\mathcal{D}(A^2)$ is dense, by Lemma 3.3. Lemma 3.2 yields that $R(\cdot, A)x$ is the Laplace transform of $e^{tA}x$ for $x \in \mathcal{D}(A^2)$. By the uniqueness of Laplace transforms [2, Section 1.7] $T(t)x = e^{tA}x$, $t \geq 0$. Since $\mathcal{D}(A^2)$ is dense and $e^{tA}$ is a closed operator, $e^{tA} = T(t)$ is a bounded operator (see Exercise 12 below). Conversely, suppose that $A$ is densely defined and $T(t)$ is a bounded operator for all $t \geq 0$. Then $T$ is a semigroup (by general functional calculus) and $\mathcal{D}(A^2)$ is dense, by Lemma 3.3. From the uniform boundedness $\sup_{t \in [0, 1]} \|T(t)\| < \infty$ one concludes easily that $(T(t))_{t \geq 0}$ is uniformly bounded on compact intervals. Lemma 3.2 and the density of $\mathcal{D}(A^2)$ imply that $(T(t))$ is strongly continuous. Its Laplace transform coincides with the resolvent of $A$ on $\mathcal{D}(A^2)$ (Lemma 3.2), hence on $X$ by density. So $A$ is the generator of $T$. □

(I am pretty sure that one cannot omit the boundedness assumption from Theorem 3.4, and I guess that one can find examples by looking into [6].)

Exercise 12: Let $A$ be a closed operator on a Banach space $X$. Suppose that there exists $D \subset \mathcal{D}(A)$ and a number $c \geq 0$ such that
\[ \|Ax\| \leq c \|x\| \quad (3.1) \]
all $x \in D$. Prove that $\mathcal{D}(A) = X$ and (3.1) holds for all $x \in X$.

4 Abstract Meromorphic Functional Calculi

Let $\Omega \subset \mathbb{C}$ be open, and let $\mathcal{E}(\Omega)$ be a subalgebra of $\mathcal{M}(\Omega)$, the meromorphic functions on $\Omega$. Let $\Phi : \mathcal{E}(\Omega) \rightarrow \mathcal{L}(X)$ be an algebra homomorphism such that the following holds:

1) The function $z := (w \mapsto w)$ is regularizable by some element of $\mathcal{E}(\Omega)$, whence the operator $A := \Phi(z)$ is well defined by the extension procedure.

2) An operator $T \in \mathcal{L}(X)$ which commutes with $A$ also commutes with each $\Phi(e)$, $e \in \mathcal{E}(\Omega)$.

In this case we call the afc $(\mathcal{E}(\Omega), \mathcal{M}(\omega), \Phi)$ a meromorphic functional calculus for $A$. We write
\[ \mathcal{M}(\Omega)_A := \mathcal{M}(\Omega)_r \quad \text{and} \quad f(A) := \Phi(f) \quad (f \in \mathcal{M}(\Omega)_A). \]

Exercise 13: Let $A$ be of half-plane type $\omega < \infty$ and let $\alpha > \omega$. Prove that the abstract functional calculus $(\mathcal{E}(L_\alpha), \mathcal{M}(L_\alpha), \Phi)$ set up in the previous chapter is in fact a meromorphic functional calculus for $A$. 

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Exercise 14: Suppose that \((\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi)\) is a meromorphic functional calculus for an operator \(A\). Let \(f \in \mathcal{M}(\Omega)\) and \(\lambda \in \mathbb{C} \setminus \mathcal{F}(\Omega)\). Then \(g := (\lambda - f)^{-1} \in H^\infty(\Omega)\). Suppose that \(g\) is regularizable. Use Theorem 3.1 c) to show that \(f\) is regularizable if and only if \(g(A)\) is injective, and that in this case \(\lambda - f(A) = g(A)^{-1}\).

The notion of meromorphic functional calculus is still too general to prove interesting results for it. In many cases one has the following additional situation:

There is a regularizer \(\psi \in \mathcal{E}(\Omega)\) such that \(\psi\) is holomorphic, \(\psi H^\infty(\Omega) \subset \mathcal{E}(\Omega)\), and there is a contour \(\Gamma \subset \Omega\) together with a continuous family \((T_z)_{z \in \Gamma} \subset \mathcal{L}(X)\) such that \((z \mapsto \psi(z)T_z) \in L^1(\Gamma, |dz|; \mathcal{L}(X))\) and \((\psi f)(A) = \int_\Gamma f(z)\psi(z)T_z \, dz\) \((f \in H^\infty(\Omega))\).

In this case we will speak of the standard situation, and \(\psi\) will be called a standard regularizer for \(H^\infty(\Omega)\). We shall also need the constant

\[ C_\psi := \int_\Gamma \|\psi(z)T_z\| \, |dz|, < \infty. \]

In the following we shall exploit this additional structure. One of the most prominent features — the so-called Convergence Lemma — will be discussed in the next chapter. Here we want to study under which assumptions an operator of the form \(f(A)\) is a semigroup generator, in analogy to Theorem 3.4 above. Before we have to provide some general facts.

4.1 Theorem Let \((\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi)\) be a standard meromorphic functional calculus for \(A\), with standard regularizer \(\psi\). Suppose that \(M\) is a set, \(c : M \to \mathbb{R}_+\) and 

\[ F : M \times \Omega \to \mathbb{C} \]

are mappings such that 

\[ |F(m, z)| \leq c(m)|\psi(z)| \quad (m \in M, z \in \Omega) \]

and such that \(F(m, \cdot) \in \mathcal{O}(\Omega)\) for each \(m \in M\). Then \(F(m, \cdot) \in \mathcal{E}(\Omega)\) for all \(m \in M\) and 

\[ \|F(m, A)\| \leq C_\psi c(m) \quad (m \in M). \]

Moreover, the following assertions hold.

a) If \(M\) is a metric space, \(c\) is locally bounded and \(F\) is continuous, then 

\( (m \mapsto F(m, A)) : M \to \mathcal{L}(X) \) is continuous.
b) If \((M, \mu)\) is a measure space, \(c \in L^1(M, \mu)\) and \(F\) is measurable, then the function \(f\) defined by

\[
f(z) := \int_M F(z, m) \mu(dm)
\]

is in \(E(\Omega)\). Furthermore, \((m \mapsto F(m, A)) \in L^1(M; L(X))\) and

\[
\int_M F(m, A) \mu(dm) = \left( \int_M F(m, z) \mu(dm) \right)(A).
\]

Proof. Note first that we may write \(F(m, z) = G(m, z)\psi(z)\), with \(G : M \times \Omega \to \mathbb{C}\), \(G(m, \cdot)\) holomorphic and bounded by \(c(m)\). Indeed, for \(z\) from a connected component of \(\Omega\) where \(\psi\) identically vanishes, let \(G(m, z) = 0\), for the other \(z\) define \(G(m, z) = F(m, z)\psi(z)^{-1}\). Consequently \(F(m, z) = G(m, z)\psi(z) \in \psi H^\infty(\Omega) \subset E(\Omega)\). Moreover,

\[
F(m, A) = \int_\Gamma G(m, z)\psi(z)T_z dz = \int_\Gamma F(m, z)T_z dz \quad (m \in M)
\]

and since \(\|F(m, z)T_z\| \leq c(m)\|\psi(z)T_z\|\) we have

\[
\|F(m, A)\| \leq c(m)C_\psi \quad (m \in M).
\]

Under the assumptions of a), if \(m_n \to m\) in \(M\), then \(F(m_n, z) \to F(m, z)\) for every \(z\). But \(\|F(m_n, z)T_z\| \leq c(m_n)\|\psi(z)T_z\|\) and \(c\) is locally bounded, whence \(F(m_n, A) \to F(m, A)\) in norm by Lebesgue’s theorem.

Now suppose that the assumptions of b) hold true. With \(C := \int_M c(m) \mu(dm)\) we obtain

\[
|f(z)| \leq C|\psi(z)| \quad (z \in \Omega).
\]

We need to know that \(f\) is holomorphic. Since \(\psi\) is holomorphic, it is locally bounded, and so Lebesgue’s theorem shows that \(f\) is continuous. Now Morera’s theorem together with an application of Fubini yields the holomorphy of \(f\). As we have done with \(F(m, \cdot)\) above, we may write \(f = g\psi\) with \(g\) holomorphic, \(|g| \leq C\), and hence \(f \in \psi H^\infty(\Omega) \subset E(\Omega)\). Therefore

\[
f(A) = \int_\Gamma g(z)\psi(z)T_z dz = \int_\Gamma f(z)T_z dz.
\]

On the other hand, \(F(z, m)T_z\) is in \(L^1(M \times \Gamma, \mu \otimes |dz|; L(X))\). Indeed, this function is jointly measurable, and

\[
\|F(z, m)T_z\| \leq c(m)\|\psi(z)T_z\| \in L^1(M) \otimes L^1(\Gamma, |dz|)
\]

by hypothesis. Now an application of Fubini’s theorem concludes the proof. \(\square\)

In the previous fact, the functions were all elementary. We now turn to a more general situation.
4.2 Corollary Let $(E(\Omega), M(\Omega), \Phi)$ be a standard meromorphic functional calculus for $A$, with standard regularizer $\psi$. Let $(M, \Sigma, \mu)$ be a measure space, let $0 \leq g \in \mathcal{L}^1(M, \Sigma, \mu)$, and let

$$F : M \times \Omega \to \mathbb{C}$$

be product measurable such that $f(m, \cdot) \in H^\infty(\Omega)$ and $|f(m, \cdot)| \leq g(m)$ for all $m \in M$. Define

$$f(z) := \int_M F(m, z) \mu(dm)$$

Then $f \in H^\infty(\Omega)$. Moreover $F(\cdot, A)x \in \mathcal{L}^1(M; X)$ and

$$f(A)x = \int_M F(m, A)x \mu(dm) \quad (4.1)$$

for all $x \in \mathcal{R}(\psi(A))$. In particular, if $\psi(A)$ has dense range and also $\|F(m, A)\| \leq g(m)$ for all $m \in M$, then $f(A) \in \mathcal{L}(X)$ and (4.1) holds for all $x \in X$.

Proof. The function $[(m, z) \mapsto F(z, m)\psi(z)T_z]$ is in $\mathcal{L}^1(M \times \Gamma; \mu \otimes |dz|)$. Hence $(m \mapsto F(m, A)\psi(A)) \in \mathcal{L}^1(M; \mathcal{L}(X))$. This yields that $F(\cdot, A)x \in \mathcal{L}^1(M; X)$ for all $x$ from the range of $\psi(A)$. Integrating and applying Fubini yields

$$\int_M F(m, A)x \mu(dm) = \int_M F(m, A)\psi(A)\psi(A)^{-1}x \mu(dm)$$

$$\quad = \int_M \int_\Gamma F(m, z)\psi(z)T_z dz \mu(dm) \psi(A)^{-1}x$$

$$\quad = \int_\gamma f(z)\psi(z)T_z dz \mu(dm) \psi(A)^{-1}x$$

$$\quad = (f\psi)(A)\psi(A)^{-1}x = f(A)x.$$

Suppose that the additional hypotheses are true. Then

$$\|f(A)x\| = \|\int_M F(m, A)x \mu(dm)\| \leq \int_M \|F(m, A)x\| \mu(dm)$$

$$\leq \int_M g(m) \mu(dm) \|x\|$$

for all $x \in \mathcal{R}(\psi(A))$. Since the latter is dense and $f(A)$ is closed, $f(A) \in \mathcal{L}(X)$. Approximating an arbitrary element $x \in X$ by a sequence $(x_n)_n \subset \mathcal{R}(\psi(A))$ concludes the proof.

We shall now apply these results in order to obtain generation theorems for semigroups. As in the theorem we suppose that $(E(\Omega), M(\Omega), \Phi)$ is a standard
meromorphic functional calculus for $A$, with standard regularizer $\psi$. Furthermore, let $f \in \mathcal{M}(\Omega)$ such that $f(\Omega) \subset \text{L}_\omega$ for some $\omega \in \mathbb{R}$. For each $t \geq 0$ we can form the operator 

$$ (e^{tf(z)})(A) $$

since $e^{tf} \in H_\infty(\Omega)$. The next theorem is the abstract analogue to Theorem 3.4 above.

4.3 Theorem Let $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi)$ be a standard meromorphic functional calculus for $A$, with standard regularizer $\psi$, and such that $\mathcal{R}(\psi(A))$ is dense in $X$. Let $f \in \mathcal{M}(\Omega)$ such that $f(\Omega) \subset \text{L}_\omega$ for some $\omega \in \mathbb{R}$. Then the following assertions are equivalent:

(i) $f \in \mathcal{M}(\Omega)A$ and $f(A)$ is the generator of a $C_0$-semigroup $T$.

(ii) For $t \in [0, 1]$, $(e^t f)(A) \in \mathcal{L}(\mathcal{X})$ and 

$$ \sup_{t \in [0,1]} \left\| (e^{tf})(A) \right\| < \infty. $$

In this case, $e^{tf}(A) = T(t)$ for all $t \geq 0$.

Proof. We first set $F(t, z) := e^{tf(z)}\psi(z)$. Then $|F(t, z)| \leq e^{t\omega} |\psi z|$. From Theorem 4.1 it follows that $t \mapsto F(t, A) = (e^t f)(A)\psi(A)$ is continuous and that $\|F(t, A)\| \leq C_\psi e^{t\omega}$. Furthermore, 

$$ \int_0^\infty e^{-\lambda t} F(t, A) \, dt = \left( \frac{\psi}{\lambda - f} \right)(A) \quad (\text{Re} \lambda > \omega). \tag{4.2} $$

Suppose that (i) holds. By c) of Theorem 3.1 $(\lambda - f)^{-1} \in \mathcal{M}(\Omega)A$ and $(\lambda - f)^{-1}(A) = R(\lambda, f(A))$ for all $\lambda$ with sufficiently large real part. By (4.2) 

$$ \int_0^\infty e^{-\lambda t} F(t, A) \, dt = \left( \frac{\psi}{\lambda - f} \right)(A) = R(\lambda, f(A))\psi(A) $$

for $\text{Re} \lambda$ large. The uniqueness of Laplace transforms yields therefore that 

$$(e^{tf})(A)\psi(A) = F(t, A) = T(t)\psi(A) \quad (t \geq 0)$$

and hence $(e^{tf})(A) = T(t)$ on the range of $\psi(A)$ for all $t \geq 0$. Since $\mathcal{R}(\psi(A))$ is dense and $(e^{tf})(A)$ is closed, $(e^{tf})(A) = T(t)$, $t \geq 0$. Thus (ii) holds.

Suppose conversely that (ii) holds. Then by multiplicativity $T(t) := (e^{tf})(A)$ is a bounded operator for every $t \geq 0$, and the semigroup law holds. As in Fact 1.1 one shows that $T$ is exponentially bounded. Since $F(\cdot, A)$ is continuous, $[t \mapsto T(t)x]$ is continuous for every $x \in \mathcal{R}(\psi(A))$. This being dense it follows that $T$ is strongly continuous. It remains to show that $f(A)$ is the generator.
of $T$. We may now apply Corollary 4.2 with $F(t,z) = e^{-\lambda t}e^{tf(z)}$, $M := [0, \infty)$, $\lambda > \omega_0(T)$. This shows that $(\lambda - f)^{-1}(A) = R(\lambda, B)$, where $B$ is the generator of $T$. Since $R(\lambda, B)$ is injective, $(\lambda - f)$ and therefore also $F$ is contained in $\mathcal{M}(\Omega)_A$, and one has

$$\lambda - f(A) = (\lambda - f)(A) = R(\lambda, B)^{-1} = \lambda - B,$$

whence $B = f(A)$ follows. □

5 Convergence Lemma and the Hille–Yosida Theorem

The Hille–Yosida theorem is one of the most fundamental results in the “elementary” theory of $C_0$-semigroups. We show that it is a straightforward consequence of a general fact of functional calculus theory, the so-called Convergence Lemma.

Suppose that $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi)$ is a meromorphic functional calculus for $A$, and $f \in H^\infty(\Omega)$ is given. Unless $f$ is elementary, there is no reason to expect that $f(A)$ is a bounded operator. However, one may find a net $(f_\iota)_\iota \subset H^\infty(\Omega)$ approximating $f$ and so that one already knows that the $f_\iota(A)$ are bounded. As an example, think of approximating

$$e^{tz} = \lim_{n \to \infty} \left(1 - \frac{tz}{n}\right)^{-n}.$$

(See below for more on that example.) The Convergence Lemma gives conditions under which one may draw the desired conclusion. We again have to suppose that the meromorphic calculus is standard in the sense of the previous chapter.

5.1 Theorem (Convergence Lemma)

Let $(\mathcal{E}(\Omega), \mathcal{M}(\Omega), \Phi)$ be a standard meromorphic functional calculus for an operator $A$ on a Banach space $X$, with standard regularizer $\psi$ such that $\psi(A)$ has dense range. Let $(f_\iota)_\iota$ be a net in $H^\infty(\Omega)$ satisfying the following conditions:

(i) $\sup_\iota \|f_\iota\|_\infty < \infty$;
(ii) $f_\iota(A) \in \mathcal{L}(X)$ for all $\iota$, and $\sup_\iota \|f_\iota(A)\| < \infty$;
(iii) $f(z) := \lim_\iota f_\iota(z)$ exists for every $z \in \Omega$.

Then $f \in H^\infty(\Omega)$, $f(A) \in \mathcal{L}(X)$, $f_\iota(A) \to f(A)$ strongly, and $\|f(A)\| \leq \lim inf_\iota \|f_\iota(A)\|$.
Proof. By Vitali’s theorem [2, Theorem A.5] \( f \) is holomorphic and the convergence of the \( f_n \) to \( f \) is uniform on compacts. Moreover, condition (i) clearly implies that \( f \) is bounded. By Lebesgue’s theorem

\[
\lim_{n \to \infty} f_n(x) = f(x) \text{ for all } x \in \mathcal{R}(\psi(\mathcal{A})). \]

Clearly \( \|f(A)x\| \leq \liminf \|f_n(A)x\| \leq \liminf \|f_n(A)\| \|x\| \). Since \( f(A) \) is a closed operator with dense domain \( \mathcal{D}(f(A)) \supset \mathcal{R}(\psi(A)) \), \( f(A) \) is bounded with \( \|f(A)\| \leq \liminf \|f_n(A)\| \). Again by the density of \( \mathcal{R}(\psi(A)) \), \( f(A) \to f(A) \) strongly.

We may now prove the Hille–Yosida Theorem.

5.2 Theorem (Hille–Yosida)

Let \( A \) be a densely defined operator on the Banach space \( X \) such that \( (0, \infty) \subset g(A) \) and \( M := \sup_{\lambda > 0} \|\lambda R(\lambda, A)\|^n < \infty \). Then \( A \) is of half-plane type 0 and \( \|e^{tA}\| \leq M \) for all \( t \geq 0 \).

Proof. First we show that \( A \) is of half-plane type 0. Fix \( \mu > 0 \). For \( \lambda > 0 \) large, more precisely for \( \lambda > \frac{|\mu|^2}{2 \Re \mu} \), one has \( |\lambda - \mu| < \lambda \). By the Laurent series expansion of the resolvent,

\[
R(\mu, A) = \sum_{n=0}^{\infty} (\mu - \lambda)^n R(\lambda, A)^{n+1}
\]

and hence \( \|R(\mu, A)\| \leq M \sum |\mu - \lambda|^n / \lambda^{n+1} = M/(\lambda - |\mu - \lambda|) \). Now let \( \lambda \to \infty \) to conclude \( \|R(\mu, A)\| \leq M/\Re \mu \). It follows that \( A \) is of half-plane type 0.

Define \( r_{n,t}(z) := (1 - tz/n)^{-n} \). For fixed \( \omega \in (0, 1) \) and large \( n \in \mathbb{N} \) we have

\[
\sup_{\Re z \leq \omega} |r_{n,t}(z)| \leq \left( \inf_{\Re z \leq \omega} \left| 1 - \frac{tz}{n} \right| \right)^{-n} \left( 1 - \frac{t\omega}{n} \right)^{-n}.
\]

Since \( (1 - t\omega/n)^{-n} \to e^{t\omega} \) as \( n \to \infty \), we have \( \sup_n \|r_{n,t}\|_\infty < \infty \). Also, by hypothesis, \( \|r_{n,t}(A)\| = \|(1 - (t/n)A)^{-n}\| = \|((n/t)R((n/t), A))^{n}\| \leq M \) for all \( n \in \mathbb{N} \). Applying the Convergence Lemma yields \( \|e^{tA}\| \leq M \), as desired.

A more careful statement of the Convergence Lemma and an equally careful analysis of the above proof would lead to the statement that for each \( x \in X \) one has \( \lim_n r_{n,t}(A)x = e^{tA}x \) uniformly in \( t \) from compact subintervals of \( [0, \infty) \).

The formula

\[
\lim_{n \to \infty} \left( I - \frac{t}{n} A \right)^{-n} x = e^{tA}x \quad (x \in X, t \geq 0)
\]
is called the **Post–Widder approximation** of the semigroup.

### 6 Trotter–Kato Approximation

In the previous chapter we have seen that the Hille–Yosida theorem, one of the cornerstones of semigroup theory, can be deduced easily (and even intuitively) within the functional calculus framework. We now turn to another important result in semigroup theory, namely the theorem(s) of Trotter–Kato type.

While in the Convergence Lemma the function is approximated and the operator is fixed, in the following we fix the function and approximate the operator. The correct setup requires that the approximants $A_n$ are “of the same type” as the operator, with the relevant constants being uniformly bounded. More precisely, a family of operators $(A_\omega)$ is called **uniformly of half-plane type** $\omega \in \mathbb{R} \cup \{-\infty\}$ if each $A_\omega$ is of half-plane type $\omega$ and $\sup \, M_\alpha(A_\omega) < \infty$ for each $\alpha > \omega$.

**Example:** Let $A$ be of half-plane type 0, and let $A_\lambda := \lambda A R(\lambda, A)$ for $\lambda \geq 1$ be the **Yosida approximants**. Then the family $(A_\lambda)_{\lambda \geq 1}$ is uniformly of half-plane type 0. Indeed, a little computation shows

$$R(\mu, A_\lambda) = \frac{\lambda^2}{(\mu + \lambda)^2} R(\lambda \mu / (\lambda + \mu), A) + \frac{1}{\lambda + \mu}$$

For the second term we have $|\lambda + \mu|^{-1} \leq \min(\lambda^{-1}, (\text{Re} \mu)^{-1})$. To estimate the first we compute

$$\text{Re} \left( \frac{\mu \lambda}{\lambda + \mu} \right) = \frac{\lambda^2 \text{Re} \mu + \lambda |\mu|^2}{|\lambda + \mu|^2}.$$

If $A$ happens to be the generator of a bounded $C_0$-semigroup, it satisfies an estimate $\|R(\mu, A)\| \leq M(\text{Re} \mu)^{-1}$ for some $M \geq 1$ and all $\text{Re} \mu > 0$. Given this, one obtains

$$\|R(\mu, A_\lambda)\| \leq \frac{\lambda^2}{|\mu + \lambda|^2} \frac{M |\lambda + \mu|^2}{\lambda^2 \text{Re} \mu + \lambda |\mu|^2} + \frac{1}{\text{Re} \mu + \lambda} \leq \frac{M + 1}{\text{Re} \mu}$$

independent of $\lambda$. In the general case, fix $\alpha > 0$, take $\text{Re} \mu \geq \alpha$ and define $\varepsilon := \alpha/(2\alpha + 1)$. It follows that $\text{Re} \mu / (2 \text{Re} \mu + 1) \geq \varepsilon$ and this implies $\lambda^2(\text{Re} \mu - \varepsilon) \geq 2\lambda \varepsilon \text{ Re} \mu$, since $\lambda \geq 1$. From this we conclude that

$$\text{Re} \left( \frac{\mu \lambda}{\lambda + \mu} \right) = \frac{\lambda^2 \text{Re} \mu + \lambda |\mu|^2}{|\lambda + \mu|^2} \geq \varepsilon,$$
and hence that
\[ \| R(\mu, A) \| \leq \frac{\lambda^2}{(\lambda + \text{Re} \mu)^2} M_\varepsilon(A) + \frac{1}{\text{Re} \mu + \lambda} \leq M_\varepsilon(A) + 1 \]

independent of \( \lambda \geq 1 \) and \( \mu \geq \alpha \).

In the example above we clearly have \( \lim_{\lambda \to \infty} R(\mu, A) = R(\mu, A) \) in norm uniformly in \( \mu \) from compact subsets of the open halfplane (\( \text{Re} \ z > 0 \)). By Theorem 6.3 from the appendix to this chapter, we know that actually convergence in a single point \( \mu \) suffices to ensure the convergence in all points.

6.1 **Theorem**  Let \( (A_i) \) be a net of operators, uniform of half-plane type \( \omega \), and let \( A \) be an operator such that \( R(\mu, A_i) \to R(\mu, A) \) in norm/strongly for one/all \( \text{Re} \mu > \omega \). Then \( A \) is also of half-plane type \( \omega \). Moreover, for \( \alpha > \omega \) and \( f \in \mathcal{E}(L_\alpha) \) one has \( f(A_i) \to f(A) \) in norm/strongly.

Suppose furthermore that \( A \) is densely defined. If \( f \in H^\infty(L_\alpha) \) and \( f(A_i) \in \mathcal{L}(X) \) for all \( i \) with \( C := \sup_i \| f(A_i) \| < \infty \), then also \( f(A) \in \mathcal{L}(X) \), and \( f(A_i) \to f(A) \) strongly.

**Proof.** In view of Theorem 6.3 below, the first two assertions are straightforward. Suppose that \( A \) is densely defined; by Lemma 3.3 this implies already that \( \mathcal{D}(A^2) \) is dense in \( X \). Take \( x \in \mathcal{D}(A^2) \), \( f \in H^\infty(L_\alpha) \), \( \lambda > \alpha \), and \( e(z) := f(z)(\lambda - z)^{-2} \in \mathcal{E}(L_\alpha) \). Then
\[ \| e(A_i)(\lambda - A)^2 x \| = \| e(A_i) R(\lambda, A_i)^2 (\lambda - A)^2 x \| \leq C \| R(\lambda, A_i)^2 (\lambda - A)^2 x \|. \]

Since we know already that \( e(A_i) \to e(A) \) and \( R(\lambda, A_i) \to R(\lambda, A) \), we conclude that
\[ \| f(A)x \| = \| e(A)(\lambda - A)^2 x \| \leq C \| R(\lambda, A)^2 (\lambda - A)^2 x \| = C \| x \|. \]

Since \( \mathcal{D}(A^2) \) is dense, it follows that \( f(A) \in \mathcal{L}(X) \) with \( \| f(A) \| \leq C \). To prove that \( f(A_i) \to f(A) \) strongly, we need only to show \( f(A_i)x \to f(A)x \) for all \( x \in \mathcal{D}(A^2) \). So take \( x \in \mathcal{D}(A^2) \) and let \( y := (\lambda - A)^2 x \). We have seen above that \( f(A_i)R(\lambda, A_i)^2 y \to f(A)x \), so we estimate the difference
\[ \| f(A_i)x - f(A_i)R(\lambda, A_i)^2 y \| = \| f(A_i)R(\lambda, A_i)^2 y - f(A_i)R(\lambda, A_i)^2 y \| \leq C \| R(\lambda, A)^2 y - R(\lambda, A_i)^2 y \| \to 0 \]

by hypothesis.

As in the Convergence Lemma, there is a version of Theorem 6.1 that yields some uniformity: suppose that \( (f_\kappa)_\kappa \subset H^\infty(L_\omega) \) is uniformly bounded and
\[ \sup_{i,\kappa} \| f_\kappa(A_i) \| < \infty. \] Then the convergence \( f_\kappa(A_i)x \to f_\kappa(A)x \) is uniform in \( \kappa \), for every \( x \in X \).

As a consequence we obtain the classical Trotter–Kato theorem.

### 6.2 Theorem (Trotter–Kato)

Suppose that \((A_i)_i\) is a net of operators such that each \( A_i \) generates a \( C_0 \)-semigroup such that for some \( M \geq 1, \omega \in \mathbb{R} \) the stability condition

\[ \| e^{tA} \| \leq Me^{\omega t} \quad (t \geq 0) \]

is satisfied. Suppose further that \( A \) is a densely defined operator and for some \( \lambda_0 > 0 \) one has \( \lambda_0 \in \rho(A) \) and \( R(\lambda_0, A_n) \to R(\lambda_0, A) \) strongly. Then \( A \) generates a \( C_0 \)-semigroup and one has \( e^{tA_i}x \to e^{tA}x \) uniformly in \( t \in [0,\tau] \), for each \( x \in X, \tau > 0 \).

**Proof.** It follows readily from the hypothesis that \((A_i)_i\) is uniformly of half-plane type \( \omega \). So the theorem is a consequence of Theorem 6.1 and the remark immediately after. \( \square \)
Remarks:

1) Assumptions on $A$ implying the resolvent convergence are the following: the operator $A$ is densely defined, $\lambda_0 - A$ has dense range, and there exists a core $D$ of $A$ such that $A, x \to Ax$ for all $x \in D$. See [4, Theorem III.4.9].

2) Sometimes the existence of the operator $A$ is not clear. Let us start with a net $(A_\iota)$ satisfying the stability condition, and merely ask that $R(\lambda, A_\iota)x \to Q \in L(X)$ strongly, for some bounded operator $Q$ with dense range. Then by analogous arguments as in the Appendix below, $R_\lambda x := \lim_{\iota} R(\lambda, A_\iota)x$ exists for all $x \in X$ and all $\Re \lambda > \omega$. Clearly $(R_\lambda)_\lambda$ satisfies the resolvent identity. Furthermore, $\|R_\lambda\| = O(\lambda^{-1})$ as $\Re \lambda \to \infty$. By the resolvent identity and the density of the range of $Q = R_{\lambda_0}$, it follows that $\lambda R_\lambda \to I$ strongly. This shows that all operators $R_\lambda$ are injective, and one can apply Fact 1.4.

3) The Trotter–Kato Theorems as presented in [4] also give a converse: If $A_\iota \sim T_\iota(t)$ satisfy the stability condition and $T_\iota(t) \to T(t)$ strongly, uniformly in $t$ from compact intervals, then $(T(t))_{t \geq 0}$ is a semigroup (trivial) and its generator $A$ satisfies $R(\lambda, A_\iota) \to R(\lambda, A)$ strongly, for all $\Re \lambda > \omega$. This follows easily by taking Laplace transforms.

Appendix: Resolvent approximation

We have to provide some general operator theory, based on [5, Appendix A.5] which in turn is inspired by [7]. Suppose one is given a net of operators $(A_\iota)_\iota$ and a point $\mu \in \bigcap_\iota \varrho(A_\iota)$ such that $R(\mu, A_\iota) \to R(\mu, A)$ for some other operator $A$ such that $\mu \in \varrho(A)$. The convergence is in norm/strongly. What can one say about other resolvent points? To answer this question let us suppose in addition that

$$C_\mu := \sup_\iota \|R(\mu, A_\iota)\| < \infty.$$ 

This is no loss of generality if the net is a sequence or the convergence is in norm. Under the boundedness assumption it then follows that $\|R(\mu, A)\| \leq C_\mu$ too. By Fact 1.3, if $|\lambda - \mu| < 1/C_\mu$ then $\lambda \in \varrho(A) \cap \bigcap_\iota \varrho(A_\iota)$ and

$$R(\lambda, A_\iota) = \sum_{n \geq 0} (\mu - \lambda)^n R(\mu, A_\iota)^{n+1},$$

hence $\|R(\lambda, A_\iota)\| \leq \frac{C_\mu}{1 - C_\mu |\lambda - \mu|}$, and the same with $A_\iota$ replaced by $A$. In particular, one has that $R(\lambda, A_\iota)$ is bounded in norm uniformly in $\iota$ and $\lambda$ close to $\mu$. We contend that actually $R(\lambda, A_\iota) \to R(\lambda, A)$ in norm/strongly for
$|\lambda - \mu| < 1/C_\mu$. To see this, compute

$$\| R(\lambda, A)x - R(\lambda, A_\iota)x \| \leq \sum_{n=0}^{N} |\lambda - \mu| \| R(\mu, A)^{n+1}x - R(\mu, A)^{n+1}x \|$$

$$+ 2 \sum_{n>N} |\mu - \lambda|^n C_\mu^{n+1} \| x \| .$$

This also shows that the convergence is even uniformly in $\lambda$ with $|\lambda - \mu| \leq 1/(2C_\mu)$.

6.3 Theorem Let $\Omega \subset \mathbb{C}$ be connected, and suppose one is given a net of operators $(A_\iota)$, such that $\Omega \subset \varrho(A_\iota)$ for all $\iota$ and $C_\lambda := \sup_\iota \| R(\lambda, A_\iota) \| < \infty$ for all $\lambda \in \Omega$. Suppose that $A$ is another operator such that there exists $\mu \in \Omega \cap \varrho(A)$ such that

$$R(\mu, A) \rightarrow R(\mu, A) \quad \text{in norm/strongly.} \quad (6.1)$$

Then $\Omega \subset \varrho(A)$ and (6.1) holds for all $\mu \in \Omega$. Moreover, the convergence is uniform in $\mu$ from compact subsets of $\Omega$.

Proof. Define $M := \{ \mu \in \Omega \mid (6.1) \text{ holds} \}$. Then $M$ is not empty, by hypothesis. We show that $M$ is open and closed in $M$. In fact, that $M$ is open follows immediately from the previous considerations, where we also proved that the convergence is locally uniform. To prove the closedness, take a sequence $(\mu_n)_n \subset M$ such that $\mu_n \rightarrow \mu \in \Omega$. Then $R(\mu_n, A_\iota) \rightarrow R(\mu, A_\iota)$ in norm, uniformly in $\iota$. To see this, use the Laurent series representation to obtain the estimate

$$\| R(\mu_n, A_\iota) - R(\mu, A_\iota) \| \leq \frac{C_\mu^2 |\mu - \mu_n|}{1 - C_\mu |\mu - \mu_n|},$$

valid if $C_\mu |\mu_n - \mu| < 1$. Now, by hypothesis, $R(\mu_n, A_\iota) \rightarrow R(\mu_n, A)$ for all $\mu_n$ in norm/strongly. By the general interchanging-of-limits result there exists an operator $Q$ such that $R(\mu_n, A) \rightarrow Q$ in norm/strongly. By Fact 1.3 $\mu \in \varrho(A)$ and using the continuity of $R(\cdot, A)$ we obtain $Q = R(\mu, A)$. □

7 Phillips Calculus

Suppose that $A$ generates a bounded $C_0$-semigroup. For a bounded measure $\mu \in M(\mathbb{R}_+)$ one may form the operator $T_\mu \in \mathcal{L}(X)$ defined by

$$T_\mu x = \int T(t)x \mu(dt) \quad (x \in X). \quad (7.1)$$
7.1 Fact Let $\mu, \nu \in M(\mathbb{R}_+)$. Then

$$T_\mu T_\nu = T_{\mu * \nu}.$$ 

Proof. This is the semigroup law together with the mere definition of convolution of measures:

$$T_\mu T_\nu = \int_{\mathbb{R}_+} T(t) x \nu(dt) \mu(ds) = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} T(s + t) x \nu(dt) \mu(ds) = \int_{\mathbb{R}_+} T(r) x (\mu * \nu)(dr) = T_{\mu * \nu} x.$$ 

□

Of course, we expect $T_\mu = f(A)$, where

$$f(z) = \mathcal{L}(\mu)(z) = \int_{\mathbb{R}} e^{tz} \mu(dt) \quad (\text{Re } z \leq 0). \quad (7.2)$$

Here we encounter a difficulty. Our functional calculus defined so far only allows functions which are meromorphic on a half-plane $L_\alpha$, with $\alpha$ strictly larger than 0. The function $f$ in (7.2) on the other hand is only defined on the closed left half-plane, holomorphic in the interior and continuous up to the imaginary axis.

7.2 Fact Let $\alpha > 0$ and $\psi \in \mathcal{E}(L_\alpha)$. Then there exists $\mu \in M(\mathbb{R}_+)$ such that $\psi = \mathcal{L}(\mu)$. More precisely, $\mu$ is given as $\mu(ds) = g(s) ds$, where

$$g(s) = \frac{1}{2\pi i} \int_{\text{Re } z = \delta} \psi(z) e^{-zs} dz \quad (s \geq 0) \quad (7.3)$$

and $\delta \in (0, \alpha)$.

Proof. Define $g$ as in the formula above. By Fact 7.1 above,

$$\psi(a) = \frac{1}{2\pi i} \int_{\text{Re } z = \delta} \frac{\psi(z)}{z - a} dz = \frac{1}{2\pi i} \int_{\text{Re } z = \delta} \psi(z) \int_0^\infty e^{(a-z)s} ds dz = \int_0^\infty e^{as} g(s) ds = \mathcal{L}(\mu)(a)$$

for $\text{Re } a \leq 0$, by Fubini’s theorem. It is easy to see that $g \in L^1(\mathbb{R}_+)$ and that Fubini is applicable. □

7.3 Theorem Suppose that $A$ generates a bounded $C_0$-semigroup on $X$. Let $\mu \in M(\mathbb{R}_+)$ be such that $f = \mathcal{L}(\mu)$ is contained in $\mathcal{M}(L_\alpha)$ for some $\alpha > 0$. Then $f(A) = T_\mu \in \mathcal{L}(X)$. 

Proof. We first suppose that \( f = \psi \in \mathcal{E}(L_\alpha) \). By Fact 7.2 \( \mu = g(s) ds \) is its pre-Laplace transform., with \( g \) as in (7.3). Hence
\[
T_\mu = \int_0^\infty T(s)g(s) \, ds = \int_0^\infty T(s) \frac{1}{2\pi i} \int_{\text{Re}=\delta} \psi(z)e^{-sz} \, dz \, ds
= \frac{1}{2\pi i} \int_{\text{Re}=\delta} \psi(z) \int_0^\infty T(s)e^{-sz} \, ds \, dz = \frac{1}{2\pi i} \int_{\text{Re}=\delta} \psi(z)R(z, A) \, dz
= \psi(A).
\]
In the general case, let \( e \in \mathcal{E} \) be a regularizer for \( f \). Choose \( \nu \in \mathcal{M}(\mathbb{R}_+) \) such that \( L(\nu) \equiv e \). Then clearly \( ef = L(\nu) L(\mu) = L(\nu * \mu) \). Hence
\[
(ef)(A) = T_{\nu*\mu} = T_\nu T_\mu = e(A)T_\mu.
\]
Multiplying by \( e(A)^{-1} \) on both sides yields \( f(A) = T_\mu \).

So the Phillips calculus and the “Cauchy calculus” are compatible, but as remarked above, the Phillips calculus is not contained in the Cauchy calculus. Of course one may use the Phillips calculus as the elementary one and then extend by the abstract extension procedure, for example within the algebra of functions meromorphic in the open halfplane \( L_0 \) and with continuous trace on \( i\mathbb{R} \). However, since the class of Laplace transforms of measures is complicated and is not characterized by growth conditions, it is in practice of no use, since regularizability of a function is difficult to check. Other drawbacks are that a Convergence Lemma does not hold for the Phillips calculus, and also general spectral mapping theorems (cf. [5, Section 2.7]) fail.

8 Sectorial and Strip-type Operators

This section is to provide an overview over two new types of operators and their corresponding functional calculi.

Step 0: The domain set

Let
\[
S_{t_\omega} := \{ z \in \mathbb{C} \mid |\text{Im} \, z| < \omega \} \quad (\omega > 0)
S_\omega := \{ z \in \mathbb{C} \setminus \{0\} \mid |\text{arg} \, z| < \omega \} \quad (\omega \in (0, \pi])
\]
with the degenerate cases \( S_{t_0} := \mathbb{R}, S_0 := (0, \infty) \).

Step 1: The type of the operator

Let \( \omega \geq 0 \). An operator \( A \) is of **strip-type** \( \omega \), if \( \pm iA \) are both of half-plane type \( \omega \), i.e., \( \sigma(A) \subset S_{t_\omega} \) and
\[
\sup\{ \|R(\lambda, A)\| \mid |\text{Im} \, \lambda| > \alpha \} < \infty
\]
for each \( \alpha > \omega \).
Let \( \omega \in [0, \pi) \). An operator is called \textbf{sectorial} of angle \( \omega \) if \( \sigma(A) \subset \overline{S_{\omega}} \) and
\[
\sup\{\|\lambda R(\lambda, A)\| \mid |\arg \lambda| \in [\alpha, \pi]\} < \infty
\]
for each \( \alpha \in (\omega, \pi] \). For the sake of simplicity we confine ourselves in this lecture course to \textit{injective} sectorial operators\(^4\).

**Step 2: The elementary function class**

On the strip \( S_{\omega} \) consider
\[
E(S_{\omega}) := \{ f : S_{\omega} \to \mathbb{C} \mid f \text{ is holomorphic and } \exists M, s > 0 : |f(z)| \leq M |z|^{-s} \text{ as } z \to \infty \}.
\]
Then it is established as in the half-plane case that the Cauchy formula holds: for \( f \in E(S_{\omega}) \)
\[
f(a) = \frac{1}{2\pi i} \int_{\Gamma} f(z) \frac{1}{z - a} \, dz, \quad 0 = \frac{1}{2\pi i} \int_{\Gamma} f(z) \, dz.
\]
Here \( \Gamma = \{ \text{Im } z = \pm \delta \} \) with \( |\text{Im } a| < \delta < \omega \).

On the sector \( S_{\omega} \) consider
\[
E(S_{\omega}) := \{ f : S_{\omega} \to \mathbb{C} \mid f \text{ is holomorphic and } \exists M, s > 0 : |f(z)| \leq M \min(|z|^s, |z|^{-s}) \text{ as } z \to 0, \infty \}.
\]
For historical reasons we write also \( H^\infty_{\omega}(S_{\omega}) \) for \( E(S_{\omega}) \). Analogously to the half-plane case one establishes that the Cauchy formula holds: for \( f \in H^\infty_{\omega}(S_{\omega}) \)
\[
f(a) = \frac{1}{2\pi i} \int_{\Gamma} f(z) \frac{1}{z - a} \, dz, \quad 0 = \frac{1}{2\pi i} \int_{\Gamma} f(z) \, dz;
\]
here, \( \Gamma = \partial S_{\delta} \) with \( |\text{arg } a| < \delta < \omega \).

**Step 3: The elementary functional calculus**

Let \( A \) be of strip-type \( \omega \geq 0, \alpha > \omega \) and \( f \in E(S_{\alpha}) \). Then we define
\[
\Phi(f) := f(A) := \frac{1}{2\pi i} \int_{\partial St_{\delta}} f(z) R(z, A) \, dz,
\]
where \( \delta \in (\omega, \alpha) \) is arbitrary(!). For the mapping \( \Phi : E(S_{\alpha}) \to \mathcal{L}(X) \) Theorem 2.3 holds \textit{mutatis mutandis}.

\(^4\)In the general case to construct the elementary calculus a single Cauchy-integral is not sufficient, but it is not hard to remedy this, see [5].
Let $A$ be sectorial of angle $\omega \in [0, \pi)$, $\alpha \in (\omega, \pi]$ and $f \in H^\infty_0(S_\alpha)$. Then we define

$$\Phi(f) := f(A) := \frac{1}{2\pi i} \int_{\partial S_\delta} f(z)R(z, A) \, dz$$

where $\delta \in (\omega, \alpha)$ is arbitrary(!). Again, for the mapping $\Phi : \mathcal{E}(S_\alpha) \rightarrow \mathcal{L}(X)$ Theorem 2.3 holds mutatis mutandis. However, functions $(\lambda - z)^{-1}(\mu - z)^{-1}$ are not contained in $H^\infty_0$, and their role is taken by the functions

$$\frac{z}{(\lambda - z)(\mu - z)} \in H^\infty(S_\alpha) \quad (\lambda, \mu \notin S_\alpha).$$

**Step 4: Extension**

For an operator $A$ of strip type $\omega \geq 0$ the mapping $\Phi$ defined above yields — for each $\alpha > \omega$ — an abstract functional calculus $(\mathcal{E}(S_\alpha), \mathcal{M}(S_\alpha), \Phi)$ which is proper, since

$$\Phi \left( \frac{1}{(\lambda - z)(\mu - z)} \right)(A) = R(\lambda, A)R(\mu, A)$$

is injective ($\lambda, \mu \notin S_\alpha$). Of course, functions $F$ that grow at most polynomially are regularizable (by rationals). In particular, each bounded function is regularizable. Prominent examples are the functions $e^{sz}$, $s \in \mathbb{R}$, and their derivates $\cos(sz), \sin(sz)$. If $\omega < 1$ then $\arctan(z)$ is regularizable. Furthermore, one can show that in the case that $\omega < \pi$, also the function $e^{z}$ is regularizable.

For a sectorial operator $A$ of angle $\omega$ the mapping $\Phi$ above yields — for each $\alpha \in (\omega, \pi]$ — an abstract functional calculus $(\mathcal{E}(S_\alpha), \mathcal{M}(S_\alpha), \Phi)$ which is proper, since

$$\Phi \left( \frac{z}{(\lambda - z)(\mu - z)} \right)(A) = AR(\lambda, A)R(\mu, A)$$

is injective ($\lambda, \mu \notin S_\alpha$). The function $z(1 + z)^{-2}$ is a regularizer for every bounded function, so $H^\infty(S_\alpha) \subset \mathcal{M}(S_\alpha)$. Furthermore, the function $\log z$ is regularizable by $z(1 + z)^{-2}$, and so the **operator logarithm**

$$\log A := (\log z)(A)$$

is defined. Powers of $z(1 + z)^{-2}$ regularize all functions that grow at most polynomially at 0 and at $\infty$. Hence the **fractional powers**

$$A^\alpha = (z^\alpha)(A) = (z^{\alpha \log z})(A) \quad (\alpha \in \mathbb{C})$$
These operators play a central role in the regularity theory of parabolic problems. For \( t \geq 0 \) the function \( e^{-tz} \) is bounded on the halfplane \( R_0 = S_{\pi/2} \). Even more,

\[
e^{-tz} - (1 + z)^{-1} \in H^\infty_0(S_\alpha)
\]

for every \( \alpha < \pi/2 \). Hence if \( \omega < \pi/2 \),

\[
e^{-tA} := (e^{-tz})(A) \quad (t \geq 0)
\]

forms a semigroup (which is strongly continuous iff \( A \) is densely defined). By performing a simple change of variables in the Cauchy-integral, one sees that the semigroup is uniformly bounded. Replacing \( t \) by \( \lambda \) with \( |\arg \lambda| < \pi/2 - \omega \) one proves that the semigroup is even analytic of angle \( \pi/2 - \omega \). This establishes a correspondence between \textbf{bounded holomorphic semigroups} and sectorial operators of angle \( < \pi/2 \). Consult [5, Section 3.4] or [2, Section 3.7] and [4, Section II.4].

\textbf{Exercise 15:} Let \( A \) be of strip-type \( \omega \geq 0 \). Let \( \alpha > 0 \). Show that the constructed meromorphic functional calculus \((\mathcal{E}(St_\alpha), \mathcal{M}(St_\alpha), \Phi)\) is a \textbf{standard functional calculus} in the sense of Chapter 4. What could be a standard regularizer \( \psi \)? Prove that \( \psi(A) \) has dense range if and only if \( A \) is densely defined. Can you formulate a Convergence Lemma and a Trotter-Kato type theorem? Do the same for the sector situation.

\section{9 Composition Rules}

In the half-plane case we already encountered the concept of a composition rule. Namely, inserting a shifted operator into a function is the same as inserting the original operator into the shifted function:

\[
[f(z)](A + \omega) = [f(z + \omega)](A).
\]

The form of this is

\[
f(g(A)) = (f \circ g)(A);
\]

to render this meaningful one needs that \( g(A) \) is defined by some functional calculus for \( A \) and that \( f(g(A)) \) is defined for some functional calculus for \( g(A) \), and that \( g \circ f \) has sense. The philosophy is that the formula is true whenever all is meaningful, but this cannot be a mathematical theorem. One has to establish composition rules in each special case.

One example is the famous sector-strip correspondence via the logarithm. A theorem of \textsc{Nollau} reads that if \( A \) is sectorial of angle \( \omega \) and injective, then
log $A$ is of strip-type $\omega$ [5, Lemma 3.5.1]. Since the logarithm maps sector to
strips, one expects

$$f(\log A) = (f \circ \log z)(A)$$

for all, say, $f \in H^\infty(St_\alpha)$, $\alpha > \omega$. This is indeed true, and the proofs are also
“generic”. We will deal with a similar situation.

9.1 Fact Suppose that $A$ is an operator on $X$ with $\sigma(A) \subset St_\omega$ such that

$$\|R(\lambda, A)\| \leq \frac{M}{|\Im \lambda| - \omega} \quad (|\Im \lambda| > \omega).$$

Then for each $\alpha > \omega$ the operator $B := \alpha^2 + A^2$ is invertible and sectorial of
some angle $\angle < \pi/2$.

Proof. We have $B = \alpha^2 + A^2 = (\alpha + iA)(-\alpha + A)$ and this is a product of
invertible operators, hence $B$ is invertible.

Let $\mu \in \mathbb{C}, \Re \mu \leq 0$. Let $\lambda = x + iy$ with $x, y \in \mathbb{R}$ such that $\lambda^2 = \mu - \alpha^2$, and $y \geq \alpha$. (This is possible: one has $\lambda^2 = (x^2 - y^2) + 2ixy = \mu - \alpha^2$, hence $y^2 - \alpha^2 = x^2 - \Re \mu \geq 0$, and so $y^2 \geq \alpha^2$.) Now

$$\mu R(\mu, B) = \mu R(\mu, \alpha^2 + A^2) = \mu R(\mu - \alpha^2, A^2) = \mu R(\alpha^2, A^2)$$

$$= - (\lambda^2 + \alpha^2) R(\lambda, A) R(-\lambda, A).$$

Estimating yields

$$\|\mu R(\mu, B)\| \leq M^2 \frac{|\lambda|^2 + \alpha^2}{(|\Im \lambda| - \omega)^2} = M^2 \frac{x^2 + y^2 + \alpha^2}{(y - \omega)^2}$$

But $x^2 + \alpha^2 = \Re \mu + y^2 \leq y^2$, and so

$$\|\mu R(\mu, B)\| \leq 2M^2 \left( \frac{y}{y - \omega} \right)^2$$

and this is uniformly bounded in $y \in [\alpha, \infty)$.

A standard Taylor series argument [5, Proposition 2.1.1] yields that the angle
of sectoriality of $B$ must be strictly less than $\pi/2$. $\square$

It is clear that one could be more precise here, and in fact determine the spectral
sector of $B$ as a function of $\alpha$. It turns out that the sector becomes smaller as $\alpha$
increases.

9.2 Theorem Let $A, \omega, \alpha, B$ as above, and let $f \in \mathcal{M}(S_{\pi/2})$ such that $f(B)$ is
defined. Then $[f(\alpha^2 + z^2)](A)$ is defined and

$$[f(\alpha^2 + z^2)](A) = f(B) = f(\alpha^2 + A^2).$$
For let $g(z) := \alpha^2 + z^2$. In a first step one makes sure that the formula holds for the resolvent of $B$, i.e., for elementary rationals $f$. That is, for $\lambda \notin \alpha^2 + St_{\omega}^2 = \lambda - g^{-1} \in H^\infty(St_\theta)$ for some $\theta > \omega$. Hence $(\lambda - g), (\lambda - g)^{-1}$ are both regularizable in the functional calculus for $A$, by Exercise 14 this yields

$$(\lambda - g)^{-1}(A) = [(\lambda - g)(A)]^{-1} = (\lambda - B)^{-1}.$$

We now look for elementary functions $f$ such that the statement holds. The idea is to choose $f$ in such a way that $f \circ g$ is also elementary, so one can manipulate Cauchy integrals. Therefore, let $f$ be elementary on $S_{\omega/2}$ such that $f = O(|z|^{-s})$ as $|z| \to \infty$, for some $s > 1/2$. Then $f \circ g \in \mathcal{E}(St_\theta)$ for $\theta \in (\omega, \alpha)$. (Note that, since $\theta < \alpha$, $g(St_\alpha) \subset S_{\pi/2}$, and so $f \circ g$ is defined on the whole strip $St_g$.)

Fix $\theta \in (\omega, \alpha)$. Then choose a contour $\partial S_\delta$ with $\delta$ “big enough”; that means bigger than the sectoriality angle of $B$ and such that $\overline{g(St_\theta)} \subset St_\theta$. This guarantees that if $z \in \partial S_\delta$ then $(z - g)^{-1} \in H^\infty(St_\theta)$. Choose any regularizer $e \in \mathcal{E}(St_\theta)$, and fix $e^\prime \in (\omega, \theta)$. Then

$$e(A)f(B) = \frac{1}{2\pi i} \int_{\partial S_\delta} f(z)e(A)R(z, B) \, dz = \frac{1}{2\pi i} \int_{\partial S_\delta} f(z) \left( \frac{e}{z - g} \right)(A) \, dz$$

$$= \frac{1}{2\pi i} \int_{\partial S_\delta} f(z) \frac{1}{2\pi i} \int_{\partial S_\delta^\prime} \left( \frac{e(w)}{z - g(w)} \right) R(w, A) \, dw \, dz$$

$$= \frac{1}{2\pi i} \int_{\partial S_\delta^\prime} e(w) \frac{1}{2\pi i} \int_{\partial S_\delta} \left( \frac{f(z)}{z - g(w)} \right) dz R(w, A) \, dw$$

$$= \frac{1}{2\pi i} \int_{\partial S_\delta^\prime} e(w)f(g(w))R(w, A) \, dw = e(A)(f \circ g)(A),$$

since $(f \circ g)(A) \in \mathcal{L}(X)$. Hence $f(B) = (f \circ g)(A)$ as required.

In the general case we use what we know already. Let $e$ be any regularizer for $f$ in the functional calculus for $B = g(A)$. Without loss of generality we may suppose that $e, ef = O(|z|^{-1})$ as $|z| \to \infty$ (replace $e$ by a power of itself, if necessary). We claim that $e \circ g$ is a regularizer for $f \circ g$ in the functional calculus for $A$. Indeed $(e \circ g)(f \circ g) = (ef) \circ g$ is elementary, and $(e \circ g)(A) = e(g(A)) = e(B)$ is injective. Hence $f \circ g$ is regularizable, and

$$(f \circ g)(A) = [(e \circ g)(A)]^{-1}[(e \circ g)(f \circ g)](A) = e(B)^{-1}[(ef) \circ g](A)$$

$$= e(B)^{-1}(ef)(B) = f(B).$$

This concludes the proof. $\square$

The proof shows that the first and the last steps are comparably simple, but the second requires good knowledge of the mapping behaviour of $g$. For more on composition rules see [5, Sections 1.3.2, 2.4, 4.2].
Exercise 16: Suppose that $A, \omega$ are as in Fact 9.1. Here is a different approach to the operators $\alpha^2 + A^2$. For $\Re \lambda > 0$ consider the operator

$$T_\lambda := (e^{-\lambda z^2})(A).$$

Note that the functions $e^{-\lambda z^2}$ are elementary on every horizontal strip, so $T_\lambda \in \mathcal{L}(X)$. It is clear that the mapping

$$\lambda \mapsto T_\lambda : \{\Re \lambda > 0\} \longrightarrow \mathcal{L}(X)$$

is a semigroup. Prove that it is holomorphic. By using the definition of $T_\lambda$ via Cauchy integrals, derive an exponential bound for $(T_\lambda)^\phi$ for each $\phi < \pi/2$. Show that $-A^2$ is the generator of this semigroups, in the sense that the Laplace transform of the semigroup yields the resolvent of $-A^2$. Using the correspondence of holomorphic semigroups and sectorial operators [5, Section 3.4], determine the generic angle of sectoriality of $\alpha^2 + A^2$, $\alpha > \omega$.

10 Transference Principles for Groups

From now on, our standard assumption is that $-iA$ is the generator of a $C_0$-group $U = (U(t))_{t \in \mathbb{R}}$. Both semigroups $(U(t))_{t \geq 0}, (U(-t))_{t \geq 0}$ are exponentially bounded, hence the group type

$$\theta(U) := \inf \{ \theta \geq 0 \mid \exists M \geq 1 : \|U(s)\| \leq Me^{\theta|s|} (s \in \mathbb{R}) \} < \infty$$

is finite. Both operators $\pm iA$ are of half-plane type $\theta(U)$. In particular, $A$ is of strip-type $\theta(U)$. Instead of the Phillips calculus for semigroups we shall use (and therefore have to define) a Phillips calculus for groups. If

$$\|U(s)\| \leq Me^{|s|} \quad (s \in \mathbb{R})$$

one considers

$$\mathcal{M}_\omega(\mathbb{R}) := \{ \mu \mid \|\nu\|_{\mathcal{M}_\omega} := \int e^{|s|} |\nu| (ds) < \infty \}$$

and for $\mu \in \mathcal{M}_\omega(\mathbb{R})$ the operator $T_\mu$ defined by

$$T_\mu x = \int U(s)x \mu(ds) \quad (x \in X).$$

It is easy to show that $\mathcal{M}_\omega(\mathbb{R})$ is a convolution algebra and $T_{\mu*\nu} = T_\mu T_\nu$ as in the semigroup case. For $\mu \in \mathcal{M}_\omega(\mathbb{R})$ its Fourier transform is

$$\hat{\mu}(z) := \int e^{-izs} \mu(ds) \quad (|\Im z| \leq \omega)$$
and \( \hat{\mu} \) is holomorphic in \( St_\omega \) and bounded and continuous on \( \overline{St_\omega} \). As in the semigroup case, if \( \alpha > \omega \) and \( f \in \mathcal{E}(St_\alpha) \), then there is \( \mu \in M_\omega \) with \( \hat{\mu} = f \). More precisely, one has the following result.

10.1 Fact Let \( \alpha > 0 \) and let \( f \in \mathcal{E}(St_\alpha) \). Define

\[
g(s) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{ists} f(t) \, dt \quad (s \in \mathbb{R})
\]

Then \( g \in L^1(\mathbb{R}) := L^1(\mathbb{R}) \cap M_\omega(\mathbb{R}) \) for all \( \omega \in [0, \theta) \), and \( \hat{g} = f \). Moreover, \( g \) is Hölder continuous of some exponent \( \gamma \in (0, 1) \).

Proof. By an application of Cauchy’s theorem, one has

\[
g(s) = \frac{1}{2\pi} \int_{\mathbb{R} + i\delta} e^{isz} f(z) \, dz
\]

for all \( \delta \in (-\theta, \theta) \). Estimating yields

\[
|g(s)| \leq \frac{1}{2\pi} e^{-\delta s} \int_{\mathbb{R}} |f(t + i\delta)| \, dt
\]

and changing \( \delta \) into \(-\delta \) yields an estimate \(|g(s)| \leq M e^{-\delta s}, s \in \mathbb{R}, \) for \( \delta \in [0, \theta) \). The identity \( \hat{g} = f \) is then usual Fourier inversion. The Hölder continuity is a bit more difficult. Suppose that \( f = O(\|t\|^{-(1+\varepsilon)}) \) as \( |t| \to \infty \). Let \( 1/2 < \alpha < \varepsilon + 1/2 \). Then \( f(t), |t|^\alpha f(t) \in L^2(\mathbb{R}) \). By Plancherel’s theorem and some interpolation theory, \( g \in W^{\alpha,2}(\mathbb{R}) \). The embedding \( W^{\alpha,2}(\mathbb{R}) \subset C^{\alpha-1/2}(\mathbb{R}) \) is a well-known Sobolev embedding, see [1]. □

Exercise 17: The last argument may not be easily accessible for some readers. Suppose that \( \varepsilon > 1/2 \) and let \( \alpha = 1 \). Prove from \( f \in L^2 \) and \( |t|f \in L^2 \) that \( f \in W^{1,2} \) by using Plancherel. Use the Cauchy–Schwarz inequality to prove the embedding \( W^{1,2}(\mathbb{R}) \subset C^{1/2}(\mathbb{R}) \).

As in the semigroup case one can now prove that \( T_\mu = \hat{\mu}(A) \) if \( \mu \in M_\theta(\mathbb{R}) \) for some \( \theta > \theta(U) \). Here is a little example.

Exercise 18: Let \(-iA\) be the generator of a \( C_0 \)-group \( U \). Prove by using the above results that \( -A^2 \) is the generator of a holomorphic semigroup \( T \) of angle \( \pi/2 \) given by

\[
T(\lambda) = \left( e^{-\lambda z^2} \right)(A) = \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} e^{-s^2/4\lambda} U(s) \, ds \quad (\text{Re} \lambda > 0).
\]
We shall now consider measures with compact supports. Let \( M > 0 \) and \( \mu \in M[-M,M] \); then \( \hat{\mu} \) is an entire function, bounded on every strip \( St_\theta, \theta > 0 \), and one has \( T_\mu = \hat{\mu}(A) \). A **transference principle** is an estimate of \( T_\mu \) in terms of a convolution operator on a vector-valued function space.

**The Transference Principle**

To be more precise, fix \( p \in [1, \infty) \) and let \( Y := L^p(\mathbb{R}; X) \). Denote by \( L_\mu := (f \mapsto \mu * f) \) the convolution operator associated with \( \mu \). Then by Young's inequality, \( L_\mu \in L(Y) \) with \( \|L_\mu\| \leq \|\mu\|_{M(\mathbb{R})} \). The aim is to establish an estimate of the form

\[
\|T_\mu\|_{L(X)} \leq C \|L_\mu\|_{L(Y)},
\]

and in order to do this we shall factorize the operator \( T_\mu \) over \( Y \), i.e., we write \( T_\mu = PL_\mu P \). This is the picture:

\[
\begin{array}{ccc}
Y & \xrightarrow{L_\mu} & Y \\
\downarrow{\iota} & & \downarrow{P} \\
X & \xrightarrow{T_\mu} & X
\end{array}
\]

where \( \iota : X \to Y \) and \( P : Y \to X \) are bounded operators. This yields the estimate \( \|T_\mu\| \leq \|\iota\| \|P\| \|L_\mu\| \). By varying \( \iota, P \) one may eliminate certain dependencies on \( \mu \), as we shall see below.

How are we to choose \( \iota \)? Fix \( N > 0, t \in [-N,N] \) and compute

\[
U(-t)T_\mu x = \int_{-M}^{M} U(s-t) x \mu(ds) = \int_{-M}^{M} U(-(t-s)) x \mu(ds).
\]

The last term can be written as

\[
... = (\mu * \iota_N x)(t) = (L_\mu \iota_N x)(t)
\]

where \( \iota_N : X \to Y \) is given by

\[
\iota_N(x) := (s \mapsto 1_{[-(M+N), (M+N)]}(s)U(-s)x).
\]

The mapping \( P = P_N : Y \to X \) has to be chosen so that \( T_\mu = P_N L_\mu \iota_N \). This is achieved by letting

\[
P_N f := \frac{1}{2N} \int_{-N}^{N} U(t)f(t) \, ds \quad (f \in Y).
\]

Indeed,

\[
P_N L_\mu \iota_N x = P_N(\mu * \iota_M x) = \frac{1}{2N} \int_{-N}^{N} U(t)U(-t)T_\mu x \, dt = T_\mu x
\]
for \( x \in X \). Let us compute the norms of \( \iota_N \) and \( P_N \). For arbitrary \( x \in X \)

\[
\| \iota_N x \|_Y = \left( \int_{-(M+N)}^{(M+N)} \| U(-s)x \|^p \, ds \right)^{1/p} \leq C_{M+N}(2(M+N))^{1/p} \| x \|_X,
\]

with \( C_{M+N} := \sup \{ \| U(s) \| \mid |s| \leq M + N \} \). On the other hand, for \( f \in Y \)

\[
\| P_N f \| \leq \frac{1}{2N} \int_{-N}^N \| U(t)f(t) \| \, dt \leq \frac{C_N}{2N} (2N)^{1/p'} \| f \|_{L^p(R;X)}
\]

\[
= C_M (2N)^{-1/p} \| f \|_Y.
\]

This shows that

\[
\| \iota_N \| \leq C_{M+N}(2(M+N))^{1/p}, \quad \| P_N \| \leq C_N(2N)^{-1/p}
\]

and thus

\[
\| T_{\mu} \| \leq C_{M+N}^2 \left( 1 + \frac{M}{N} \right)^{1/p} \| L_\mu \|.
\]

(10.1)

Specializing \( M = N = 1 \) yields the first conclusion.

10.2 Theorem Let \( \mu \in M[-1,1] \) and let \(-iA \sim U\). Then

\[
\| \hat{\mu}(A) \| \leq C^2 2^{1/p} \| L_\mu \|_{\mathcal{L}(L^p(R;X))}
\]

where \( C = \sup_{|s| \leq 2} \| U(s) \| \).

For bounded groups one may go much further. This is a classical result of Coifman–Weiss and Berkson–Gillespie–Muhly.

10.3 Theorem Let \(-iA \) be the generator of a bounded group \( U \). Then

\[
\| T_{\mu} \| \leq C^2 \| L_\mu \|_{\mathcal{L}(L^p(R;X))} \quad (\mu \in M(R))
\]

with \( C := \sup_{s \in R} \| U(s) \| \).

Proof. The measures with compact support are dense within \( M(R) \) with respect to the total variation norm. Since one has also \( \| L_\mu \|_{\mathcal{L}(L^p(R;X))} \leq \| \mu \|_{M(R)} \) by Young’s inequality, it suffices to prove the estimate for \( \mu \) with \( \text{supp}(\mu) \subset [-M,M] \) for some \( M > 0 \). Letting \( N \to \infty \) in (10.1) concludes the proof. \( \square \)

The last theorem has an interesting consequence for functional calculus and evolution equations. However, we are interested in general (unbounded) groups and so will focus on applications of Theorem 10.2.
11 Vector-valued Harmonic Analysis and UMD Spaces

To make effective use of the above transference results, one needs information about the norm of the convolution operators $L_\mu$ on the vector-valued $L^p$-spaces $L^p(\mathbb{R}; X)$. A convenient road to go is to make an hypothesis on (the geometry of) the Banach space $X$. For example, if $X = H$ is a Hilbert space and $p = 2$ then $\|L_\mu\|_{L^p(\mathbb{R}; X)} = \|\hat{\mu}\|_{L^\infty(\mathbb{R})}$, by the (vector-valued) Plancherel theorem. The wish to be more general leads to the class of UMD spaces, which we now introduce.

Let $X$ be a Banach space. The Fourier transform $\mathcal{F}: L^1(\mathbb{R}; X) \rightarrow C_0(\mathbb{R}; X)$ is defined by

$$(\mathcal{F}f)(t) = \hat{f}(t) = \int_{\mathbb{R}} e^{-ist} f(s) \, ds \quad (t \in \mathbb{R}).$$

(One proves $\mathcal{F}L^1(\mathbb{R}; X) \subset C_0(\mathbb{R}; X)$ as in the scalar case, or reduces to it by noting that $L^1(\mathbb{R} \otimes X)$ is dense in $L^1(\mathbb{R}; X)$.) The Fourier transform is injective, and its inverse is given by

$$\mathcal{F}^{-1}g(s) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ist} g(t) \, dt \quad (s \in \mathbb{R}).$$

In particular, $\mathcal{F}$ is an isomorphism on the class of Schwartz functions $S(\mathbb{R}; X)$. If $\mu \in \mathcal{M}(\mathbb{R})$ and $f \in L^1(\mathbb{R}; X)$ then their convolution is defined as

$$\mu * f = \int_{\mathbb{R}} f(\cdot - s) \mu(ds) \in L^1(\mathbb{R}; X),$$

i.e., one integrates the bounded and continuous function $(s \mapsto f(\cdot - s)) \in C^b(\mathbb{R}; L^1(\mathbb{R}; X))$ against $\mu$. It is then easily established that

$$\mathcal{F}(\mu * f)(t) = \hat{\mu}(t)\hat{f}(t) \quad (t \in \mathbb{R}).$$

So one has

$$L_\mu f = \mathcal{F}^{-1}(\hat{\mu} \mathcal{F}f) \quad (f \in S(\mathbb{R}; X)).$$

Since $\|\mu * f\|_{L^p} \leq \|\mu\|_M \|f\|_{L^q}$ (Young’s inequality) one says that $L_\mu$ is a bounded $L^p$–Fourier multiplier with symbol $m$.

More generally, let $m \in L^\infty(\mathbb{R})$ and consider the operator $T_m: S(\mathbb{R}; X) \rightarrow C_0(\mathbb{R}; X)$ defined by

$$T_m f := \mathcal{F}^{-1}(mf) \quad (f \in S(\mathbb{R}; X)).$$
Let $1 \leq p < \infty$. One says that $T_m$ is a bounded $L^p(\mathbb{R}; X)$–Fourier multiplier with symbol $m$ if there is a constant $c = c(p, m)$ such that

$$\|T_m f\|_{L^p} \leq c \|f\|_{L^p} \quad (f \in \mathcal{S}(\mathbb{R}; X)).$$

Since $\mathcal{S}(\mathbb{R}; X)$ is dense in $L^p(\mathbb{R}; X)$, in this case $T_m$ extends uniquely to a bounded operator on $L^p(\mathbb{R}; X)$. One defines

$$\mathcal{M}_p(X) := \{m \in L^\infty(\mathbb{R}) \mid T_m \text{ is a bounded } L^p(\mathbb{R}; X)\text{-Fourier multiplier}\}$$

with norm $\|m\|_{\mathcal{M}_p} = \|T_m\|_{L(L^p(\mathbb{R}; X))}$. Young’s inequality (see above) shows that $\mathcal{F}M(\mathbb{R}) \subset \mathcal{M}_p(X)$ for every $p$ and every $X$. One also can show that $\mathcal{M}_1(X) = \mathcal{F}M(\mathbb{R})$ for every Banach space $X$. Plancherel’s theorem says that $\mathcal{F}$ extends to an (almost isometric) isomorphism on $L^2(\mathbb{R}; X)$ when $X$ is a Hilbert space. Consequently, if $X = H$ is Hilbert

$$\mathcal{M}_2(H) = L^\infty(\mathbb{R}).$$

All results not covered by these cases are very difficult.

The Hilbert Transform and UMD Spaces

The Hilbert transform is the Fourier multiplier operator $H = T_m$ with $m(t) = -i \text{sgn}(t)$. On Schwartz functions this operator is given by the singular convolution

$$Hf(t) = \frac{1}{\pi} \int_\mathbb{R} \frac{f(t-s)}{s} ds \quad (t \in \mathbb{R}, f \in \mathcal{S}(\mathbb{R}; X)).$$

(This is not obvious.) A space $X$ is called a UMD space if $H$ is a bounded operator on $L^2(\mathbb{R}; X)$, i.e. if $\text{sgn} t \in \mathcal{M}_2(X)$. Here are some known facts about this notion.

1) $X$ is UMD iff $\text{sgn} t \in \mathcal{M}_p(X)$ for one/all $p \in (1, \infty)$.
2) Every Hilbert space is UMD.
3) If $(\Omega, \mu)$ is a measure space and $X$ is UMD, then also $L^p(\Omega, \mu; X)$ is UMD, for every $p \in (1, \infty)$.
4) If $X$ is UMD then $X$ is (super)reflexive.

In the next section we will need the following characterization.

11.1 Fact Let $X$ be a Banach space. For $\varepsilon \in (0, 1)$ let the operator $H_\varepsilon$ be defined by

$$H_\varepsilon f = \int_{|s| \leq 1} \frac{f(t-s)}{s} ds = \mu_\varepsilon * f$$

with $\mu_\varepsilon = 1_{|s| \leq 1(s)}ds/s$. Then the following assertions are equivalent:
(i) \( X \) is UMD.

(ii) \( \sup_{\varepsilon \in (0,1)} \| H_\varepsilon \|_{L^2(\mathbb{R};X)} < \infty. \)

(iii) \( H_0 f := \lim_{\varepsilon \downarrow 0} H_\varepsilon f \) exists for all \( f \in L^2(\mathbb{R};X) \).

**Proof.** We skip the proof. However, no really deep facts are required. \( \square \)

## 12 Functional Calculus for Unbounded Groups on UMD Spaces

Again we suppose that \(-iA\) generates a \( C_0\)-group \( U \) on the Banach space \( X \). We shall establish the boundedness of certain operators \( f(A) \) under the assumption that \( X \) is a UMD space. The first function is the crucial point.

Define
\[
h(z) := 2 \int_0^z \sin(sz) \frac{ds}{s} \quad (z \in \mathbb{C}).
\]

Here are some of its properties:

### 12.1 Fact

a) \( h' = 2\frac{\sin z}{z} = \mathcal{F}(\mathfrak{1}_{(-1,1)}) \) and
\[
h(z) = 2 \int_0^z \frac{\sin w}{w} dw \quad (z \in \mathbb{C}).
\]

b) For all \( \theta > 0 : h \in H^\infty(St_\theta) \) and
\[
h(z) \to \pm \pi \quad \text{as} \quad \text{Re} z \to \pm \infty, |\text{Im} z| < \theta.
\]

c) \( h(z) = i \lim_{\varepsilon \downarrow 0} \int_{\varepsilon \leq |s| \leq 1} e^{-isz} \frac{ds}{s} = i \lim_{\varepsilon \downarrow 0} \hat{\mu}_\varepsilon(z) \)

**Proof.** a) is straightforward:
\[
h'(z) = 2 \int_0^1 \cos(sz) \, ds = 2 \frac{\sin(sz)}{z} \bigg|_0^1 = 2 \frac{\sin z}{z}.
\]

To prove b) we use a) and write
\[
\frac{1}{2} h(z) = \int_0^z \frac{\sin w}{w} dw = \int_0^{\text{Re} z} \frac{\sin s}{s} ds + \int_{\text{Re} z}^z \frac{\sin w}{w} dw
\]
By classical analysis, the first summand tends to $\pm \pi/2$, and the second to 0 as long as has $|\text{Im } z| < \theta$. The proof of c) is again simple:

$$i \int_{|s| \leq 1} e^{-isz} ds = i \int_{-1}^1 e^{-isz} ds + i \int_1 e^{-isz} ds$$

$$= -2 \int_1 \frac{e^{-isz} - e^{isz}}{2is} ds = 2 \int_1 \sin sz \frac{ds}{s} \rightarrow h(z)$$
as $\varepsilon \to 0$. $\square$

In general $h(A) \notin L(X)$. An example is $U$ being the shift group on $X = L^1(\mathbb{R})$. One can show that if $h(A)$ is bounded in this case, $h$ must be the Fourier transform of a bounded measure. However, if $f = \hat{\mu}$ is the Fourier transform of a bounded measure $\mu$ then $f$ has finite Cesaro limits in $\pm \infty$, and these Cesaro limits are both equal to $\mu\{0\}$, see [5, Proposition E.4.3]. In particular, if $f$ has ordinary limits, then these limits must coincide. This is not the case with $h$.

12.2 Theorem Let $-iA$ be the generator of a $C_0$-group $U$ on a UMD space $X$. Then $h(A) \in L(X)$ and

$$h(A)x = i \lim_{\varepsilon \to 0} \int_{|s| \leq 1} U(s)x \frac{ds}{s} \quad (x \in X).$$

Moreover, one has

$$\sup_{r \in \mathbb{R}} \|h(A + r)\| < \infty.$$

Proof. First of all, note that $-i(A + r)$ generates the group $(e^{-isr}U(s))_{s \in \mathbb{R}}$. Define $h_\varepsilon := i\mathcal{F}(\mu_\varepsilon)$. Then

$$h_\varepsilon(z) = 2 \int_\varepsilon^1 \sin(sz) \frac{ds}{s} = h(z) - 2 \int_0^\varepsilon \sin(sz) \frac{ds}{s} = h(z) - h(\varepsilon z).$$

This shows that $h_\varepsilon \rightharpoonup h$ uniformly on compact sets, and that $\sup_{\varepsilon} \|h_\varepsilon\|_{H^{\infty}(St_\theta)} < \infty$ for each $\theta > 0$. The transference principle yields

$$\|h_\varepsilon(A)\| \leq C^2 \sqrt{2} \|H_\varepsilon\|_{L^2(\mathbb{R};X)}$$

for all $\varepsilon \in (0,1)$, where $C = \sup_{|s| \leq 1} \|U(s)\|$. Since $X$ is UMD, it follows from Fact 11.1 that

$$K := \sup_{\varepsilon \in (0,1)} \|H_\varepsilon\|_{L^2(\mathbb{R};X)} < \infty.$$

One may now apply the Convergence Lemma (in its version for groups) to conclude that $h(A) \in L(X)$, $h_\varepsilon(A) \rightharpoonup h(A)$ strongly and

$$\|h(A)\| \leq KC^2 \sqrt{2}.$$
Since $C$ is the same for all groups $(e^{-isr}U(s))_{s \in \mathbb{R}, r \in \mathbb{R}}$, the required boundedness is clear.

Suppose a function $f \in H^\infty(St_\theta)$ can be written as

$$f = \hat{g} + ch + d,$$

(12.1)

where $g \in L^1_\omega(\mathbb{R})$ and $c, d \in \mathbb{C}$. Then $f$ has limits at $\text{Re} \ z = \pm \infty$ with

$$f(\infty) = \pi c + d, \quad f(-\infty) = -\pi c + d.$$

(This shows that the representation (12.1) is unique.) Furthermore, there exists a function $k$ that is contained in each $L^1_\omega(\mathbb{R})$, $\theta \in [0, \omega)$, such that $f' = \hat{k}$. Indeed, $k = (-is)g + c \mathbf{1}_{(-1,1)}$.

12.3 Theorem Let $-iA$ be the generator of a $C_0$-group $U$ on a UMD space $X$, $\theta > \theta(U)$, and let $f \in H^\infty(St_\theta)$. Suppose that $f$ admits a decomposition

$$f = \hat{g} + ch + d,$$

(12.2)

with $g \in L^1_\omega(\mathbb{R})$ for some $\omega > \theta(U)$ and $c, d \in \mathbb{C}$. Then

$$\sup_{r \in \mathbb{R}} \|f(A + r)\| < \infty$$

and

$$f(A)x = dx + \lim_{\varepsilon \downarrow 0} \int_{|s| \geq \varepsilon} ik(s)U(s)x \frac{ds}{s} \quad (x \in X)$$

(12.3)

where $\hat{k} = f'$ and $d = |f(\infty) + f(-\infty)|/2$.

The condition on $f$ is satisfied in particular if $f' \in \mathcal{E}(\theta)$.

**Proof.** The first statement follows directly from the previous proposition. To prove the representation formula (12.3) we fix $x \in X$ and compute

$$f(A)x = dx + \hat{g}(A)x + ch(A)x$$

$$= dx + \int_{\mathbb{R}} g(s)U(s)x ds + ic \text{PV} - \int_{-1}^{1} U(s)x \frac{ds}{s}$$

$$= dx + \lim_{\varepsilon \downarrow 0} \int_{|s| \geq \varepsilon} [sg(s) + ic \mathbf{1}_{(-1,1)}]U(s)x \frac{ds}{s}$$

$$= dx + \lim_{\varepsilon \downarrow 0} \int_{|s| \geq \varepsilon} ik(s)U(s)x \frac{ds}{s}$$

where $k(s) = (-is)g + c \mathbf{1}_{(-1,1)}$. It was observed above that $\hat{k} = f'$. The formula for $d$ is also clear from the above remarks. Suppose now that $f' \in \mathcal{E}(\theta)$. We apply Fact 10.1 to find a function $k$ such that $\hat{k} = f'$. Let

$$g(s) := \frac{i}{s} k(s) \mathbf{1}_{|s| \geq 1}(s) + \frac{k(s) - k(0)}{s} \mathbf{1}_{(-1,1)}(s) \quad (s \in \mathbb{R}).$$
Since $k$ is Hölder continuous at zero, $g \in L^1_\omega(\mathbb{R})$ for all $\omega \in [0, \theta)$. Writing $c := k(0)$ and recalling that $h' = \mathcal{F}(1_{(-1,1)})$, one obtains

\[(\hat{g} + ch)' = \mathcal{F}((\text{is})g) + ch' = \mathcal{F}((-\text{is})g + c1_{(-1,1)}) = \hat{k} = f'.\]

Hence there is a constant $d \in \mathbb{C}$ such that $f = \hat{g} + ch + d$. This finishes the proof. \( \square \)

**Two examples**

The function $f(z) = \arctan z$ is holomorphic and bounded on each strip $St_\theta$, $\theta < 1$. Moreover, $f'(z) = (1+z^2)^{-1} \in \mathcal{E}(\theta)$. Hence if $-iA$ generates a $C_0$-group on a UMD space $X$ such that $\theta(U) < 1$, then $\arctan(A) \in \mathcal{L}(X)$. Moreover, one has the representation

\[
\arctan(A)x = \frac{i}{2} \text{PV} - \int_{\mathbb{R}} e^{-|s|}U(s)x \frac{ds}{s} \quad (x \in X).
\]

This follows from Theorem 12.3 since

\[
f'(z) = \frac{1}{1 + z^2} = \mathcal{F}\left(\frac{1}{2} e^{-|s|}\right).
\]

A second example is

\[
f(z) := (1 - iz)(1 + z^2)^{-1/2} = (1 + z^2)^{1/2}(1 + iz)^{-1} = \left(\frac{1 - iz}{1 + iz}\right)^{1/2} = e^{-i \arctan z}.
\]

The function $f$ is bounded and $f' = -if/(1 + z^2)$ is elementary on every strip $St_\theta$, $\theta \in (0, 1)$. Hence $f(A)$ is bounded, when $-iA$ generates a group $U$ such that $\theta(U) < 1$. General functional calculus rules and the composition rule of Theorem 9.2 yield that

\[(1 - iA)(1 + A^2)^{-1/2} = (1 + A^2)^{1/2}(1 + iA)^{-1}\]

is bounded, and this implies that

\[D(A) = D\left(1 + A^2\right)^{1/2}\]

This identity is equivalent to Fattorini’s famous theorem on cosine functions on UMD spaces. This will be the topic of a forthcoming lecture course.
References


