Lectures on Functional Analysis

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Preface

The present text forms the basis of a one-semester course which I have given over the past three years at Delft University of Technology. The students usually are in the first semester of their fourth year at university, after having obtained a Bachelor's degree. The course is designed for 14 weeks, each comprising 90 minutes of lectures plus a 45 minutes problem class.

Due to my special audience this is not a conventional functional analysis course. Students here are well-trained in applied maths, with a good background in computing, modelling, differential equations. On the other hand, they miss some central theoretical notions — for instance metric spaces or measure-theoretical integration theory — and many of them have difficulties to assume abstract concepts, to formulate mathematical statements correctly, and to give rigorous proofs. In many cases, the basic mathematical skill to look at a concrete example through the glasses of abstract concepts is underdeveloped.

It is clear that under these circumstances not only the material of a beginning functional analysis course had to be adapted, but also the way of its presentation. On the contents' side, classical material had to be dropped, like the treatment of dual spaces and the Hahn-Banach theorem, the Baire theorem and its consequences (open mapping theorem, uniform boundedness principle), the Arzelá-Ascoli and the abstract Stone-Weierstrass theorem, compact operators other than on Hilbert spaces, ℓ^p-spaces other than $p = 1, 2, \infty$. Furthermore, the lack of knowledge in measure theory forbids a treatment of really decent applications in PDE, so one is reduced to toy examples showing the essential features.

Practically none of the existing textbooks on functional analysis meets these requirements, not even the books [**15**], [**10**] and [**14**], which aim at an undergraduate level. ("Graduate level" and "undergraduate level" appear to be quite relative notions anyway.) The book which comes closest in spirit to what seems feasible here are the excellent books by Nicholas Young [**18**] and Ward Cheney [**4**], from which I profitted most during writing these notes. Although they go far beyond what we do here, I recommend them to any student interested in applicable analysis.

On the didactical side, beginning with abstract topological notions would just knock out the majority of my students already in the first lecture. The whole classical top down approach from general to concrete appears to be not feasible. On the contrary, it is one of the major didactical challenges of this course to build up basic conceptual mathematical skills in the first place. By passing through elementary functional analysis, the students should also learn to use proper mathematical language, to appreciate and practice rigour, to "embrace the pleasure and acknowledge the power of abstraction".

Such a goal, if ever, can only be reached if the students do not get into contact with the material only in a superficial way. We cannot be content with the students just "having seen some functional analysis". What should be the merit of that? Only *depth* can really make a difference. It is therefore no harm that many classical topics are not covered, as long as one knows why the students should study those that remain and makes sure that these are treated thoroughly. Now, here are my main points:

- viewing a function/sequence as a *point* in a certain space (abstracting from its internal structure);
- the concept of an abstract *distance* (metric/norm) to be able to treat *approximations* (convergence);
- the *diversity* of such distances and the associated notions of convergence;
- the role of *orthogonality* in Hilbert spaces for an efficient way of approximation (Fourier series);
- *completeness* as a necessary condition for finding solutions to minimization and fixed-point problems;
- the notion of a *weak derivative* to facilitate the search for solutions of differential equations;
- the concept of an *operator*, emerging as a unified method of producing solutions to a problem with varying initial data.

In composing the course, I tried to obey the rule that a new abstract notion can only be introduced *after* the students have seen concrete examples. Also, not too many recognizably difficult things — for instance abstract topology *and* abstract linear algebra — should be novel to them at the same time. (These principles imply that one can neither start with metric spaces nor with normed linear spaces right away.) Here is a brief synopsis of the course, with the major difficulties attached in square brackets.

1) inner product spaces; no topology/convergence whatsoever, only basic algebraic manipulations. But: examples; the analogy between $\mathbb{K}^d$ and $\mathrm{C}[a, b]$.
   [The concept of a vector space of functions, functions as points.]

2) Cauchy–Schwarz and the triangle inequality for the norm associated with an inner product; the example ℓ^2; then general notion of a norm and other examples, e.g., sup-norm, 1-norm; bounded linear mappings with (very few) examples, operator norm; still no convergence whatsoever.
   [The wealth of examples, sequences as points.]

3) metric spaces and convergence; different metrics lead to different notions of convergence: uniform, pointwise and 2-norm convergence as examples; closure is defined via sequences; then density, with Weierstrass as example. No other topological notions.
   [Different convergence notions; sequences of functions, sequences of sequences.]

4) closed sets (more important than open ones) defined via sequences; continuity, defined via sequences (the $\epsilon - \delta$ definition is mentioned but never used); examples: continuity of the norm, of addition etc.; continuity=boundedness for linear maps; compactness (very briefly, hardly ever used), equivalence of norms.
   [Wealth of the material, sequence definition of convergence.]

5) Cauchy sequences and completeness; Hilbert space and the completeness of ℓ^2; non-completeness of $\mathrm{C}[a, b]$ with the 2-norm; Banach spaces, completeness of $\mathcal{B}(\Omega)$, ℓ^∞, $\mathrm{C}[a, b]$ with the sup-norm; series, absolute convergence, orthogonal series in Hilbert spaces.
   [Completeness proofs, again sequences of sequences]

5a) Density principles; is not to be presented in the course, just used as a reference later.

6) a round-up in Lebesgue integration, almost no proofs; central notions: Lebesgue measure, null set, equality almost everywhere, the spaces L^1 and L^2. Dominated Covergence and completeness, density of $\mathrm{C}[a, b]$.
   [Superficiality of the presentation, wealth of material, equivalence classes of functions]

7) best approximations in Hilbert space; orthogonal projections and decomposition; Riesz-Fréchet; abstract Fourier expansions.
[Abstract nature of the material, few concrete examples.]

8) weak derivatives of $L^2(a,b)$ functions, $H^1(a,b)$, the integration operator J, FTC for H^1-functions, higher order Sobolev spaces, Poincaré inequality (1-dim), the solution of Poisson's problem via Riesz–Fréchet.
[Weak derivatives.]

9) J as an integral operator, Fubini round-up, Green's function for the Poisson problem on $[a,b]$; Hilbert–Schmidt integral operators, the Laplace transform as a non-Hilbert–Schmidt integral operator; the operator norm, examples; completeness of the space of bounded linear operators; the Neumann series and abstract Volterra operators; Hilbert space adjoints, examples; compact operators on Hilbert spaces (defined as limits of finite rank operators); Hilbert–Schmidt integral operators are compact; subsequence criterion for compactness (diagonal argument in the proof).
[To work with the integration operator; operator norm convergence (vs. strong convergence), definition of the adjoint, diagonal proof]

10) eigenvalues and approximate eigenvalues, are basically the same for compact operators, self-adjoint operators, equality of numerical radius and norm; the spectral theorem for compact self-adjoint operators, spectral decomposition; solution theory of the eigenvalue problem.
[Abstract treatment, no examples here.]

11) applications of the spectral theorem: the Dirichlet-Laplacian, one-dimensional Schrödinger operator with positive potential, the associated Sturm-Liouville problem with Green's functions; the heat equation for the Schrödinger operator; the operator norm of J, leads to the Laplace operator with mixed boundary conditions; the best constant in the Poincaré inequality, needs the spectral decomposition of the Dirichlet-Laplacian.
[The need of all the foregoing material.]

Every chapter is accompanied by a set of exercises (about ten per week), some of them utmost elementary. I do not claim that it is necessary to do all of them, as what's necessary does strongly depend on the student. However, being able to solve these, the student is strongly expected to pass the exam. Besides these "quasi-obligatory" exercises, I also collected some others, for further training or just for the fun of it. In later lectures I sometimes refer to one of those.

The offical name of this course is "Applied Functional Analysis". Now, the functional analysis presented here is certainly applied every day, just not by us in this course. To make my point clear: this is not a failure of the course, only of its name. Cheney has called his own book "Analysis *for* Applied Mathematics" and he views it as *prolegomena* for applications. That's pretty much what this course is supposed to be, too. Only that we are even further away, given that this course makes up less than a quarter of his.

These lecture notes are not in their final version, and who knows whether they will ever be. They undergo changes according to my experience with students and hopefully improve over the years.

This is the second (2010) edition, and I am very grateful to all my students who helped me finding mistakes in the first version and improve on them. And I am grateful for their patience, because in the first version there were so many mistakes that at some points reading was quite an unpleasant experience. I dearly hope that the current version shows much fewer shortcomings, but anyone who is writing him- or herself knows how incredibly blind one is towards self-made errors. So I again have to ask for patience, and apologize in advance for all inconvenience that is still hidden in the text.

Finally, I have to admit that some of the plans from last year for the current version could not be realized: there are still no diagrams and additional sections on the Banach contraction principle and the uniform boundednesss theorem are still missing, too. I am not happy about this, especially concerning the diagrams. But I am full of hope for the future ...

Delft, 9 July 2010 Markus Haase

Chapter 1

Inner Product Spaces

The main objects of study in functional analysis are *function spaces*, i.e., vector spaces of real or complex-valued functions on certain sets. Although much of the theory can be done in the context of real vector spaces, at certain points it is very convenient to have vector spaces over $\mathbb{C}$. So we introduce the generic notation $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$. Background on linear algebra is collected in Appendix A.7.

We begin by introducing two of the main players.

The space $\mathbb{K}^d$. This is the set of all tuples $x = (x_1, \dots, x_d)$ with components $x_1, \dots, x_d \in \mathbb{K}$:

$$\mathbb{K}^d := \{x = (x_1, \dots, x_d) \mid x_1, \dots, x_d \in \mathbb{K}\}.$$

It is a vector space over $\mathbb{K}$ with the obvious (i.e., componentwise) operations:

$$(x_1, \dots, x_d) + (y_1, \dots, y_d) := (x_1 + y_1, \dots, x_d + y_d),$$
$$\lambda(x_1, \dots, x_d) := (\lambda x_1, \dots, \lambda x_d).$$

The space $\mathrm{C}[a, b]$. We let $[a, b]$ be any closed interval of $\mathbb{R}$ of positive length. Let us define

$$\mathcal{F}[a, b] := \{f \mid f : [a, b] \longrightarrow \mathbb{K}\}$$
$$\mathrm{C}[a, b] := \{f \mid f : [a, b] \longrightarrow \mathbb{K}, \text{continuous}\}.$$

If $\mathbb{K} = \mathbb{C}$ then $f : [a, b] \longrightarrow \mathbb{C}$ can be written as $f = \operatorname{Re} f + \mathrm{i} \operatorname{Im} f$ with $\operatorname{Re} f, \operatorname{Im} f$ being real-valued functions; and f is continuous if and only if both $\operatorname{Re} f, \operatorname{Im} f$ are continuous.

Let us define the sum and the scalar multiple of functions pointwise, i.e.,

$$(f + g)(t) := f(t) + g(t), \quad (\lambda f)(t) := \lambda f(t)$$

where $f, g : [a, b] \longrightarrow \mathbb{K}$ are functions, $\lambda \in \mathbb{K}$ and $t \in [a, b]$. This turns the set $\mathcal{F}[a, b]$ into a vector space over $\mathbb{K}$ (see also Appendix A.7).

If $f, g \in \mathrm{C}[a, b]$ then we know from elementary analysis that $f + g, \lambda f \in \mathrm{C}[a, b]$ again, and so $\mathrm{C}[a, b]$ is a subspace of $\mathcal{F}[a, b]$, hence a vector space in its own right.

Advice/Comment:
We use the notation $\mathrm{C}[a, b]$ for the generic case, leaving open whether $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. If we want to stress a particular choice of $\mathbb{K}$, we write $\mathrm{C}([a, b]; \mathbb{C})$ or $\mathrm{C}([a, b]; \mathbb{R})$. If we use the notation $\mathrm{C}[a, b]$ in concrete situations, it is always tacitly assumed that we have the more general case $\mathbb{K} = \mathbb{C}$.

Similar remarks apply to $\mathcal{F}[a, b]$ and all other function spaces we will encounter in this lecture.

There is an analogy between these two examples. Namely, note that each vector $(x_1, \ldots, x_d) \in \mathbb{K}^d$ defines a map

$$x : \{1, \ldots, d\} \longrightarrow \mathbb{K} \quad \text{by} \quad x(j) := x_j \quad (j = 1, \ldots, d).$$

Conversely, each such function x determines exactly one vector $(x(1), \ldots, x(d))$. Apart from a set-theoretical point of view, there is no difference between the vector and the corresponding function, and we will henceforth identify them. So we may write

$$\mathbb{K}^d = \mathcal{F}(\{1, \ldots, d\}; \mathbb{K}).$$

A short look will convince you that the addition and scalar multiplication in vector notation coincides precisely with the pointwise sum and scalar multiplication of functions.

How far can we push the analogy between $\mathbb{K}^d$ and $\mathrm{C}[a, b]$? Well, the first result is negative:

Theorem 1.1. *The space* $\mathbb{K}^d$ *has a basis consisting of precisely* d *vectors, hence is finite-dimensional. The space* $\mathrm{C}[a, b]$ *is not finite-dimensional. For example, the set of monomials* $\{1, t, t^2, \ldots\}$ *is an infinite linearly independent subset of* $\mathrm{C}[a, b]$.

Proof. The first assertion is known from linear algebra. Let us turn to the second. Let

$$p(t) := a_n t^n + \cdots + a_1 t + a_0$$

be a finite linear combination of monomials, i.e., $a_0, \ldots, a_n \in \mathbb{K}$. We suppose that not all coefficients a_j are zero, and we have to show that then p cannot be the zero function.

Now, if $p(c) = 0$ then by long division we can find a polynomial q such that $p(t) = (t - c)q(t)$ and $\deg q < \deg p$. If one applies this successively, one may write

$$p(t) = (t - c_1)(t - c_2)\dots(t - c_k)\, q(t)$$

for some $k \le n$ and some polynomial q that has no zeroes in $[a, b]$. But that means that p can have only finitely many zeroes in $[a, b]$. Since the interval $[a, b]$ has infinitely many points, we are done. □

(See Exercise 1.1 for an alternative proof.) Ex.1.1

1.1. Inner Products

We now come to a positive result. Recall that on $\mathbb{K}^d$ we can define

$$\langle x, y\rangle := x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_d\overline{y_d} = \sum_{j=1}^{d} x_j\overline{y_j}$$

for $x, y \in \mathbb{K}^d$. We call this the **standard inner product** of the two vectors x, y. If $\mathbb{K} = \mathbb{R}$, this is the usual scalar product you know from undergraduate courses; for $\mathbb{K} = \mathbb{C}$ this is a natural extension of it.

Analogously, we define the **standard inner product** on $\mathrm{C}[a, b]$ by

$$\langle f, g\rangle := \int_a^b f(t)\overline{g(t)}\,\mathrm{d}t$$

for $f, g \in \mathrm{C}[a, b]$. There is a general notion behind these examples.

Definition 1.2. Let E be a vector space. A mapping

$$E \times E \longrightarrow \mathbb{K}, \quad (f, g) \longmapsto \langle f, g\rangle$$

is called an **inner product** or a **scalar product** if it is *sesquilinear*:

$$\langle \lambda f + \mu g, h\rangle = \lambda \langle f, h\rangle + \mu \langle g, h\rangle$$
$$\langle h, \lambda f + \mu g\rangle = \overline{\lambda} \langle h, f\rangle + \overline{\mu} \langle h, g\rangle \qquad (f, g, h \in E, \lambda, \mu \in \mathbb{K}),$$

symmetric:

$$\langle f, g\rangle = \overline{\langle g, f\rangle} \qquad (f, g \in E),$$

positive:

$$\langle f, f\rangle \ge 0 \qquad (f \in E),$$

and *definite*:

$$\langle f, f\rangle = 0 \quad \Longrightarrow \quad f = 0 \qquad (f \in E).$$

A vector space E together with an inner product on it is called an **inner product space** or a **pre-Hilbert space**.

Advice/Comment:
There are different conventions to denote generic inner products, for example

$$\langle f, g\rangle, \quad (f \mid g), \quad \langle f|g\rangle \quad \text{or simply} \quad (f, g).$$

The latter convention has the disadvantage that it is the same as for the *ordered pair* (f, g) and this may cause considerable confusion. In these notes we stick to the notation $\langle f, g\rangle$.

The proof that the standard inner product on $\mathrm{C}[a, b]$ is sesquilinear,
Ex.1.2 symmetric and positive is an exercise. The definiteness is more interesting and derives from the following fact.

Lemma 1.3. *Let $f \in \mathrm{C}[a, b]$, $f \geq 0$. If $\int_a^b f(t)\,\mathrm{d}t = 0$ then $f = 0$.*

Proof. To prove the statement, suppose towards a contradiction that $f \neq 0$. Then there is $t_0 \in (a, b)$ where $f(t_0) \neq 0$, i.e. $f(t_0) > 0$. By continuity, there are $\epsilon, \delta > 0$ such that

$$|t - t_0| \leq \delta \quad \Rightarrow \quad f(t) \geq \epsilon.$$

But then

$$\int_a^b f(t)\,\mathrm{d}t \geq \int_{t_0-\delta}^{t_0+\delta} f(t)\,\mathrm{d}t \geq 2\delta\epsilon > 0,$$

which contradicts the hypothesis. □

Using this lemma, we prove definiteness as follows: Suppose that $f \in \mathrm{C}[a, b]$ is such that $\langle f, f\rangle = 0$. Then

$$0 = \langle f, f\rangle = \int_a^b f(t)\overline{f(t)}\,\mathrm{d}t = \int_a^b |f(t)|^2\,\mathrm{d}t$$

Since $|f|^2$ is also a continuous function, the previous lemma applies and yields $|f|^2 = 0$. But this is equivalent to $f = 0$.

Advice/Comment:
Note that by "$f = 0$" we actually mean "$f(t) = 0$ for all $t \in [a, b]$". Also we use $|f|$ as an abbreviation of the *function* $t \mapsto |f(t)|$.

Inner products endow a vector space with a geometric structure that allows to measure lengths and angles. If $(E, \langle\cdot,\cdot\rangle)$ is an inner product space then the **length** of $x \in E$ is given by

$$\|x\| := \sqrt{\langle x, x\rangle}.$$

The following properties are straightforward from the definition:

$$\|x\| \geq 0, \quad \|\lambda x\| = |\lambda| \, \|x\| \,, \quad \|x\| = 0 \iff x = 0.$$

The mapping $\|\cdot\| : E \longrightarrow \mathbb{R}$ is called the (natural) **norm** on the inner product space E. We will see more about norms in the next chapter.

Ex.1.3

Example 1.4. For the standard inner product on $\mathbb{K}^d$, the associated norm is

$$\|x\|_2 := \sqrt{\langle x, x\rangle} = \Bigl(\sum\nolimits_{j=1}^{d} x_j \overline{x_j}\Bigr)^{1/2} = \Bigl(\sum\nolimits_{j=1}^{d} |x_j|^2\Bigr)^{1/2}$$

and is called the 2-**norm** or **Euclidean norm** on $\mathbb{K}^d$.

For the standard inner product on $\mathrm{C}[a, b]$ the associated norm is given by

$$\|f\|_2 := \sqrt{\langle f, f\rangle} = \Bigl(\int_a^b f(t)\overline{f(t)} \, \mathrm{d}t\Bigr)^{1/2} = \Bigl(\int_a^b |f(t)|^2 \, \mathrm{d}t\Bigr)^{1/2}$$

and is called the 2-norm.

> **Advice/Comment:**
> We shall soon consider other norms different from the 2-norm, on $\mathbb{K}^d$ as well as on $\mathrm{C}[a, b]$.

Ex.1.4

Let us turn to some "geometric properties" of the norm.

Lemma 1.5. *Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space. Then the following identities hold for all $f, g \in E$:*

a) $\|f + g\|^2 = \|f\|^2 + 2 \operatorname{Re} \langle f, g\rangle + \|g\|^2$

b) $\|f + g\|^2 - \|f - g\|^2 = 4 \operatorname{Re} \langle f, g\rangle$ **(polarization identity)**

c) $\|f + g\|^2 + \|f - g\|^2 = 2 \|f\|^2 + 2 \|g\|^2$ **(parallelogram law).**

Proof. The sesquilinearity and symmetry of the inner product yields

$$\begin{aligned}\|f + g\|^2 &= \langle f + g, f + g\rangle = \langle f, f\rangle + \langle f, g\rangle + \langle g, f\rangle + \langle g, g\rangle \\ &= \|f\|^2 + \langle f, g\rangle + \overline{\langle f, g\rangle} + \|g\|^2 = \|f\|^2 + 2 \operatorname{Re} \langle f, g\rangle + \|g\|^2 .\end{aligned}$$

since $z + \overline{z} = 2 \operatorname{Re} z$ for every complex number $z \in \mathbb{C}$. This is a). Replacing g by $-g$ yields

$$\|f - g\|^2 = \|f\|^2 - 2 \operatorname{Re} \langle f, g\rangle + \|g\|^2$$

and addding this to a) yields c). Subtracting it leads to b). □

Ex.1.5

See Exercise 1.14 for a useful comment on the polarization identity.

1.2. Orthogonality

As in the case of 3-space geometry we could use the inner product to define angles between vectors. However, we shall not need other angles than right ones, so we confine ourselves to this case.

Definition 1.6. Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space. Two elements $f, g \in E$ are called **orthogonal**, written $f \perp g$, if $\langle f, g \rangle = 0$. For a subset $S \subseteq E$ we let

$$S^{\perp} := \{f \in E \mid f \perp g \text{ for all } g \in S\}.$$

Note that by symmetry of the inner product, $f \perp g$ if and only if $g \perp f$. The definiteness of the inner product translates into the following useful fact:

$$x \perp E \quad \Longleftrightarrow \quad x = 0$$

or in short: $E^{\perp} = \{0\}$.

Example 1.7. In the (standard) inner product space $\mathrm{C}[a, b]$ we denote by $\mathbf{1}$ the function which is constantly equal to 1, i.e., $\mathbf{1}(t) := 1$, $t \in [a, b]$. Then for $f \in \mathrm{C}[a, b]$ one has

$$\langle f, \mathbf{1} \rangle = \int_a^b f(t)\overline{\mathbf{1}(t)}\, \mathrm{d}t = \int_a^b f(t)\, \mathrm{d}t.$$

Hence $\{\mathbf{1}\}^{\perp} = \{f \in \mathrm{C}[a, b] \mid \int_a^b f(t)\, \mathrm{d}t = 0\}$.

Let us note a useful lemma.

Lemma 1.8. *Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space, and let $S \subseteq E$ be any subset. Then $S^{\perp}$ is a linear subspace of E.*

Proof. Clearly $0 \in S^{\perp}$. Let $x, y \in S^{\perp}$, $\lambda \in \mathbb{K}$ we have to show that $\lambda x + y \in S^{\perp}$. Then

$$\langle \lambda x + y, s \rangle = \lambda \langle x, s \rangle + \langle y, s \rangle = \lambda \cdot 0 + 0 = 0$$

for arbitrary $s \in S$, and this was to show. □

The following should seem familiar from elementary geometry.

Lemma 1.9 (Pythagoras' Theorem). *Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space with associated norm $\|\cdot\|$. Let $f_1, \dots, f_d \in E$ be pairwise orthogonal, i.e., $f_i \perp f_j$ whenever $i \neq j$. Then*

$$\|f_1 + \cdots + f_d\|^2 = \|f_1\|^2 + \cdots + \|f_d\|^2 .$$

Proof. For $d = 2$ this follows from Lemma 1.5.a). The rest is induction. □

Ex.1.6

A collection of vectors $(e_i)_{i\in I} \subseteq E$ (I an arbitrary index set) in an inner product space E is called an **orthonormal system** (ONS) if

$$\langle e_i, e_j\rangle = \delta_{ij} := \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

Let $(e_i)_{i\in I}$ be an ONS in the inner product space H. For a vector $x \in H$ we call the scalar

$$\langle x, e_i\rangle$$

the i-th **abstract Fourier coefficient** and the formal(!) series

$$\sum\nolimits_{i\in I} \langle x, e_i\rangle\, e_i$$

its **abstract Fourier series** with respect to the given ONS. It will be one of the major tasks in these lectures to make sense of such expressions when I is not finite. Until then there is still much work to do, so let us for now confine our study to *finite* ONS's.

Lemma 1.10. *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space with induced norm $\|\cdot\|$, and let $e_1, \dots, e_n \in E$ be a finite orthonormal system. Then the following assertions hold:*

a) *Let $g = \sum_{j=1}^n \lambda_j e_j$ (with $\lambda_1, \dots, \lambda_n \in \mathbb{K}$) be any linear combination of the e_j. Then*

$$\langle g, e_k\rangle = \sum\nolimits_{j=1}^n \lambda_j \langle e_j, e_k\rangle = \lambda_k \qquad (k = 1, \dots, n)$$

$$\text{and} \qquad \|g\|^2 = \sum\nolimits_{j=1}^n |\lambda_j|^2 = \sum\nolimits_{j=1}^n |\langle g, e_j\rangle|^2 .$$

b) *For $f \in E$ define $Pf := \sum_{j=1}^n \langle f, e_j\rangle\, e_j$. Then*

$$f - Pf \perp \operatorname{span}\{e_1, \dots, e_n\} \quad \text{and} \quad \|Pf\| \le \|f\| .$$

Proof. a) is just sesquilinearity and Pythagoras' theorem. For the proof of b) note that by a) we have $\langle Pf, e_j\rangle = \langle f, e_j\rangle$, i.e.,

$$\langle f - Pf, e_j\rangle = \langle f, e_j\rangle - \langle Pf, e_j\rangle = 0$$

for all $j = 1, \dots, n$. By Lemma 1.8 it follows that $f - Pf \perp \operatorname{span}\{e_j \mid j = 1, \dots, n\} =: F$. In particular, since $Pf \in F$ we have $f - Pf \perp Px$ and

$$\|f\|^2 = \|(f - Pf) + Pf\|^2 = \|f - Pf\|^2 + \|Pf\|^2 \ge \|Pf\|^2$$

by Pythagoras' theorem. □

The mapping

$$P : E \longrightarrow E \qquad Pf = \sum_{j=1}^{n} \langle f, e_j \rangle\, e_j$$

is called the **orthogonal projection** onto the subspace span$\{e_1, \dots, e_n\}$. It is *linear*, i.e., it satisfies

$$P(f+g) = Pf + Pg, \quad P(\lambda f) = \lambda Pf \qquad (f, g \in H, \lambda \in \mathbb{K}).$$

Ex.1.7 Exercise 1.7 provides more information about this mapping.

Advice/Comment:
It is strongly recommended to do Exercise 1.7. The proofs are quite straightforward, and an optimal way to train the notions of this chapter.

Orthogonal projections are an indispensable tool in Hilbert space theory and its applications. We shall see in Chapter 7 how to construct them in the case that the range space F is not finite-dimensional any more.

By Lemma 1.10.a) each ONS is a linearly independent set. So it is a *basis* for its linear span. Assume for the moment that this span is already everything, i.e.,

$$E := \operatorname{span}\{e_1, \dots, e_n\}.$$

Now consider the (linear!) mapping

$$T : E \longrightarrow \mathbb{K}^n, \qquad Tf := (\langle f, e_1 \rangle, \dots, \langle f, e_n \rangle).$$

By Lemma 1.10.a) T is exactly the coordinatization mapping associated with the algebraic basis $\{e_1, \dots, e_n\}$. Hence it is an *algebraic isomorphism.* However, more is true:

(1.1) $$\langle Tf, Tg \rangle_{\mathbb{K}^n} = \langle f, g \rangle_E \qquad (f, g \in E)$$

where $\langle \cdot, \cdot \rangle_{\mathbb{K}^n}$ denotes the standard inner product on $\mathbb{K}^n$.

Ex.1.8

As a consequence, one obtains that

$$\|Tf\|_{2,\mathbb{K}^n} = \|f\|_E \qquad \text{for all } f \in E.$$

This means that T maps members of E onto members of $\mathbb{K}^n$ of equal length, whence is called a (linear) **isometry**.

Ex.1.9

The next, probably already well-known result shows that one can always find an orthonormal basis in an inner product space with finite or countable algebraic basis.

Lemma 1.11 (Gram–Schmidt). *Let $N \in \mathbb{N} \cup \{\infty\}$ and let $(f_n)_{1\le n<N}$ be a linearly independent set of vectors in an inner product space E. Then there is an ONS $(e_n)_{1\le n<N}$ of E such that*

$$\operatorname{span}\{e_j \mid 0 \le j < n\} = \operatorname{span}\{f_j \mid 0 \le j < n\}$$

for all $n \le N$.

Proof. The construction is recursive. By the linear independence, f_1 cannot be the zero vector, so $e_1 := f_1/\|f_1\|$ has norm one. Let $g_2 := f_2 - \langle f_2, e_1\rangle e_1$. Then $g_2 \perp e_1$. Since f_1, f_2 are linear independent, $g_2 \neq 0$ and so $e_2 := g_2/\|g_2\|$ is the next unit vector.

Suppose that we have already constructed an ONS $\{e_1, \dots, e_{n-1}\}$ such that $\operatorname{span}\{e_1, \dots, e_{n-1}\} = \operatorname{span}\{f_1, \dots, f_{n-1}\}$. If $n = N$, we are done. Else let

$$g_n := f_n - \sum\nolimits_{j=1}^{n-1} \langle f_n, e_j\rangle e_j.$$

Then $g_n \perp e_j$ for all $1 \le j < n$ (Lemma 1.10). Moreover, by the linear independence of the f_j and the construction of the e_j so far, $g_n \neq 0$. Hence $e_n := g_n/\|g_n\|$ is the next unit vector in the ONS. □

Ex.1.10

As a corollary we obtain that for each finite-dimensional subspace $G = F$ of an inner product space E, there exists the orthogonal projection from E onto F. The extension of this statement to the infinite-dimensional case will occupy us in Chapter 7.

1.3. The Trigonometric System

We now come to an important example of an ONS in the Pre-Hilbert space $E = \mathrm{C}([0,1];\mathbb{C})$. Consider

$$e_n(t) := e^{2n\pi \mathrm{i} t} \qquad (t \in [0,1], n \in \mathbb{Z}).$$

If $n \neq m$

$$\begin{aligned}\langle e_n, e_m\rangle &= \int_0^1 e_n(t)\overline{e_m(t)}\,\mathrm{d}t = \int_0^1 e^{2\pi \mathrm{i}(n-m)t}\,\mathrm{d}t \\ &= \frac{e^{2\pi \mathrm{i}(n-m)t}}{2\pi \mathrm{i}(n-m)}\Big|_0^1 = \frac{1-1}{2\pi \mathrm{i}(n-m)} = 0\end{aligned}$$

by the fundamental theorem of calculus. On the other hand

$$\|e_n\|^2 = \int_0^1 \left|e^{2\pi \mathrm{i} n}\right|^2 \,\mathrm{d}t = \int_0^1 \mathbf{1}\,\mathrm{d}t = 1.$$

This shows that $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal system in the complex space $\mathrm{C}([0,1];\mathbb{C})$, the so-called **trigonometric system**.

One can construct from this an ONS in the real space $\mathrm{C}([0,1];\mathbb{R})$, see Exercise 1.15. For $f \in \mathrm{C}[0,1]$ the abstract Fourier series

$$f \sim \sum\nolimits_{n=-\infty}^{\infty} \langle f, e_n \rangle \, e^{2\pi i n t}$$

with respect to the trigonometric system is called its **Fourier series**. We shall have to say more about this in later chapters.

Exercises

Exercise 1.1. Here is a different way of proving Theorem 1.1. Suppose first that 0 is in the interior of $[a,b]$. Then prove the theorem by considering the derivatives $p^{(j)}(0)$ for $j = 0, \dots, n$. In the general case, find $a < c < b$ and use the change of variables $y = x - c$.

Exercise 1.2. Show that $\langle \cdot, \cdot \rangle : \mathrm{C}[a,b] \times \mathrm{C}[a,b] \longrightarrow \mathbb{K}$ defined above is indeed sesquilinear, positive and symmetric on $\mathrm{C}[a,b]$.

Exercise 1.3. Show that in an inner product space $\|\lambda x\| = |\lambda| \, \|x\|$ for every $x \in E$. Treat complex scalars explicitly!

Exercise 1.4. a) Compute the 2-norm of the monomials t^n, $n \in \mathbb{N}$, in the inner product space $\mathrm{C}[a,b]$ with standard inner product.

b) Let $E := \mathrm{P}[0,\infty)$ be the space of all polynomials, considered as functions on the half-line $[0,\infty)$. Define $\|p\|$ by

$$\|p\|^2 = \int_0^\infty |p(t)|^2 \, \mathrm{e}^{-t} \, \mathrm{d}t$$

for $p \in E$. Show that $\|p\|$ is the norm induced by an inner product on E. Prove all your claims.

Exercise 1.5. Make a drawing that helps you understanding why the parallelogram law carries its name.

Exercise 1.6. Work out the induction proof of Pythagoras' Theorem.

Exercise 1.7. Let $\{e_1, \dots, e_n\}$ be a finite ONS in an inner product space $(E, \langle \cdot, \cdot \rangle)$, let $F := \operatorname{span}\{e_1, \dots, e_n\}$ and let $P : E \longrightarrow F$ the orthogonal projection onto F. Show that the following assertions hold:

a) $PPf = Pf$ for all $f \in E$.

b) If $f, g \in E$ are such that $g \in F$ and $f - g \perp F$, then $g = Pf$.

c) Each $f \in E$ has a *unique* representation as a sum $f = u + v$, where $u \in F$ and $v \in F^\perp$. (In fact, $u = Pf$.)

d) If $f \in E$ is such that $f \perp F^\perp$, then $f \in F$. (Put differently: $(F^\perp)^\perp = F$.)

e) Let $Qf := f - Pf$, $f \in E$. Show that $QQf = Qf$ and $\|Qf\| \le \|f\|$ for all $f \in E$.

Exercise 1.8. Prove the identity (1.1).

Exercise 1.9. Let E, F be real inner product spaces, let $T : E \longrightarrow F$ be an isometry, i.e., T is linear and $\|Tf\|_F = \|f\|_E$ for all $f \in E$. Show that $\langle Tf, Tg\rangle_F = \langle f, g\rangle_E$ for all $f, g \in E$. (Hint: polarization identity.)

Exercise 1.10. Apply the Gram-Schmidt procedure to the polynomials $1, t, t^2$ in the inner product space $\mathrm{C}[-1, 1]$ to construct an orthonormal basis of $F = \{p \in \mathrm{P}[-1, 1] \mid \deg p \le 2\}$. (Continuing this for $t^3, t^4 \ldots$ would yield the sequence of so-called **Legendre polynomials**.)

Further Exercises

Exercise 1.11. Apply the Gram–Schmidt procedure to the monomials $1, t, t^2$ in the inner product space $\mathrm{P}[0, \infty)$ with inner product

$$\langle f, g\rangle := \int_0^\infty f(t)\overline{g(t)}e^{-t}\,\mathrm{d}t.$$

Exercise 1.12. Let us call a function $f : [1, \infty) \longrightarrow \mathbb{K}$ *mildly decreasing* if there is a constant $c = c(f)$ such that $|f(t)| \le ct^{-1}$ for all $t \ge 1$. Let $E := \{f : [1, \infty) \longrightarrow \mathbb{K} \mid f \text{ is continuous and mildly decreasing}\}$.

a) Show that E is a linear subspace of $\mathrm{C}[1, \infty)$.

b) Show that

$$\langle f, g\rangle := \int_1^\infty f(t)\overline{g(t)}\,\mathrm{d}t$$

defines an inner product on E.

c) Apply the Gram-Schmidt procedure to the functions t^{-1}, t^{-2}.

Exercise 1.13. Let E be the space of polynomials of degree at most 2. On E define

$$\langle f, g\rangle := f(-1)\overline{g(-1)} + f(0)\overline{g(0)} + f(1)\overline{g(1)} \qquad (f, g \in E).$$

a) Show that this defines an inner product on E.

b) Describe $\{t^2 - 1\}^\perp$.

c) Show that the polynomials $t^2 - 1, t^2 - t$ are orthogonal, and find a nonzero polynomial $p \in E$ that is orthogonal to both of them.

Exercise 1.14. Use the polarization identity to show that an inner product is competely determined by its induced norm. More precisely, show that if

$\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ are two inner products with $\|f\|_1^2 = \langle f, f \rangle_1 = \langle f, f \rangle_2 = \|f\|_2^2$ for *all* $f \in E$, then $\langle f, g \rangle_1 = \langle f, g \rangle_2$ for all $f, g \in E$. (Hint: start with real scalars first. In the complex case, how is $\langle \cdot, \cdot \rangle$ determined by $\operatorname{Re} \langle \cdot, \cdot \rangle$?)

Exercise 1.15. Let $(e_n)_{n \in \mathbb{Z}}$ be any ONS in $\mathrm{C}([a, b]; \mathbb{C})$, with standard inner product. Suppose further that $e_{-n} = \overline{e_n}$ for all $n \in \mathbb{Z}$. Show that

$$\{e_0\} \cup \{\sqrt{2} \operatorname{Re} e_n \mid n \in \mathbb{N}\} \cup \{\sqrt{2} \operatorname{Im} e_n \mid n \in \mathbb{N}\}$$

is an ONS in the *real* inner product space $\mathrm{C}([a, b]; \mathbb{R})$.

Chapter 2

Normed Spaces

On an inner product space $(E, \langle \cdot, \cdot \rangle)$ we have defined the norm as $\|f\| := \langle f, f \rangle^{1/2}$. In this chapter we examine further properties of this mapping, leading to the abstract definition of a norm on a vector space. Then we shall see many examples of normed spaces, which are *not* inner product spaces.

2.1. The Cauchy–Schwarz Inequality and the space ℓ^2

The following is a cornerstone in the theory of inner product spaces.

Theorem 2.1 (Cauchy–Schwarz Inequality). *Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space with induced norm*

$$\|f\| := \sqrt{\langle f, f \rangle} \qquad (f \in E).$$

Then

$$|\langle f, g \rangle| \le \|f\| \, \|g\| \qquad (f, g \in E),$$

with equality if and only if f and g are linearly dependent.

Proof. If $g = 0$ or $f = 0$ then the inequality reduces to the trivial equality $0 = 0$. So we may suppose that $f, g \neq 0$. Hence $\|g\|^2 > 0$. We define $\alpha := \langle f, g \rangle / \|g\|^2$ and

$$h := f - \alpha g = f - \frac{\langle f, g \rangle}{\|g\|^2} g.$$

Then by linearity in the first component

$$\langle h, g \rangle = \langle f, g \rangle - \frac{\langle f, g \rangle}{\|g\|^2} \langle g, g \rangle = \langle f, g \rangle - \langle f, g \rangle = 0.$$

Now Lemma 1.5, part a) yields

$$\|f\|^2 = \|\alpha g + h\|^2 = \|\alpha g\|^2 + 2\operatorname{Re}\langle h, \alpha g\rangle + \|h\|^2 = |\alpha|^2 \|g\|^2 + \|h\|^2 .$$

Multiplying this by $\|g\|^2$ yields

$$\|f\|^2 \|g\|^2 = |\langle f, g\rangle|^2 + \|g\|^2 \|h\|^2 \geq |\langle f, g\rangle|^2$$

with equality if and only if $\|h\|^2 = 0$. □

Example 2.2. On $\mathbb{K}^d$ the Cauchy–Schwarz inequality takes the form

$$\left|\sum_{j=1}^d x_j \overline{y_j}\right| \leq \left(\sum_{j=1}^d |x_j|^2\right)^{1/2} \left(\sum_{j=1}^d |y_j|^2\right)^{1/2}$$

and on $\mathrm{C}[a, b]$ it is

$$\left|\int_a^b f(t)\overline{g(t)}\,\mathrm{d}t\right| \leq \left(\int_a^b |f(t)|^2\,\mathrm{d}t\right)^{1/2} \left(\int_a^b |g(t)|^2\,\mathrm{d}t\right)^{1/2} .$$

With the help of the Cauchy–Schwarz inequality we can establish an important fact about the norm in an inner product space: the so-called **triangle inequality**.

Corollary 2.3. *The norm $\|\cdot\|$ induced by an inner product $\langle\cdot,\cdot\rangle$ on a vector space E satisfies*

(2.1) $$\|f + g\| \leq \|f\| + \|g\| \qquad (f, g \in E).$$

Proof. Let $f, g \in E$. Then, using Cauchy–Schwarz,

$$\begin{aligned}\|f + g\|^2 &= \|f\|^2 + 2\operatorname{Re}\langle f, g\rangle + \|g\|^2 \leq \|f\|^2 + 2|\langle f, g\rangle| + \|g\|^2 \\ &\leq \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 = (\|f\| + \|g\|)^2.\end{aligned}$$

Ex.2.1 Ex.2.2 Taking square roots proves the claim. □

The Cauchy–Schwarz inequality also helps in constructing new examples of inner product spaces.

Example 2.4. A scalar sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ is called **square summable** if $\sum_{n=1}^\infty |x_n|^2 < \infty$. We let

$$\ell^2 = \ell^2(\mathbb{N}) := \left\{(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{C} \mid \sum_{n\geq 1} |x_n|^2 < \infty\right\}$$

the set of all square-summable sequences. We claim: *The set ℓ^2 is a vector space and*

$$\langle x, y\rangle_{\ell^2} := \sum_{j=1}^\infty x_j \overline{y_j} \qquad (x = (x_j)_{j\in\mathbb{N}}, y = (y_j)_{j\in\mathbb{N}} \in \ell^2)$$

is a well-defined inner product on ℓ^2 with induced norm

$$\|x\|_{\ell^2} = \Big(\sum_{j=1}^{\infty} |x_j|^2\Big)^{1/2} \qquad (x = (x_j)_{j\in\mathbb{N}} \in \ell^2).$$

Proof. Let $x, y \in \ell^2$. We first prove that the scalar series $\sum_{j=1}^{\infty} x_j\overline{y_j}$ converges absolutely. To this aim, fix $N \in \mathbb{N}$. Then the Cauchy–Schwarz inequality for the (standard) inner product space $\mathbb{K}^N$ yields

$$\begin{aligned}\sum_{j=1}^{N} |x_j\overline{y_j}| &\le \Big(\sum_{j=1}^{N} |x_j|^2\Big)^{1/2} \Big(\sum_{j=1}^{N} |y_j|^2\Big)^{1/2} \\ &\le \Big(\sum_{j=1}^{\infty} |x_j|^2\Big)^{1/2} \Big(\sum_{j=1}^{\infty} |y_j|^2\Big)^{1/2} =: M.\end{aligned}$$

The right-hand side is a finite number $M < \infty$ (by hypothesis $x, y \in \ell^2$) and is independent of N. Taking the supremum with respect to N yields

$$\sum_{j=1}^{\infty} |x_j\overline{y_j}| \le M < \infty,$$

hence the series converges absolutely. Since every absolutely convergent series in $\mathbb{C}$ converges ordinarily, our first claim is proved.

To show that ℓ^2 is a vector space, take $x, y \in \ell^2$ and $\lambda \in \mathbb{K}$. Then $\lambda x = (\lambda x_j)_j$ is again square-summable because

$$\sum_{j\ge1} |\lambda x_j|^2 = \sum_{j\ge1} |\lambda|^2 |x_j|^2 = |\lambda|^2 \sum_{j\ge1} |x_j|^2 < \infty.$$

Note that $x + y = (x_j + y_j)_{j\in\mathbb{N}}$. Fixing $N \in \mathbb{N}$ and using the triangle inequality in $\mathbb{K}^N$ we estimate

$$\begin{aligned}\Big(\sum_{j=1}^{N} |x_j + y_j|^2\Big)^{1/2} &\le \Big(\sum_{j=1}^{N} |x_j|^2\Big)^{1/2} + \Big(\sum_{j=1}^{N} |y_j|^2\Big)^{1/2} \\ &\le \Big(\sum_{j=1}^{\infty} |x_j|^2\Big)^{1/2} + \Big(\sum_{j=1}^{\infty} |y_j|^2\Big)^{1/2} \\ &= \|x\|_{\ell^2} + \|y\|_{\ell^2}.\end{aligned}$$

The right-hand side is finite and independent of N, so taking the supremum with respect to N yields

$$\sum_{j=1}^{\infty} |x_j + y_j|^2 \le (\|x\|_{\ell^2} + \|y\|_{\ell^2})^2 < \infty.$$

This proves that $x + y \in \ell^2$, thus ℓ^2 is a vector space. The proof that $\langle\cdot,\cdot\rangle_{\ell^2}$ is an inner product, is left as an exercise. □ Ex.2.3
Ex.2.4

2.2. Norms

We have seen that inner product spaces allow to assign a length to each of their elements. This length is positive (as lengths should be), scales nicely if you multiply the vector by a scalar and obeys the triangle inequality, which in colloquial terms just says that the direct way is always shorter (or at least not longer) than making a detour via a third stop. Let us put these properties into an abstract definition.

Definition 2.5. Let E be a vector space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A mapping

$$\|\cdot\| : E \longrightarrow \mathbb{R}_+ := [0, \infty)$$

is called a **norm** on E if it has the following properties:

1) $\|x\| = 0 \iff x = 0$ for all $x \in E$ **(definiteness)**
2) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in E$, $\lambda \in \mathbb{K}$ **(homogeneity)**
3) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in E$ **(triangle inequality)**.

A **normed (linear) space** is a pair $(E; \|\cdot\|)$ where E is a vector space and $\|\cdot\|$ is a norm on it.

We have seen that the natural length function on an inner product space satisfies the axioms of a norm, so we were justified to call it "norm" in the first place. So any inner product space becomes a normed space in a canonical way. However, there are many norms *not* coming from inner products, and we shall exhibit some of them in the following. The first is maybe well-known, for example from undergraduate numerical analysis.

Example 2.6. On $\mathbb{K}^d$ we consider the mappings $\|\cdot\|_1, \|\cdot\|_\infty$ defined by

$$\|x\|_1 := \sum\nolimits_{j=1}^{d} |x_j| \,, \quad \|x\|_\infty := \max\{|x_j| \mid j = 1, \dots, d\}$$

for $x = (x_1, \dots, x_d) \in \mathbb{K}^d$. The proof that these mapping (called the **1-norm** and the **max-norm**) are indeed norms, is left as an exercise.

The following is a perfect analogue.

Example 2.7. On $\mathrm{C}[a, b]$ we consider the mappings $\|\cdot\|_1, \|\cdot\|_\infty$ defined by

$$\|f\|_1 := \int_a^b |f(t)| \, \mathrm{d}t, \quad \|f\|_\infty := \sup\{|f(t)| \mid t \in [a, b]\}$$

for $f \in \mathrm{C}[a, b]$. We sketch a proof of the triangle inequality, leaving the other properties of a norm as an exercise. Let $f, g \in \mathrm{C}[a, b]$. Then for each

$t \in [a, b]$ one has

$$|(f+g)(t)| = |f(t) + g(t)| \leq |f(t)| + |g(t)| \,.$$

For the 1-norm, one simply integrates this inequality, leading to

$$\begin{aligned} \|f+g\|_1 &= \int_a^b |(f+g)(t)| \, \mathrm{d}t \leq \int_a^b |f(t)| + |g(t)| \, \mathrm{d}t \\ &= \int_a^b |f(t)| \, \mathrm{d}t + \int_a^b |g(t)| \, \mathrm{d}t = \|f\|_1 + \|g\|_1 \,. \end{aligned}$$

For the sup-norm, note that one can estimate $|f(t)| \leq \|f\|_\infty$ and $|g(t)| \leq \|g\|_\infty$ for all $t \in [a, b]$, by definition of the sup-norm. This leads to

$$|(f+g)(t)| \leq |f(t)| + |g(t)| \leq \|f\|_\infty + \|g\|_\infty$$

for all $t \in [a, b]$. Taking the supremum over $t \in [a, b]$ we obtain

$$\|f+g\|_\infty = \sup_{t \in [a,b]} |(f+g)(t)| \leq \|f\|_\infty + \|g\|_\infty \,.$$

Ex.2.5

Advice/Comment:
Certainly you recall that a continuous positive function on a compact interval has a maximum, i.e., attains its supremum. So we can write $\|f\|_\infty = \max\{|f(t)| \mid t \in [a, b]\}$. However, in the proofs above this property is never used. With a view towards more general situations below, it is better to use the supremum rather than the maximum.

Now, experience shows that many students have problems with the notion of supremum. We have collected the relevant definitions in Appendix A.3. It may consolate you that in these lectures only suprema and infima over sets of positive real numbers occur. However, they will occur quite frequently, hence in case you have difficulties, you have to get to terms with it quickly.

Examining the sup-norm for $\mathrm{C}[a, b]$ we may realize that in the proof of the norm properties the continuity of the functions actually does not play any role whatsoever. The only property that was used, was that $\sup\{|f(t)| \mid t \in [a, b]\}$ was a finite number. For continuous functions this is automatically satisfied, since $[a, b]$ is a compact interval. If we leave compact domains, we must include this into the definition of the function space.

Example 2.8. Let Ω be any (non-empty) set. A function $f : \Omega \longrightarrow \mathbb{K}$ is called **bounded** if there is a finite number $c = c(f) \geq 0$ such that $|f(t)| \leq c$ for all $t \in \Omega$. For a bounded function f, the number

$$\|f\|_\infty := \sup\{|f(t)| \mid t \in \Omega\}$$

is finite. Let

$$\mathcal{B}(\Omega) := \{f : \Omega \longrightarrow \mathbb{K} \mid f \text{ is bounded}\}$$

be the set of bounded functions on Ω. Then $E = \mathcal{B}(\Omega)$ is a linear subspace
Ex.2.6 of $\mathcal{F}(\Omega)$ and $\|\cdot\|_\infty$ is a norm on it. (Proof as exercise.) The norm $\|\cdot\|_\infty$ is usually called the **sup-norm**.

A special instance of $\mathcal{B}(\Omega)$ occurs when $\Omega = \mathbb{N}$. For this we use the symbol

$$\ell^\infty := \mathcal{B}(\mathbb{N}) = \Big\{(x_j)_{j\in\mathbb{N}} \mid \sup_{j\in\mathbb{N}} |x_j| < \infty\Big\}.$$

Recall that a scalar sequence *is the same* as a scalar function on $\mathbb{N}$!

Last, but not least (see exercises below), we treat a sequence analogue of the 1-norm.

Example 2.9. A scalar sequence $x = (x_j)_{j\in\mathbb{N}}$ is called **absolutely summable** if the series $\sum_{j=1}^\infty x_j$ converges absolutely, i.e., if

$$\|x\|_1 := \sum_{j=1}^\infty |x_j| < \infty.$$

We denote by

$$\ell^1 := \Big\{(x_j)_{j\in\mathbb{N}} \mid \sum_{j=1}^\infty |x_j| < \infty\Big\}$$

Ex.2.7 the set of all absolutely summable sequences.

Remark 2.10. We claimed in the beginning that these norms do not come from any inner product on the underlying space. How can one be so sure about that? Well, we know that a norm coming from an inner product satisfies the parallelogram law. A stunning theorem of von Neumann actually states the converse: *a norm on a vector space comes from an inner product if and only if it satisfies the parallelogram law.* See [**4**, p.65/66] for a proof. And spaces such as $\mathrm{C}[a, b]$ with the sup-norm or the 1-norm do not satisfy the parallelogram law, see Exercise 2.13.

2.3. Bounded Linear Mappings

We shall now learn how normed spaces give rise to new normed spaces. Suppose that E, F are normed spaces. Then we can consider mappings $T : E \longrightarrow F$ which are **linear**, i.e., which satisfy

$$T(f + g) = Tf + Tg \quad \text{and} \quad T(\lambda f) = \lambda Tf$$

for all $f, g \in E$, $\lambda \in \mathbb{K}$. A linear mapping is also called linear **operator**, and if the codomain space $F = \mathbb{K}$ is one-dimensional, they are called (linear) **functionals**.

One can add linear mappings and multiply them by scalars by

$$(T+S)f := Tf + Sf, \qquad (\lambda T)f := \lambda(Tf) \qquad (f \in E,\ \lambda \in \mathbb{K})$$

and in this way the set of all linear mappings from E to F becomes a new vector space, see Lemma A.7. Is there a natural norm on that space?

> **Advice/Comment:**
> Think a moment about the finite-dimensional situation. If $E = \mathbb{K}^n$ and $F = \mathbb{K}^m$, then the linear mappings from E to F are basically the $n \times m$-matrices. As a vector space this is isomorphic to $\mathbb{K}^{m \cdot n}$, and we know already several norms here. But which of them relates naturally to the chosen norms on E and F?

It turns out that if E is infinite-dimensional, then there is no chance to define a norm on the space of *all* linear mappings. However, there is an important *subspace* of linear mappings that allows for a norm.

Definition 2.11. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed spaces. A linear mapping $T : E \longrightarrow F$ is called **bounded** if there is a real constant $c \geq 0$ such that

$$\|Tf\|_F \leq c \|f\|_E \qquad \text{for all} \quad f \in E.$$

We denote by $\mathcal{L}(E;F)$ the **space of bounded linear operators** from E to F. If $E = F$ we simply write $\mathcal{L}(E)$.

> **Advice/Comment:**
> When considering linear mappings $T : E \longrightarrow F$, we should distinguish the norms on E and on F. We have done this in the above definition explicitly by writing $\|\cdot\|_E$ and $\|\cdot\|_F$, but one often drops the subscripts when there is no danger of confusion.

Remark 2.12. *A linear mapping $T : E \longrightarrow F$ is bounded if and only if its* **operator norm**

$$\|T\|_{\mathcal{L}(E;F)} := \sup_{\|f\|_E \leq 1} \|Tf\|_F \tag{2.2}$$

is a finite number.

Proof. Indeed, if $\|Tf\| \leq c\|f\|$ for all $f \in E$, then obviously $\|T\| \leq c$ is finite. And if $\|T\| < \infty$ then for a general $f \in E, f \neq 0$ we write $\lambda := \|f\|$ and compute

$$\|Tf\| = \left\|T(\lambda\lambda^{-1}f)\right\| = \lambda \left\|T(\lambda^{-1}f)\right\| \leq \lambda \|T\| = \|T\| \|f\|,$$

since $\left\|\lambda^{-1}f\right\| = \lambda^{-1}\left\|f\right\| = 1$. And for $f = 0$ the inequality $\left\|Tf\right\| \le \left\|T\right\|\left\|f\right\|$ holds trivially, since $T(0) = 0$. □

Sometimes one write $\left\|T\right\|_{E\to F}$ is place of $\left\|T\right\|_{\mathcal{L}(E;F)}$, but if there is no danger of confusion, one omits subscripts. The following result shows that the name "operator norm" is justified.

Theorem 2.13. *Let E, F be normed spaces. Then $\mathcal{L}(E;F)$ is a vector space, and the operator norm defined by* (2.2) *is a norm on it. Furthermore, one has*

$$\left\|Tf\right\|_F \le \left\|T\right\|_{E\to F}\left\|f\right\|_E \qquad (f \in E). \tag{2.3}$$

Proof. The inequality (2.3) has been established above. Suppose $S, T \in \mathcal{L}(E;F)$. Then, for all $f \in E$ we have

$$\begin{aligned}\left\|(T+S)f\right\| &= \left\|Tf + Sf\right\| \le \left\|Tf\right\| + \left\|Sf\right\| \le \left\|T\right\|\left\|f\right\| + \left\|S\right\|\left\|f\right\| \\ &= (\left\|T\right\| + \left\|S\right\|)\left\|f\right\|.\end{aligned}$$

But this means that $S + T$ is bounded, and $\left\|T + S\right\| \le \left\|T\right\| + \left\|S\right\|$. If $T \in \mathcal{L}(E;F)$ and $\lambda \in \mathbb{K}$, then

$$\left\|(\lambda T)f\right\| = \left\|\lambda Tf\right\| = |\lambda|\left\|Tf\right\| \le |\lambda|\left\|T\right\|\left\|f\right\|$$

for all $f \in E$; and this means that λT is bounded, with $\left\|\lambda T\right\| \le |\lambda|\left\|T\right\|$. Replacing T by $\lambda^{-1}T$ here yields $\left\|\lambda T\right\| = |\lambda|\left\|T\right\|$. Finally, if $\left\|T\right\| = 0$ then $\left\|Tf\right\| \le \left\|T\right\|\left\|f\right\| = 0$ and hence $Tf = 0$ for all $f \in E$, i.e., $T = 0$. □

We write ST in place of $S \circ T$ whenever the composition of the operators makes sense. The next lemma shows that it is safe to compose bounded linear operators.

Lemma 2.14. *Let E, F, G be normed spaces, and let $T : E \longrightarrow F$ and $S : F \longrightarrow G$ be bounded linear operators. Then $ST := S \circ T$ is again a bounded linear operator, and one has*

$$\left\|ST\right\|_{E\to G} \le \left\|S\right\|_{F\to G} \cdot \left\|T\right\|_{E\to F} \tag{2.4}$$

Proof. It is clear that ST is again linear. For $f \in E$ we have

$$\left\|(ST)f\right\| = \left\|S(Tf)\right\| \le \left\|S\right\|\left\|Tf\right\| \le \left\|S\right\|\left\|T\right\|\left\|f\right\| = (\left\|S\right\|\left\|T\right\|)\left\|f\right\|.$$

This shows that $ST \in \mathcal{L}(E;G)$ and establishes (2.4). □

We shall encounter many bounded linear mappings in these lectures, but a more thourough study is postponed until Chapter 9. At this point we look only at a few very simple examples.

Example 2.15. A linear mapping $T : E \longrightarrow F$ is called an **isometry** if

$$\|Tf\|_F = \|f\|_E \qquad \text{for all} \quad f \in E.$$

An isometry is obviously bounded. It has trivial kernel (only the zero vector is mapped to 0), and hence is injective. If it is also surjective, i.e., if $\operatorname{ran}(T) = F$, we call T an **isometric isomorphism**. In this case, T is invertible and also $T^{-1} : F \longrightarrow E$ is an isometric isomorphism.

Let E be a finite-dimensional inner product space with orthonormal *basis* $\{e_1, \dots, e_d\}$. Then we have seen that the coordinatization with respect to this basis is an isometric isomorphism $T : E \longrightarrow \mathbb{K}^d$, where $\mathbb{K}^d$ carries the standard inner product.

Example 2.16. *Any linear mapping $T : \mathbb{K}^d \longrightarrow F$ is bounded, where F is an arbitrary normed space and on $\mathbb{K}^d$ we consider the standard (Euclidean) norm.*

Proof. Let $e_1, \dots, e_d$ denote the canonical basis of $\mathbb{K}^d$. Then for arbitrary $x = (x_1, \dots, x_d)$

$$\begin{aligned} \|Tx\|_F &= \Big\| T\big(\sum\nolimits_{j=1}^d x_j e_j\big) \Big\|_F \le \sum\nolimits_{j=1}^d \|x_j T(e_j)\|_F \\ &= \sum\nolimits_{j=1}^d |x_j| \, \|T(e_j)\|_F \le c \, \|x\|_\infty \le c \, \|x\|_2 \end{aligned}$$

where $c := \sum_{j=1}^d \|T(e_j)\|_F$. □

Example 2.17 (Point evaluation)**.** Let $E = \ell^p$ with $p = 1, 2, \infty$, and let $j_0 \in \mathbb{N}$. The **point evaluation** at j_0 is

$$\ell^p \longrightarrow \mathbb{K}, \quad x = (x_j)_{j \in \mathbb{N}} \mapsto x_{j_0}$$

is linear and bounded with $|x_{j_0}| \le \|x\|_p$. Ex.2.8 Ex.2.9 Ex.2.10

Let us close this chapter with an equivalent description of boundedness of linear mappings. This hinges on the following general concept.

Definition 2.18. A subset A of a normed space $(E, \|\cdot\|)$ is called **bounded** if there is $c \ge 0$ such that

$$\|f\| \le c \qquad \text{for all } f \in A.$$

The **closed unit ball** of a normed space E is

$$B_E := \{f \in E \mid \|f\| \le 1\}.$$

Then B_E is obviously a bounded set, and a subset $A \subseteq E$ is bounded iff there is a constant $c > 0$ such that $A \subseteq cB_E$.

Lemma 2.19. *A linear mapping $T : E \longrightarrow F$ between normed spaces E and F is bounded if and only if the set $T(B_E)$ is bounded if and only if T maps bounded sets from E into bounded sets from F.* Ex.2.11

Exercises

Exercise 2.1. Make a picture illustrating the name "triangle inequality".

Exercise 2.2. Write down the instances of the triangle inequality in the standard inner product spaces $\mathbb{K}^d$ and $\mathrm{C}[a, b]$.

Exercise 2.3. Complete the proof of the claim in the Example 2.4.

Exercise 2.4. Mimic the proof in Example 2.4 to show that the set

$$E := \{f \in \mathrm{C}[0, \infty) \mid \int_0^\infty |f(t)|^2 \,\mathrm{d}t < \infty\}$$

is a vector space and that

$$\langle f, g\rangle_{\mathrm{L}^2} := \int_0^\infty f(t)\overline{g(t)} \,\mathrm{d}t$$

defines an inner product on it.

Exercise 2.5. Show that 1-norm and max-norm on $\mathbb{K}^d$ are indeed norms. Complete the proof of the norm properties of 1-norm and sup-norm on $\mathrm{C}[a, b]$. Where is continuity actually needed?

Exercise 2.6. Show that $(\mathcal{B}(\Omega), \|\cdot\|_\infty)$ is indeed a normed vector space. (Mimic the proof in the $\mathrm{C}[a, b]$-case.)

Exercise 2.7. Show that ℓ^1 is a vector space and that $\|\cdot\|_1$ is a norm on it. (Mimic the ℓ^2-case treated in Example 2.4)

Exercise 2.8. Let $E := \mathrm{C}[a, b]$ with the sup-norm and let $t_0 \in [a, b]$. Show that point evaluation $f \longmapsto f(t_0)$ is a bounded linear mapping from $(\mathrm{C}[a, b], \|\cdot\|_\infty) \longrightarrow (\mathbb{K}, |\cdot|)$.

Is this still true when one replaces $\|\cdot\|_\infty$ by $\|\cdot\|_2$ or $\|\cdot\|_1$?

Exercise 2.9. For a continuous function $f \in \mathrm{C}[a, b]$ let

$$(Jf)(t) := \int_a^t f(s) \,\mathrm{d}s \qquad (t \in [a, b]).$$

Show that

$$J : (\mathrm{C}[a, b], \|\cdot\|_1) \longrightarrow (\mathrm{C}[a, b], \|\cdot\|_\infty)$$

is a bounded linear mapping. What is its kernel, what is its range?

Exercise 2.10. Let E consist of all functions $f : \mathbb{R}_+ \longrightarrow \mathbb{K}$ constant on each interval $[n-1, n)$, $n \in \mathbb{N}$, and such that

$$\|f\|_1 := \int_0^\infty |f(t)| \,\mathrm{d}t < \infty.$$

Show that E is a vector space, $\|\cdot\|_1$ is a norm on it and describe an isometric isomorphism $T : \ell^1 \longrightarrow E$.

Exercise 2.11. Prove Lemma 2.19.

Further Exercises

Exercise 2.12. Make a sketch of the unit balls of $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^2$.

Exercise 2.13. In the examples, $\mathbb{K}^d$ and $\mathrm{C}[a, b]$, find pairs of vectors violating the parallelogram law for the 1-norm, and the ∞-norm, respectively.

Exercise 2.14. Mimic the proof of Example 2.4 to show that the set

$$E := \{f \in \mathrm{C}(\mathbb{R}) \mid \int_{-\infty}^{\infty} |f(t)|^2 \,\mathrm{d}t < \infty\}$$

is a vector space and that

$$\langle f, g\rangle_{\mathrm{L}^2} := \int_{-\infty}^{\infty} f(t)\overline{g(t)} \,\mathrm{d}t$$

defines an inner product on it.

Exercise 2.15. Mimic the proof of Example 2.4 to show that the set

$$E := \{f \in \mathrm{C}[0, \infty) \mid \int_0^\infty |f(t)|^2 e^{-t} \,\mathrm{d}t < \infty\}$$

is a vector space and that

$$\langle f, g\rangle := \int_0^\infty f(t)\overline{g(t)} e^{-t} \,\mathrm{d}t$$

defines an inner product on it.

Exercise 2.16. Let $\mathrm{c}_0 := \{x = (x_j)_{j\in\mathbb{N}} \subseteq \mathbb{K} \mid \lim_{j\to\infty} x_j = 0\}$ be the set of scalar **null sequences**. Show that c_0 is a linear subspace of ℓ^∞, containing ℓ^2.

Exercise 2.17. Show that $\ell^1 \subseteq \ell^2$ and that this inclusion is proper.

Exercise 2.18. Let $w : (0, \infty) \longrightarrow \mathbb{R}$ be continuous, $w(t) > 0$ for all $t > 0$. Show that the set

$$E := \left\{f \in \mathrm{C}(0, \infty) \mid \int_0^\infty |f(t)| \, w(t) \,\mathrm{d}t < \infty\right\}$$

is a vector space and

$$\|f\|_1 := \int_0^\infty |f(t)|\, w(t)\, \mathrm{d}t$$

is a norm on it.

Exercise 2.19. Let $w : (0, \infty) \longrightarrow \mathbb{R}$ be continuous, $w(t) > 0$ for all $t > 0$. Show that the set

$$E := \Big\{ f \in \mathrm{C}(0, \infty) \mid \sup_{t>0} |f(t)|\, w(t) < \infty \Big\}$$

is a vector space and

$$\|f\|_\infty := \sup_{t>0} |f(t)|\, w(t)$$

defines a norm on it.

Exercise 2.20. Fix a non-trivial interval $[a, b] \subseteq \mathbb{R}$, and a number $\alpha \in (0, 1)$. A function $f : [a, b] \longrightarrow \mathbb{K}$ is called **Hölder continuous** of order α if there is a finite number $c = c(f) \geq 0$ such that

$$|f(t) - f(s)| \leq c\, |t - s|^\alpha$$

for all $s, t \in [a, b]$. Let

$$\mathrm{C}^\alpha[a, b] := \{ f : [a, b] \longrightarrow \mathbb{K} \mid f \text{ is Hölder continuous of order } \alpha \}.$$

Show that $\mathrm{C}^\alpha[a, b]$ is a linear subspace of $\mathrm{C}[a, b]$, and that

$$\|f\|_{(\alpha)} := \sup \{ |f(t) - f(s)| \,/\, |s - t|^\alpha \;\mid s, t \in [a, b], s \neq t \}$$

satisfies the triangle inequality and is homogeneous. Is it a norm? How about

$$\|f\| := |f(a)| + \|f\|_{(\alpha)}\, ?$$

Exercise 2.21. Let $(F, \|\cdot\|_F)$ be a normed space, let E be any vector space and $T : E \longrightarrow F$ an *injective* linear mapping. Show that

$$\|f\|_E := \|Tf\|_F \qquad (f \in E)$$

defines a norm on E, and T becomes an isometry with respect to this norm.

Chapter 3

Distance and Approximation

Approximation is at the heart of analysis. In this section we shall see how a norm induces naturally a notion of distance, and how this leads to the notion of convergent sequences.

3.1. Metric Spaces

Originally, i.e., in three-dimensional geometry, a "vector" is a translation of (affine) three-space, and it abstracts a physical motion or displacement. So the length of a vector is just the length of this displacement. By introducing a coordinate system in three-space, points can be identified with vectors: with each point P you associate the vector which "moves" the origin O to P. Given this identification, vectors "are" points, and the length of the vector x becomes the *distance* of the point x to the origin. More generally, the distance of the two points x and y is the length of the vector $x - y$.

Now let $(E, \|\cdot\|)$ be a normed space, and let $f, g \in E$. Then we call

$$d_E(f, g) := \|f - g\|$$

the **distance** of f and g in the norm $\|\cdot\|$. The function

$$d_E : E \times E \longrightarrow \mathbb{R}_+, \qquad (f, g) \longmapsto \|f - g\|$$

is called the **induced metric** and has the following properties:

$$\begin{aligned} &d_E(f, g) = 0 \iff f = g, \\ &d_E(f, g) = d_E(g, f), \\ &d_E(f, g) \le d_E(f, h) + d_E(h, g) \end{aligned}$$

Ex.3.1 with f, g, h being arbitrary elements of E.

> **Advice/Comment:**
> We should better write $d_{\|\cdot\|}(x, y)$ instead of $d_E(x, y)$, since the distance depends evidently on the norm. However our notation is more convenient, and we shall take care that no confusion arises.

Definition 3.1. A **metric** on a set Ω is a mapping $d : \Omega \times \Omega \longrightarrow [0, \infty)$ satisfying the following three conditions:

1) $d(x, y) = 0$ if and only if $x = y$ (**definiteness**)
2) $d(x, y) = d(y, x)$ (**symmetry**)
3) $d(x, y) \le d(x, z) + d(z, y)$ (**triangle inequality**)

for all $x, y, z \in \Omega$. A **metric space** is a pair (Ω, d) with Ω being a set and d a metric on it. If $x \in \Omega$ and $r > 0$ the set

$$\mathrm{B}_r(x) := \{y \in \Omega \mid d(x, y) < r\}$$

is called the (open) **ball** of radius r around x.

We immediately note important examples.

Example 3.2. Every normed space $(E, \|\cdot\|)$ is a metric space under the induced metric $d_E(x, y) = \|x - y\|$, $x, y \in E$.

Example 3.3. Every set Ω becomes a metric space under the **discrete metric**, defined by

$$d(x, y) := \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \ne y. \end{cases}$$

(Check that the axioms of a metric are indeed satisfied!)

Example 3.4. The interval $[0, \infty]$ becomes a metric space under

$$d(x, y) := \left| \frac{1}{1 + x} - \frac{1}{1 + y} \right| \qquad (x, y \in [0, \infty])$$

where we use the convention that $1/\infty = 0$.

Example 3.5. If (Ω, d) is a metric space and $A \subseteq \Omega$ is an arbitrary subset, then A becomes a metric space by just restricting the metric d to $A \times A$. This metric on A is called the **induced metric**. For example, the interval $(0, 1]$

is a metric space in its own right by setting $d(x, y) := |x - y|$, $x, y \in (0, 1]$. This metric is induced by the usual metric on $\mathbb{R}$.

3.2. Convergence

Recall from your undergraduate analysis course that a sequence of real numbers $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converges to a real number $x \in \mathbb{R}$ if

$$\forall\, \epsilon > 0 \;\; \exists\, N \in \mathbb{N} \text{ such that } \; |x_n - x| < \epsilon \quad (n \geq N).$$

We now generalize this concept of convergence to general metric spaces. Note that $(x_n)_{n\in\mathbb{N}}$ converges to x if and only if $\lim_{n\to\infty} |x_n - x| = 0$, and $|x_n - x| = d_{\mathbb{R}}(x_n, x)$ is the natural distance (metric) on $\mathbb{R}$.

Definition 3.6. Let (Ω, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq \Omega$ in Ω **converges** to an element $x \in \Omega$ (in symbols: $x_n \to x$, $\lim_{n\to\infty} x_n = x$) if

$$\forall\, \epsilon > 0 \;\; \exists\, N \in \mathbb{N} \text{ such that } d(x_n, x) < \epsilon \quad (n \geq N).$$

If $(x_n)_{n\in\mathbb{N}}$ converges to x, we call x a **limit** of the sequence $(x_n)_{n\in\mathbb{N}}$.

We say that a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \Omega$ is **convergent** (in Ω) if it has a limit, i.e., if there is $x \in \Omega$ such that $x_n \to x$.

We can rephrase the convergence $x_n \to x$ also in the following way: for every $r > 0$ one has $x_n \in \mathrm{B}_r(x)$ *eventually*, (i.e., for all sufficiently large $n \in \mathbb{N}$).

> **Advice/Comment:**
> Compare the general definition with the one for real numbers, you will realize that $x_n \to x$ (in the metric space Ω) is equivalent to $d(x_n, x) \to 0$ (in $\mathbb{R}$). This *reduces* the general concept of convergence to a special case, convergence in $\mathbb{R}$ to 0.

Example 3.7. In a discrete metric space, a sequence is convergent if and only if it is *eventually constant*.

Example 3.8. If $(E, \|\cdot\|)$ is a normed space, then $x_n \to x$ in E is equivalent to $\|x_n - x\| \to 0$. For instance, if $E = \mathbb{K}^d$ with the Euclidean metric and writing $x_n = (x_{n1}, \dots, x_{nd})$ for all $n \in \mathbb{N}$ then $(x_n)_{n\in\mathbb{N}}$ is convergent in $\mathbb{K}^d$ if and only if each coordinate sequence $(x_{nj})_{n\in\mathbb{N}}$, $j = 1, \dots, d$, is convergent in $\mathbb{K}$.

We shall see more examples of convergent and non-convergent sequences shortly. Let us return to the theory.

Lemma 3.9. *Limits are unique. More precisely, let* $(x_n)_{n\in\mathbb{N}} \subseteq \Omega$ *such that* $x_n \to x \in \Omega$ *and* $x_n \to x' \in \Omega$. *Then* $x = x'$.

Proof. By the triangle inequality we have

$$d(x, x') \leq d(x, x_n) + d(x_n, x')$$

for every $n \in \mathbb{N}$. Since both $d(x, x_n) \to 0$ and $d(x_n, x') \to 0$, it follows that $d(x, y) = 0$. By definiteness, $x = x'$. □

By the lemma, we shall henceforth speak of "the" limit of a convergent sequence.

Advice/Comment:
Note that the concept of convergence is always *relative* to a given metric space. The assertion "The sequence $(x_n)_{n\in\mathbb{N}}$ is convergent" to be meaningful (and answerable) requires a metric *and* a space Ω where this metric is defined. When the reference space is in doubt, we may say for clarification that $(x_n)_{n\in\mathbb{N}}$ is convergent *in* Ω. This remark is not as trivial as it looks, see the following example.

Example 3.10. Consider the sequence $x_n = 1/n$, $n \in \mathbb{N}$. Does it converge? The answer is: it depends. If you consider this sequence within $\mathbb{R}$ with the standard metric, the answer is yes, and 0 is the limit. If you consider this sequence in the metric space $(0, 1]$ (again with the standard metric), the answer is no. If you consider it in $(0, 1]$ with the discrete metric, the answer is again no. If you take the metric(!)

$$d(x, y) := \left| e^{2\pi \mathrm{i} x} - e^{2\pi \mathrm{i} y} \right| \qquad (x, y \in (0, 1])$$

the answer is yes, and 1 is the limit.

3.3. Uniform, Pointwise and 2-Norm Convergence

We have seen above that each metric induces its own notion of convergence. The same sequence may or may not converge, depending on the metric one considers. In $\mathbb{K}^d$ the convergence with respect to the Euclidean norm is equivalent to convergence in every component. The following example shows that the infinite-dimensional analogon of this statement is false.

Example 3.11. Consider on the space $E = \ell^2$ the sequence of **standard unit vectors** $(e_n)_{n\in\mathbb{N}}$, defined by

$$e_n = (0, 0, \dots, 0, 1, 0, \dots)$$

where the 1 is located at the nth place. (One could write $e_n = (\delta_{nj})_{j\in\mathbb{N}}$, where δ_{ij} is the **Kronecker delta**.) Then we have $\|e_n\|_2 = 1$ for every $n \in \mathbb{N}$.

Suppose that $(e_n)_{n\in\mathbb{N}}$ converges in the 2-norm, and let $f \in \ell^2$ be its limit. Fix a component $k \in \mathbb{N}$; then trivially

$$|f(k) - e_n(k)| \le \Big(\sum\nolimits_{j=1}^{\infty} |f(j) - e_n(j)|^2\Big)^{1/2} = \|f - e_n\|_2 \to 0.$$

Since $e_n(k) = 0$ for $n > k$, this forces $f(k) = 0$. As $k \in \mathbb{N}$ was arbitary, $f = 0$. But this implies

$$1 = \|e_n\|_2 = \|e_n - 0\|_2 = \|e_n - f\|_2 \to 0$$

and this is clearly false. Hence our assumption was wrong and the sequence $(e_n)_{n\in\mathbb{N}}$ of standard unit vectors *does not converge in norm*. However, it obviously converges "componentwise" (i.e., in each component).

Advice/Comment:
Make sure that you understand the above examples completely. We are dealing here with a *sequence of sequences*: each vector e_n *is* a scalar sequence, and so $(e_n)_{n\in\mathbb{N}}$ is a sequence whose elements are scalar sequences.

Dealing with sequences of sequences is a notational challenge. To avoid unnecessary confusion we shall often write points of a sequence space in function notation $x = (x(j))_{j\in\mathbb{N}}$, or simply $x : \mathbb{N} \longrightarrow \mathbb{K}$. A sequence of sequences is then $(x_n)_{n\in\mathbb{N}} = ((x_n(j)_{j\in\mathbb{N}})_{n\in\mathbb{N}}$, avoiding double indices in this way.

The example above shows, roughly speaking, that convergence in every component does not imply the convergence in the (2-)norm. Clearly one can replace the 2-norm here by the 1-norm or the sup-norm. On the other hand, the trivial estimate

$$|f(k)| \le \Big(\sum\nolimits_{j=1}^{\infty} |f(j)|^2\Big)^{1/2} = \|f\|_2 \qquad (k \in \mathbb{N}, f \in \ell^2)$$

yields as in the example that 2-norm convergence implies componentwise convergence. Ex.3.2 Ex.3.3

We now turn to the sup-norm $\|\cdot\|_\infty$ on the space $\mathcal{B}(\Omega)$ of bounded functions on some non-empty set Ω (introduced in Example 2.8). Recall the definition

$$\|f\|_\infty = \sup\{|f(t)| \mid t \in \Omega\}$$

of the sup-norm of a bounded function $f : \Omega \longrightarrow \mathbb{K}$. Then $d(f, g) = \|f - g\|_\infty \le \epsilon$ is equivalent to

$$|f(t) - g(t)| \le \epsilon$$

being true for *all* $t \in \Omega$. So $f_n \to f$ in the norm $\|\cdot\|_\infty$ may be written as

$$\forall \epsilon > 0\ \exists\, N = N(\epsilon) : |f_n(t) - f(t)| \le \epsilon \quad (n \ge N, t \in \Omega).$$

Note that the chosen N may depend on ϵ but it is the same (= "uniform") for every $t \in \Omega$. Therefore we say that $(f_n)_{n\in\mathbb{N}}$ converges to f **uniformly** (on Ω) and call the norm $\|\cdot\|_\infty$ sometimes the **uniform norm**.

A weaker notion of convergence (and the analogue of componentwise convergence above) is the notion of **pointwise convergence**. We say that a sequence $(f_n)_{n\in\mathbb{N}}$ of functions on Ω **converges pointwise** to a function $f : \Omega \longrightarrow \mathbb{K}$, if

$$f_n(t) \to f(t) \qquad \text{in } \mathbb{K} \quad \text{as } n \to \infty$$

for every $t \in \Omega$. In logical notation

$$\forall\, t \in \Omega\ \forall \epsilon > 0\ \exists\, N = N(t, \epsilon) : |f_n(t) - f(t)| \le \epsilon \quad (n \ge N).$$

Here the N may depend on ϵ *and* the point $t \in \Omega$.

Clearly, uniform convergence implies pointwise convergence, as follows also from the (trivial) estimate

$$|f(t)| \le \sup_{s\in\Omega} |f(s)| = \|f\|_\infty$$

for each $t \in \Omega$. The converse is not true.

Example 3.12. Consider $\Omega = (0, \infty)$ and the functions $f_n(x) = \mathrm{e}^{-nx}$. Then clearly $f_n(x) \to 0$ as $n \to \infty$ for each $x \in \Omega$. However,

$$\|f_n\|_\infty = \sup_{x>0} \left|\mathrm{e}^{-nx}\right| = 1$$

for each $n \in \mathbb{N}$, and thus $f_n \not\to 0$ uniformly.

If we consider $g_n := nf_n$, then $\|g_n\|_\infty = n \to \infty$, but still $g_n(x) \to 0$ for each $x > 0$. This shows that pointwise convergence of a sequence does not even imply its boundedness in the sup-norm, let alone convergence in this norm. Ex.3.4

Clearly the "obstacle" against uniform convergence is the behaviour of the functions f_n at 0. If we multiply f_n by a bounded function $h \in \mathcal{B}(\Omega)$ such that $\lim_{x\to 0} h(x) = 0$, then their "bad" behaviour at 0 is tempered, and we have indeed $\|hf_n\|_\infty \to 0$. (We ask for a formal proof in Exercise 3.5.) Ex.3.5

Example 3.13. For each interval $[a, b] \subseteq \mathbb{R}$ we have

$$\|f\|_1 \le \sqrt{b-a}\,\|f\|_2 \quad \text{and} \quad \|f\|_2 \le \sqrt{b-a}\,\|f\|_\infty \tag{3.4}$$

for all $f \in \mathrm{C}[a,b]$. The first inequality follows from Cauchy-Schwarz and

$$\|f\|_1 = \int_a^b |f| = \langle |f|, \mathbf{1}\rangle \le \|f\|_2 \|\mathbf{1}\|_2 = \sqrt{b-a}\,\|f\|_2$$

where we have written $\mathbf{1}$ for the function which is constantly equal to 1. The second inequality follows from

$$\|f\|_2^2 = \int_a^b |f|^2 \le \int_a^b \|f\|_\infty^2 \mathbf{1} = (b-a)\,\|f\|_\infty^2 .$$

The inequalities (3.4) imply that uniform convergence implies 2-norm convergence and 2-norm convergence implies 1-norm convergence. Ex.3.6

Now we are asking for the converse implications. Consider as an example $[a,b] = [0,1]$ and in $\mathrm{C}[0,1]$ the sequence of functions $f_n(t) := t^n$, for $t \in [0,1]$ and $n \in \mathbb{N}$. Then $f_n(t) \to 0$ as $n \to \infty$ for each $t < 1$, but $f_n(1) = 1$ for all $n \in \mathbb{N}$. This means that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the function f given by

$$f(t) := \begin{cases} 0 & (0 \le t < 1) \\ 1 & (t = 1) \end{cases}$$

which is, however, not contained in $\mathrm{C}[0,1]$. A fortiori, $(f_n)_{n\in\mathbb{N}}$ has also no uniform (i.e., sup-norm) limit in $\mathrm{C}[0,1]$.

Now let us consider the 1-norm and the 2-norm instead. For $n \in \mathbb{N}$ we compute

$$\|f_n\|_1 = \int_0^1 t^n \,\mathrm{d}t = \frac{1}{n+1} \to 0 \qquad \text{and}$$
$$\|f_n\|_2^2 = \int_0^1 |f_n(t)|^2 \,\mathrm{d}t = \int_0^1 t^{2n} \,\mathrm{d}t = \frac{1}{2n+1} \to 0.$$

This shows that $f_n \to 0$ in $\mathrm{C}[a,b]$ with the standard inner product *and* in $\mathrm{C}[a,b]$ with respect to the 1-norm.

To complete the picture, let $g_n := \sqrt{2n+1} f_n$. Then

$$\|g_n\|_2 = \frac{\sqrt{2n+1}}{\sqrt{2n+1}} = 1, \quad \text{but} \quad \|g_n\|_1 = \frac{\sqrt{2n+1}}{n+1} \to 0.$$

Hence 1-norm convergence does not imply 2-norm convergence.

Advice/Comment:
The previous example shows also that 2-norm-convergence on an interval $[a,b]$ does not imply pointwise convergence.

3.4. The Closure of a Subset

One of the major goals of analysis is to describe how a complicated object may be *approximated* by simple ones. To approximate an object x is nothing else than to find a sequence $(x_n)_{n\in\mathbb{N}}$ converging to it; this is of course only meaningful if one has a surrounding metric space in which one wants the convergence to happen.

The classical example is the approximation of real numbers by rationals in the standard metric of $\mathbb{R}$. Here is a similar example in infinite dimensions.

Example 3.14. The space of **finite sequences** is defined as

$$\mathrm{c}_{00} := \{(x_j)_{j\in\mathbb{N}} \mid x_j = 0 \text{ eventually}\} = \operatorname{span}\{e_j \mid j \in \mathbb{N}\},$$

where $\{e_j \mid j \in \mathbb{N}\}$ are the standard unit vectors introduced in Example 3.11. Clearly, it is a subspace of ℓ^2.

Claim: *Every element $f \in \ell^2$ is the $\|\cdot\|_2$-limit of a sequence $(f_n)_{n\in\mathbb{N}}$ in c_{00}.*

Proof. Fix

$$f = (x_j)_{j\in\mathbb{N}} = (x_1, x_2, x_3, \dots) \in \ell^2.$$

Then it is natural to try as approximants the finite sequences created from f by "cutting off the tail"; i.e., we define

$$f_n := (x_1, x_2, \dots, x_n, 0, 0, \dots) \in \mathrm{c}_{00}$$

for $n \in \mathbb{N}$. Now

$$\|f - f_n\|_2^2 = \sum_{j=1}^{\infty} |f_n(j) - f_n(j)|^2 = \sum_{j=n+1}^{\infty} |x_j|^2 \to 0 \qquad (n \to \infty)$$

since $f \in \ell^2$. This yields $\|f - f_n\|_2 \to 0$ as claimed. □

These considerations motivate the following definition.

Definition 3.15. Let (Ω, d) be a metric space, and let $A \subseteq \Omega$. The **closure** of A (in Ω) is the set

$$\overline{A} := \{x \in \Omega \mid \exists\, (x_n)_{n\in\mathbb{N}} \subseteq A : x_n \to x\}$$

of all the points in Ω that can be approximated by elements from A.

Advice/Comment:
This is analogous to the linear span of a subset B of a vector space E: In $\operatorname{span}(A)$ one collects all the vectors that can be produced by performing the operation "form a finite linear combination" on elements of A. And in $\overline{A}$ one collects all elements of the metric space Ω that can be produced

by performing the operation "take the limit of a convergent sequence" on members of A.

Lemma 3.16. *Let (Ω, d) be a metric space, and let $A \subseteq \Omega$. Then the following assertions are equivalent for $x \in \Omega$:*

(i) $x \in \overline{A}$;

(ii) *There exists $(x_n)_{n\in\mathbb{N}} \subseteq A$ such that $x_n \to x$;*

(iii) $\mathrm{B}_r(x) \cap A \neq \emptyset$ *for all* $r > 0$.

Proof. The equivalence of (i) and (ii) is just the definition. If (ii) holds and $r > 0$ then if for large n we have $d(x_n, x) < r$, i.e., $x_n \in \mathrm{B}_r(x) \cap A$. This proves (iii).

Conversely, if (iii) holds, then for each $n \in \mathbb{N}$ we take $r = 1/n$ and conclude that there is $x_n \in A \cap \mathrm{B}_{1/n}(x)$. This means that $(x_n)_{n\in\mathbb{N}} \subseteq A$ and $d(x_n, x) < 1/n$, which implies that $x_n \to x$. And this is (ii). □

Note that Example 3.14 can be reformulated as $\overline{c_{00}} = \ell^2$ with respect to the 2-norm.

Advice/Comment:
As with convergence, the closure $\overline{A}$ of a set is taken with respect to a surrounding metric space. Closures of the same set in different metric spaces usually differ.

Lemma 3.17. *The closure operation has the following properties:*

a) $\overline{\emptyset} = \emptyset$, $\overline{\Omega} = \Omega$;

b) $A \subseteq \overline{A}$;

c) $A \subseteq B \quad \Longrightarrow \quad \overline{A} \subseteq \overline{B}$;

d) $\overline{A \cup B} = \overline{A} \cup \overline{B}$;

e) $\overline{\overline{A}} = \overline{A}$.

Proof. Assertions a)–c) are pretty obvious, so we leave them as exercise. For the proof of d), note that since $A \subseteq A \cup B$ it follows from c) that $\overline{A} \subseteq \overline{A \cup B}$, and likewise for B. This yields the inclusion "$\supseteq$". To prove the converse inclusion, take $x \in \overline{A \cup B}$; then by definition there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$ and $x_n \in A \cup B$. One of the sets $\{n \in \mathbb{N} \mid x_n \in A\}$ and $\{n \in \mathbb{N} \mid x_n \in B\}$ is infinite (as they partition $\mathbb{N}$), and without loss of generality we may suppose it is the first one. Then this defines a subsequence $(x_{n_k})_{k\in\mathbb{N}} \subseteq A$. Since also $x = \lim_{k\to\infty} x_{n_k}$, we conclude that $x \in \overline{A}$.

For the proof of e), note that $\overline{A} \subseteq \overline{\overline{A}}$ follows from b). For the converse, suppose that $x \in \overline{\overline{A}}$, and let $r > 0$. By Lemma 3.16 there is $y \in \overline{A} \cap \mathrm{B}_{r/2}(x)$; again by Lemma 3.16 there is $z \in A \cap \mathrm{B}_{r/2}(y)$. Then $d(x, z) \le d(x, y) + d(y, z) < r/2 + r/2 = r$ and so $z \in A \cap \mathrm{B}_r(x)$. So $x \in \overline{A}$, by Lemma 3.16.

Ex.3.7
Ex.3.8 □

3.5. Dense Subsets and Weierstrass' Theorem

We have seen above that the 2-norm closure of c_{00} is the whole space ℓ^2. This receives a special name.

Definition 3.18. Let (Ω, d) be a metric space. A subset $A \subseteq \Omega$ is called **dense** (in Ω) if $\overline{A} = \Omega$.

Ex.3.9
Ex.3.10 So c_{00} is dense in ℓ^2 (with respect to the 2-norm).

Advice/Comment:
Note that to say that "A is dense in Ω" always presupposes that A is *contained* in Ω.

Advice/Comment:
The concept of a dense subset is one of the most important concepts in analysis. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, every real number can be arbitrarily well approximated by rational numbers, and when the difference is small it will not matter in practice. This reasoning is at the heart of scientific computing.

Passing from numbers to functions, a smooth curved shape may be arbitrarily well approximated by polygonal shapes, and in practice it may suffice to use the approximation instead of the original.

In the course of our lectures we shall need two famous and important density results with respect to the uniform norm. The first is due to Weierstrass and its proof can be found in Appendix B.1.

Theorem 3.19 (Weierstrass). *Let $[a, b]$ be a compact interval in $\mathbb{R}$. Then the space of polynomials $\mathrm{P}[a, b]$ is dense in $\mathrm{C}[a, b]$ with respect to the sup-norm.*

We let, for $k \in \mathbb{N}$ or $k = \infty$,

$$\mathrm{C}^k[a,b] := \{f \longrightarrow \mathbb{K} \mid f \text{ is } k\text{-times continuously differentiable}\}.$$

Since polynomials are infinitely differentiable, the Weierstrass theorem shows that $\mathrm{C}^\infty[a,b]$ is dense in $\mathrm{C}[a,b]$.

Advice/Comment:
It is recommended that you work yourself through the proof of Weierstrass' theorem at a later stage.

The second density result we quote without proof. It is a consequence of Fejér's theorem in Fourier analysis. We call it the "trigonometric Weierstrass" because there is a formal analogy to Weierstrass' theorem above. (Actually, both theorems are special cases of the so-called Stone–Weierstrass theorem; this is not part of these lectures.)

Recall the definition of the trigonometric system

$$e_n(t) := \mathrm{e}^{2\pi i n \cdot t} \qquad (t \in [0,1], n \in \mathbb{Z})$$

from Chapter 1. Any linear combination of these functions e_n is called a **trigonometric polynomial**. Clearly, each function e_n is 1-periodic, i.e., satisfies $e_n(0) = e_n(1)$, and hence every trigonometric polynomial is periodic too. Let

$$\mathrm{C}_{\mathrm{per}}[0,1] := \{f \in \mathrm{C}[0,1] \mid f(0) = f(1)\}$$

be the space of 1-periodic continuous functions on $[0,1]$. This is a linear subspace of $\mathrm{C}[0,1]$ with respect to the sup-norm.

Theorem 3.20 ("Trigonometric Weierstrass"). *The space of trigonometric polynomials*

$$\operatorname{span}\{e_n \mid n \in \mathbb{Z}\}$$

is dense in $\mathrm{C}_{\mathrm{per}}[0,1]$ *with respect to the sup-norm.*

Advice/Comment:
Theorem 3.20 is *not* due to Weierstrass, and we chose the name "Trigonometric Weierstrass" just because of the resemblance to Weierstrass' theorem 3.19.

Exercises

Exercise 3.1. Let $(E, \|\cdot\|)$ be a normed space, with associated distance function d_E defined by $d_E(f,g) = \|f-g\|$. Show that d_E has the three properties claimed for it on page 25.

Exercise 3.2. Show that

$$\|f\|_\infty \le \|f\|_2 \le \|f\|_1$$

for any sequence $f : \mathbb{N} \longrightarrow \mathbb{K}$. Conclude that

$$\ell^1 \subseteq \ell^2 \subseteq \ell^\infty.$$

Conclude also that 1-norm convergence implies 2-norm convergence, and that 2-norm convergence implies sup-norm convergence. Show that the inclusions above are all strict.

Exercise 3.3. Give an example of a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \ell^1$ with $\|f_n\|_\infty \to 0$ and $\|f_n\|_2 \to \infty$.

Exercise 3.4. Let $f_n(t) := (1+nt)^{-1}$ for $t \in (0,\infty)$ and $n \in \mathbb{N}$.

a) Show that for each $\epsilon > 0$, $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent on $[\epsilon, \infty)$ (to which function?).

b) Show that $(f_n)_{n\in\mathbb{N}}$ is not uniformly convergent on $(0,\infty)$.

Exercise 3.5. Prove the claim of Example 3.12, that is: Let $h \in \mathcal{B}(0,\infty)$ such that $\lim_{t\searrow 0} h(t) = 0$. Define $g_n(t) := e^{-nt}h(t)$ for $t > 0$. Show that $f_n \to 0$ uniformly on $(0,\infty)$.

Exercise 3.6. Show with the help of the inequalities (3.4) that uniform convergence implies 2-norm convergence, and 2-norm convergence implies 1-norm convergence.

Exercise 3.7. Find an example of a metric space (Ω, d) and subsets $A, B \subseteq \Omega$ such that

$$\overline{A} \cap \overline{B} \ne \overline{A \cap B}$$

Exercise 3.8. Let (Ω, d) be a metric space, and let $A \subseteq \Omega$ be dense in Ω. Suppose that $B \subseteq \Omega$ such that $A \subseteq \overline{B}$. Show that B is dense in Ω, too.

Exercise 3.9. Show that c_{00} is dense in ℓ^1 (with respect to the 1-norm). Let $\mathbf{1} = (1,1,1,\dots)$ be the sequence with all entries equal to 1. Show that $\|\mathbf{1} - f\|_\infty \ge 1$ for every $f \in \mathrm{c}_{00}$. Conclude that $\mathbf{1} \notin \overline{\mathrm{c}_{00}}$ with respect to the sup-norm. Then show that

$$\overline{\mathrm{c}_{00}} = \mathrm{c}_0 := \{(x_j)_{j\in\mathbb{N}} \subseteq \mathbb{K} \mid \lim_{j\to\infty} x_j = 0\}$$

(closure with respect to the sup-norm) is the space of all null sequences.

Exercise 3.10. Show that

$$\mathrm{C}_0[a,b] := \{f \in \mathrm{C}[a,b] \mid f(a) = f(b) = 0\}$$

is 2-norm dense in $\mathrm{C}[a,b]$. Is it also sup-norm dense?

Further Exercises

Exercise 3.11. Let (Ω, d) be a metric space, and let $x, y, z, w \in \Omega$. Prove the "complete second triangle inequality":

$$|d(x,z) - d(y,w)| \le d(x,y) + d(z,w)$$

Exercise 3.12. Let (Ω, d) be a metric space, $(x_n)_{n\in\mathbb{N}} \subseteq \Omega$, $x \in \Omega$. Show that the following assertions are equivalent:

(i) $x_n \not\to x$.

(ii) There is $\epsilon > 0$ and a *subsequence* $(x_{n_k})_k$ such that $d(x, x_{n_k}) \ge \epsilon$ for all $k \in \mathbb{N}$.

Conclude that the following assertions are equivalent:

(i) $x_n \to x$

(ii) Each subsequence of $(x_n)_{n\in\mathbb{N}}$ has a subsequence that converges to x.

Exercise 3.13. Let (Ω, d) be a metric space. A subset $A \subseteq \Omega$ is called bounded if

$$\operatorname{diam}(A) := \sup\{d(x,y) \mid x, y \in A\} < \infty.$$

Show that a subset A of a normed space $(E, \|\cdot\|)$ is bounded in this sense if and only if $\sup\{\|x\| \mid x \in A\} < \infty$.

Exercise 3.14. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the metric space (Ω, d) and let $x \in \Omega$. Show that there is a subsequence of $(x_n)_{n\in\mathbb{N}}$ converging to x if and only if

$$x \in \bigcap_{n\in\mathbb{N}} \overline{\{x_k \mid k \ge n\}}.$$

Exercise 3.15. Give a prove of Weierstrass' theorem by using its trigonometric version. (Hint: use power series expansions for the trigonometric polynomials.)

Exercise 3.16. Let $f \in \mathrm{C}[0,1]$ such that $f(0) = 0$. Define for $n \in \mathbb{N}$

$$(T_n f)(t) := \int_0^t \mathrm{e}^{-n(t-s)} f(s)\, ds \qquad (t \in [0,1])$$

a) Show that $\|T_n f\|_\infty \le (1 - \mathrm{e}^{-n}) \|f\|_\infty / n$.

b) Show that

$$nT_nf(t) - f(t) = \int_0^t n\mathrm{e}^{-ns}(f(t-s) - f(t))\,\mathrm{d}s + \mathrm{e}^{-nt}f(t)$$

for all $t \in [0,1]$ and $n \in \mathbb{N}$.

c) Show that $\lim_{n\to\infty} nT_nf = f$ uniformly on $[0,1]$. (Hint: b) and Example 3.12.)

d) Conclude that $\mathrm{C}^1[0,1]$ is dense in $\mathrm{C}[0,1]$.

Chapter 4

Continuity

We continue our introduction to metric topology with the fundamental concepts of open and closed sets, continuity of mappings and compactness of metric spaces. Finally, we discuss equivalence of metrics.

4.1. Open and Closed Sets

You may know the concepts of open and closed set from undergraduate analysis. Here is a definition valid for general metric spaces.

Definition 4.1. Let (Ω, d) be a metric space. A subset $O \subseteq \Omega$ is called **open** if the following holds:

$$\forall\, x \in O\ \exists\, \epsilon > 0 \text{ such that } \mathrm{B}_\epsilon(x) \subseteq O.$$

A subset $A \subseteq \Omega$ is **closed** if $\overline{A} \subseteq A$.

Openness means that for each point x there is some critical distance $\epsilon = \epsilon(x)$ with the property that if you deviate from x not more than ϵ, you will remain inside the set. Closedness means that every point that you can approximate out of A is already contained in A. Note that since always $A \subseteq \overline{A}$, the set A is closed iff $A = \overline{A}$.

Examples 4.2. 1) In every metric space both sets $\emptyset$ and Ω are both open and closed!

2) *The closure $\overline{A}$ of a subset $A \subseteq \Omega$ is always closed*, since $\overline{\overline{A}} = \overline{A}$, by Lemma 3.17. If $B \subseteq \Omega$ is closed and $A \subseteq B$, then $\overline{A} \subseteq B$. Hence $\overline{A}$ is the "smallest" closed set that contains A.

3) *In every metric space, any (open) ball $\mathrm{B}_\epsilon(x)$ is indeed open.*

Proof. Take $y \in \mathrm{B}_\epsilon(x)$. Let $0 < r := d(x,y) < \epsilon$. By the triangle inequality, for each $z \in \mathrm{B}_{\epsilon - r}(y)$

$$d(x,z) \le d(x,y) + d(y,z) = r + d(y,z) < r + (\epsilon - r) = \epsilon.$$

Hence $z \in \mathrm{B}_\epsilon(x)$, which shows that $\mathrm{B}_{\epsilon-r}(y) \subseteq \mathrm{B}_\epsilon(x)$. □

4) In a discrete metric space, *every* subset is both open and closed.
5) Consider $\Omega = \ell^\infty$ with the metric induced by the norm $\|\cdot\|_\infty$. The open ball with radius 1 around 0 is

$$\begin{aligned} \mathrm{B}_1(0) &= \{x \in \ell^\infty \mid \|x\|_\infty < 1\} \\ &= \{x = (x_j)_{j\in\mathbb{N}} \mid \exists\, \delta \in (0,1) : \forall\, j \in \mathbb{N}\, |x_j| \le \delta\}. \end{aligned}$$

Ex.4.1

Advice/Comment:
As with convergence, openness and closedness of a set are notions *relative* to a metric space. To clarify this dependence on the surrounding space, we often say that O is open *in* Ω, A is closed *in* Ω.

The following example illustrates this and it shows that in general a set is *neither* open *nor* closed.

Example 4.3. Consider the set $(0,1]$. Is it open or closed? As with convergence, the answer is: it depends. In the metric space $\mathbb{R}$ with standard metric, it is neither open nor closed. In the metric space $[0,1]$ with the standard metric, it is open, but not closed! In the metric space $(0,1]$ (with any metric), it is both open and closed. In the metric space $[-1,1] \setminus \{0\}$
Ex.4.2 with the standard metric, it is also both open and closed.

The following lemma tells us how openness and closedness are connected.

Lemma 4.4. *Let (Ω, d) be a metric space. A subset $O \subseteq \Omega$ is open if and*
Ex.4.3 *only if its complement $O^c := \Omega \setminus O$ is closed.*

We have the following "permanence properties" for open and closed sets.

Theorem 4.5. *Let (Ω, d) be a metric space. The collection of closed subsets of Ω has the following properties:*

a) *$\emptyset, \Omega$ are closed.*
b) *If $(A_\iota)_\iota$ is any nonempty collection of closed sets, then $\bigcap_\iota A_\iota$ is closed.*
c) *If A, B are closed then $A \cup B$ is closed.*

The collection of open subsets has the following properties.

d) $\emptyset, \Omega$ *are open.*

e) *If* $(O_\iota)_\iota$ *is any nonempty collection of open sets, then* $\bigcup_\iota O_\iota$ *is open.*

f) *If* O, W *are open then* $O \cap W$ *is open.*

Proof. The assertions about open sets follow from the ones about closed sets by De Morgan's laws from set theory. So we prove only the latter ones. Assertion a) and b) follow directly from Lemma 3.17, a) and d) and the definition of a closed set. For c), let $A = \bigcap_\iota A_\iota$ with closed sets $A_\iota \subseteq \Omega$. To show that A is closed, let $(x_n)_{n\in\mathbb{N}} \subseteq A$ and suppose that $x_n \to x \in \Omega$. For every ι, $(x_n)_{n\in\mathbb{N}} \subseteq A_\iota$, and as A_ι is closed, $x \in A_\iota$. As ι was arbitrary it follows that $x \in A$, as was to show. □ Ex.4.4

4.2. Continuity

Continuity of a mapping $f : \Omega \longrightarrow \Omega'$ between metric spaces $(\Omega, d), (\Omega', d')$ can be defined in many ways. We choose the one most convenient for functional analysis.

Definition 4.6. Let $(\Omega', d'), (\Omega, d)$ be two metric spaces. A mapping $f : \Omega \longrightarrow \Omega'$ is called **continuous at** $x \in \Omega$, if for *every* sequence $(x_n)_{n\in\mathbb{N}} \subseteq \Omega$ the implication

$$x_n \to x \qquad \Rightarrow \qquad f(x_n) \to f(x)$$

holds. The mapping f is simply called **continuous** if it is continuous at every point $x \in \Omega$.

Example 4.7 (Continuity of the Norm)**.** *Let* $(E, \|\cdot\|)$ *be a normed space. Then the norm-mapping* $E \longrightarrow \mathbb{R}_+$, $f \longmapsto \|f\|$ *is continuous, i.e.,*

$$f_n \to f \quad \text{in } E \quad \Rightarrow \quad \|f_n\| \to \|f\| \quad \text{in } \mathbb{R}.$$

Proof. We first establish the **second triangle inequality**

$$|\|f\| - \|g\|| \le \|f - g\| \tag{4.1}$$

for all $f, g \in E$. The triangle inequality yields $\|f\| = \|f - g + g\| \le \|f - g\| + \|g\|$, whence

$$\|f\| - \|g\| \le \|f - g\|.$$

Reversing the roles of f and g we have

$$-(\|f\| - \|g\|) = \|g\| - \|f\| \le \|g - f\| = \|f - g\|.$$

Taking the maximum establishes (4.1). Now, if $f_n \to f$ then

$$\big| \|f_n\| - \|f\| \big| \le \|f_n - f\| \to 0,$$

and hence $\|f_n\| \to \|f\|$. □

Lemma 4.8. *For a mapping $f : \Omega \longrightarrow \Omega'$ the following assertions are equivalent:*

(i) *f is continuous.*

(ii) *$f^{-1}(U)$ is open in Ω for each open set $U \subseteq \Omega'$.*

(iii) *$f^{-1}(A)$ is closed in Ω for each closed set $A \subseteq \Omega'$.*

(iv) *For each $x \in \Omega$ we have*

$$\forall\, \epsilon > 0 \;\exists\, \delta > 0 \;\forall\, y \in \Omega : d(x,y) < \delta \implies d(f(x), f(y)) < \epsilon.$$

Proof. We only prove that (i) implies (iii), the remaining implications we leave as exercise. Suppose that f is continuous and that $A \subseteq \Omega'$ is closed. Take a sequence $(x_n)_{n\in\mathbb{N}} \subseteq f^{-1}(A)$ such that $x_n \to x \in \Omega$. Then $f(x_n) \in A$ for all $n \in \mathbb{N}$, and $f(x_n) \to f(x)$ by continuity. As A is closed, $f(x) \in A$,
Ex.4.5 i.e., $x \in f^{-1}(A)$. This was to prove. □

Example 4.9. *The unit ball $B_E = \{f \in E \mid \|f\| \le 1\}$ is closed.* (It is the inverse image of $[0,1]$ under the norm mapping.)

Advice/Comment:
Advice: if you want to show continuity of a mapping between metric spaces, try first our definition via sequences. Try to avoid the (equivalent) $\epsilon - \delta$ formulation.

In normed spaces, the following theorem yields many continuous mappings.

Theorem 4.10. *Let $(E, \|\cdot\|)$ be a normed space. Then the addition mapping and the scalar multiplication are continuous.*

More explicitly, suppose that $f_n \to f$ and $g_n \to g$ in E and $\lambda_n \to \lambda$ in $\mathbb{K}$. Then

$$f_n + g_n \to f + g \quad \text{and} \quad \lambda_n f_n \to \lambda f.$$

If in addition the norm is induced by an inner product $\langle \cdot, \cdot \rangle$, then

$$\langle f_n, g_n \rangle \to \langle f, g \rangle \qquad \text{in } \mathbb{K}.$$

Proof. Continuity of addition follows from

$$\begin{aligned} d_E(f_n + g_n, f + g) &= \|(f_n + g_n) - (f + g)\| = \|(f_n - f) + (g_n - g)\| \\ &\le \|f_n - f\| + \|g_n - g\| = d_E(f_n, f) + d_E(g_n, g) \to 0. \end{aligned}$$

For the scalar multiplication note that

$$\lambda_n f_n - \lambda f = (\lambda_n - \lambda)(f_n - f) + \lambda(f_n - f) + (\lambda_n - \lambda) f;$$

taking norms and using the triangle inequality yields

$$d_E(\lambda_n f_n, \lambda f) \le |\lambda_n - \lambda| \, \|f_n - f\| + |\lambda| \, \|f_n - f\| + |\lambda_n - \lambda| \, \|f\| \to 0.$$

Suppose that the norm is induced by an inner product. Then

$$\langle f_n, g_n \rangle - \langle f, g \rangle = \langle f_n - f, g_n - g \rangle + \langle f, g_n - g \rangle + \langle f_n - f, g \rangle .$$

Taking absolute values and estimating with the triangle and the Cauchy–Schwarz inequality yields

$$\begin{aligned} |\langle f_n, g_n \rangle - \langle f, g \rangle| &= |\langle f_n - f, g_n - g \rangle| + |\langle f, g_n - g \rangle| + |\langle f_n - f, g \rangle| \\ &\le \|f_n - f\| \, \|g_n - g\| + \|f\| \, \|g_n - g\| + \|f_n - f\| \, \|g\| . \end{aligned}$$

As each of these summands tend to 0 as $n \to \infty$, so does the left-hand side. □ Ex.4.6

Advice/Comment:
The previous proof is paradigmatic for convergence proofs in analysis. One needs to prove that $x_n \to x$. To achieve this, one tries to estimate $d(x_n, x) \le a_n$ with some sequence of real numbers $(a_n)_{n \in \mathbb{N}}$ converging to 0. The *sandwich theorem* from undergraduate real analysis yields that $d(x_n, x) \to 0$, i.e., $x_n \to x$ by definition. Note that in this way tedious $\epsilon - N$ arguments are avoided.

Corollary 4.11. *The following assertions hold:*

a) *Let $(E, \|\cdot\|)$ be a normed space, and let $F \subseteq E$ be a linear subspace of E. Then the closure $\overline{F}$ of F in E is also a linear subspace of E.*

b) *Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space and $S \subseteq E$. Then $S^\perp$ is a closed linear subspace of E and $S^\perp = \overline{S}^\perp$.*

Proof. The proof is an exercise. □ Ex.4.7
Ex.4.8

Here is a useful application of Theorem 4.10.

Corollary 4.12. *Let* $\mathrm{C}_0[a,b] := \{f \in \mathrm{C}[a,b] \mid f(a) = f(b) = 0\}$ *and*

$$\mathrm{C}_0^1[a,b] := \mathrm{C}^1[a,b] \cap \mathrm{C}_0[a,b] = \{f \in \mathrm{C}^1[a,b] \mid f(a) = f(b) = 0\}.$$

Then $\mathrm{C}_0^1[a,b]$ *is dense in* $\mathrm{C}_0[a,b]$ *with respect to the uniform norm.*

Proof. Let $f \in \mathrm{C}[a,b]$ with $f(a) = f(b) = 0$. By the Weierstrass theorem we find a sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials such that $p_n \to f$ uniformly on $[a,b]$. Since uniform convergence implies pointwise convergence, $a_n := p_n(a) \to f(a) = 0$ and $b_n := p_n(b) \to f(b) = 0$. We subtract from p_n a linear polynomial to make it zero at the boundary:

$$q_n(t) := p_n(t) - a_n \frac{b-t}{b-a} - b_n \frac{t-a}{b-a}$$

Then $q_n(a) = 0 = q_n(b)$ and q_n is still a polynomial. But since $a_n, b_n \to 0$, $\lim_{n\to\infty} q_n = \lim_{n\to\infty} p_n = f$ uniformly. □

Advice/Comment:
The results of Theorem 4.10 are so natural that we usually do not explicitly mention when we use them.

Other examples are continuous *linear* mappings between normed spaces.

Theorem 4.13. *A linear mapping* $T : E \longrightarrow F$ *between two normed spaces* $(E, \|\cdot\|_E)$ *and* $(F, \|\cdot\|_F)$ *is continuous if and only if it is bounded.*

Proof. Suppose that T is bounded. Then, if $f_n \to f$ in E is an arbitrary convergent sequence in E,

$$\|Tf_n - Tf\|_F = \|T(f_n - f)\|_F \le \|T\|_{E\to F} \|f_n - f\|_E \to 0$$

as $n \to \infty$. So $Tf_n \to Tf$, and T is shown to be continuous.

For the converse, suppose that T is *not* bounded. Then $\|T\|$ as defined in (2.2) is not finite. Hence there is a sequence of vectors $(f_n)_{n\in\mathbb{N}} \subseteq E$ such that

$$\|g_n\| \le 1 \quad \text{and} \quad \|Tg_n\| \ge n \qquad (n \in \mathbb{N}).$$

Define $f_n := (1/n)g_n$. Then $\|f_n\| = \|g_n\|/n \le 1/n \to 0$, but

$$\|Tf_n\| = \left\|T(n^{-1}g_n)\right\| = n^{-1}\|Tg_n\| > 1$$

for all $n \in \mathbb{N}$. Hence $Tf_n \not\to 0$ and therefore T is not continuous. □

Advice/Comment:
Although boundedness and continuity are the same for linear mappings between normed spaces, functional analysts prefer using the term "bounded linear mapping" to "continuous linear mapping".

If $T : E \longrightarrow F$ is a bounded linear mapping between normed spaces, its **kernel**

$$\ker T = \{f \in E \mid Tf = 0\}$$

is a *closed* linear subspace of E, since $\ker T = T^{-1}\{0\}$ is the inverse image of the (closed!) singleton set $\{0\}$.

Example 4.14. *The spaces* $\mathrm{C}_0[a, b]$ *and* $\mathrm{C}_{\mathrm{per}}[a, b]$ *are closed subspaces of* $\mathrm{C}[a, b]$ *with respect to the uniform norm.*

Proof. $\mathrm{C}_0[a, b]$ is the kernel of the bounded linear mapping

$$T : \mathrm{C}[a, b] \longrightarrow \mathbb{K}^2, \qquad Tf := (f(a), f(b)).$$

and $\mathrm{C}_{\mathrm{per}}[a, b]$ is the kernel of the bounded linear mapping

$$S : \mathrm{C}[a, b] \longrightarrow \mathbb{K}, \qquad Tf := f(a) - f(b).$$

□ Ex.4.9

On the other hand, the **range**

$$\operatorname{ran} T = \{Tf \mid f \in E\}$$

of a bounded linear mapping T need not be closed in F. One often writes $\overline{\operatorname{ran}}(T)$ in place of $\overline{\operatorname{ran}(T)}$.

Example 4.15. Let $E = \mathrm{c}_{00}$ with the 2-norm, let $F = \ell^2$ with the 2-norm, and let $T : E \longrightarrow F$, $Tx := x$ for $x \in E$. Since T is an isometry, it is bounded. Its range is not closed, since $\overline{\operatorname{ran}}(T) = \overline{\mathrm{c}_{00}} = \ell^2$ is the whole of ℓ^2 (see Example 3.14).

4.3. (Sequential) Compactness

Already from undergraduate courses you know that compactness is an important feature of certain sets in finite dimensions. We extend the concept to general metric spaces.

Definition 4.16. A subset A of a metric space (Ω, d) is called **(sequentially) compact** if every sequence in A has a subsequence that converges to a point in A.

Advice/Comment:
It can be shown that for metric spaces sequential compactness is the same as general compactness, a topological notion that we do not define here. We therefore often use the word "compact" instead of "sequentially compact".

From elementary analysis courses the reader knows already a wealth of examples of sequentially compact metric spaces.

Theorem 4.17 (Bolzano–Weierstrass). *With respect to the Euclidean metric on $\mathbb{K}^d$ a subset $A \subseteq \mathbb{K}^d$ is (sequentially) compact if and only if it is closed and bounded.*

This theorem is very close to the *axioms* of the real numbers. For the reader's convenience we have included a discussion of these axioms and a proof of the Bolzano–Weierstrass theorem in Appendix A.5. The following example shows that the finite-dimensionality in Theorem 4.17 is *essential*.

Example 4.18. The closed unit ball of ℓ^2 is not compact. Indeed, we have seen that the canonical unit vectors $(e_n)_{n\in\mathbb{N}}$ satisfy $d(e_n, e_m) = \sqrt{2}\delta_{nm}$. Hence no subsequence of this sequence can be convergent.

Let us return to the general theory.

Theorem 4.19. *Let (Ω, d) be a metric space and let $A \subseteq \Omega$ be compact. Then the following assertions hold.*

a) *A is closed in Ω.*
b) *If $f : \Omega \longrightarrow \Omega'$ is a continuous mapping into another metric space, the image set $f(A)$ is a compact subset of Ω'.*

Proof. a) Let $(x_n)_{n\in\mathbb{N}} \subseteq A$ and $x_n \to x \in \Omega$. By compactness, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converging to some element $y \in A$. But also $x_{n_k} \to x$, and as limits are unique, $x = y \in A$.

b) Let $(y_n)_{n\in\mathbb{N}} \subseteq f(A)$. By definition, for each $n \in \mathbb{N}$ there is $x_n \in A$ such that $f(x_n) = y_n$. By compactness of A there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converging to some $x \in A$. By continuity $y_{n_k} = f(x_{n_k}) \to f(x) \in f(A)$. □

Here is a well-known corollary.

Corollary 4.20. *Let $f : \Omega \longrightarrow \mathbb{R}$ is continuous and $A \subseteq \Omega$ is compact. Then f is bounded on A and indeed attains its supremum $\sup_{x\in A} f(x)$ and infimum $\inf_{x\in A} f(x)$.*

Proof. By Theorem 4.19, $f(A) \subseteq \mathbb{R}$ is compact, hence bounded and closed. In particular, it must contain its supremum and its infimum. □

4.4. Equivalence of Norms and Metrics

Almost all of the concepts of metric space theory considered so far depend only on convergence of sequences, but not on actual distances. This is true for the concepts of closure of a set, of closed/open/compact set and of continuity of mappings. That means that they are *qualitative* notions, in contrast to quantitative ones, where one asks actually "how fast" a given sequence converges. This motivates the following definition.

Definition 4.21. Two metrics d, d' on a set Ω are called **equivalent** if one has

$$d(x_n, x) \to 0 \quad \Longleftrightarrow \quad d'(x_n, x) \to 0$$

for *every* sequence $(x_n)_{n\in\mathbb{N}} \subseteq \Omega$ and every point $x \in \Omega$.

In short: two metrics are equivalent if they produce the same convergent sequences. From our consideration above it is clear that with respect to two equivalent metrics the same subsets are closed/open/compact, the closure of a set is the same in either metric and a mapping is continuous with respect to either metric or none of them.

Equivalence of the metrics d and d' is the same as the mappings

$$I_1 : (\Omega, d) \longrightarrow (\Omega, d') \qquad x \longmapsto x$$
$$I_2 : (\Omega, d') \longrightarrow (\Omega, d) \qquad x \longmapsto x$$

being both continuous. (Check it!)

If $\Omega = E$ is a normed space and the metrics are induced by norms $\|\cdot\|, |\!|\!|\cdot|\!|\!|$, by Theorem 4.13 this amounts to the existence of constants m_1, m_2 such that

$$|\!|\!| f |\!|\!| \le m_1 \|f\| \quad \text{and} \quad \|f\| \le m_2 |\!|\!| f |\!|\!|$$

for all $f \in E$. If this is the case, we say that $\|\cdot\|, |\!|\!|\cdot|\!|\!|$ are **equivalent norms**. Hence the induced metrics are equivalent if and only if the norms are equivalent.

Example 4.22. Consider the 1-norm and max-norm on $E = \mathbb{K}^d$. We have $|x_j| \le \|x\|_1$ for all $j = 1, \dots, d$, and hence $\|x\|_\infty \le \|x\|_1$. On the other hand, we have

$$\|x\|_1 = \sum_{j=1}^{d} |x_j| \le \sum_{j=1}^{d} \|x\|_\infty = d \cdot \|x\|_\infty .$$

Hence the two norms are equivalent.

The previous example is no coincidence, as the following result shows.

Theorem 4.23. *Let E be a finite dimensional linear space. Then all norms on E are equivalent.*

Proof. By choosing an algebraic basis in E we may suppose that $E = \mathbb{K}^d$. Let $\|\cdot\|$ be any norm on $\mathbb{K}^d$. We want to find constants $m_1, m_2 > 0$ such that

$$\|x\| \le m_1 \|x\|_2 \quad \text{and} \quad \|x\|_2 \le m_2 \|x\|$$

for all $x \in \mathbb{K}^d$. The first inequality says that

$$\mathrm{I} : (\mathbb{K}^d, \|\cdot\|_2) \longrightarrow (\mathbb{K}^d, \|\cdot\|), \qquad x \longmapsto x$$

is bounded, and this has been already shown in Example 2.16.

A consequence of the first inequality is that the mapping

$$(\mathbb{K}^d, \|\cdot\|_2) \longrightarrow \mathbb{R}_+, \qquad x \longmapsto \|x\|$$

is continuous. Indeed, continuity is immediate from

$$|\|x\| - \|y\|| \le \|x - y\| \le m_1 \|x - y\|_2 \qquad (x, y \in \mathbb{K}^d)$$

which holds by the second triangle inequality for $\|\cdot\|$. As the Euclidean unit sphere is compact (Theorem 4.17) we apply Corollary 4.20 to this continuous mapping to conclude that it must attain its infimum there. That is, there is $x' \in \mathbb{K}^d$ such that

$$\|x'\|_2 = 1 \quad \text{and} \quad \|x'\| = \inf\{\|x\| \mid x \in \mathbb{K}^d, \|x\|_2 = 1\}.$$

Now, because $\|x'\|_2 = 1$ we must have $x' \neq 0$ and since $\|\cdot\|$ is norm, also $\|x'\| \neq 0$. Define $m_2 := 1/\|x'\|$. Then for arbitrary $0 \neq x \in \mathbb{K}^d$ we have

$$1/m_2 = \|x'\| \le \|x/\|x\|_2\| = \|x\| / \|x\|_2$$

since $\|x/\|x\|_2\|_2 = 1$. And this is equivalent to

$$\|x\|_2 \le m_2 \|x\|.$$

As this inequality is trivially true if $x = 0$, we are done. □

Corollary 4.24. *Let E be a finite-dimensional normed space, and let F be an arbitrary normed space. Then all norms on E are equivalent and every linear mapping $T : E \longrightarrow F$ is bounded.*

Proof. This follows from the previous results via choosing a basis in E. □

Example 4.25. No two of the norms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$ are equivalent on
Ex.4.10 $\mathrm{C}[0, 1]$. This follows from Example 3.13.

Exercises

Exercise 4.1. Prove the assertions about the spaces in Examples 4.2.

Exercise 4.2. Show that the set

$$A := \{f \in \ell^\infty \mid \exists k \in \mathbb{N} \mid \; |f(k)| = \|f\|_\infty\}$$

is not closed in ℓ^∞.

Exercise 4.3. Give a proof of Lemma 4.4; i.e., prove that a subset A of a metric space (Ω, d) is closed if and only if its complement $A^c = \Omega \setminus A$ is open.

Exercise 4.4. Prove assertions d)–f) of Theorem 4.5 directly from the definition of an open set.

Exercise 4.5. Prove the remaining implications of Lemma 4.8.

Exercise 4.6. Let Ω be any set. Suppose that $f_n \to f$ and $g_n \to g$ uniformly, with all functions being contained in the space $\mathcal{B}(\Omega)$. Show that $f_n g_n \to fg$ uniformly as well.

Exercise 4.7. Let F be a subset of a normed space $(E, \|\cdot\|)$. Suppose that F is a *linear* sub*space*. Show that $\overline{F}$ is a linear subspace as well.

Exercise 4.8. Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space and $S \subseteq E$. Then $S^\perp$ is a closed linear subspace of E and $S^\perp = \overline{S}^\perp$.

Exercise 4.9. Consider the set $E := \{f \in \mathrm{C}[a,b] \mid \int_a^b f(t)\,\mathrm{d}t = 0\}$. Show that E is a closed linear subspace of $\mathrm{C}[a,b]$ and prove that the space $\mathrm{P}[a,b] \cap E$ is dense in E. (Here, $\mathrm{P}[a,b]$ is again the space of polynomials, and all assertions are to be understood with respect to the sup-norm.)

Exercise 4.10. Show that no two of the norms $\|\cdot\|_1, \|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent on c_{00}. (Hint: Consider the vectors $f_n := e_1 + \cdots + e_n$, $n \in \mathbb{N}$, where e_n denotes the n-th canonical unit vector.) See also Exercise 3.2.

Exercise 4.11. We define

$$\mathrm{c} := \{(x_j)_{j\in\mathbb{N}} \subseteq \mathbb{K} \mid \lim_{j\to\infty} x_j \text{ exists in } \mathbb{K}\}.$$

Show that c is closed in ℓ^∞ (with respect to the sup-norm).

Further Exercises

Exercise 4.12. Use the "trigonometric Weierstrass" theorem to show that $\{f \mid f \text{ is a trig. pol. and } f(0) = 0\}$ is sup-norm dense in $\mathrm{C}_0[0, 1]$.

Exercise 4.13. Show that the discrete metric and the Euclidean metric on $\mathbb{R}^d$ are not equivalent. Give an example of a set $A \subseteq \mathbb{R}^d$ that, with respect to the discrete metric, is closed and bounded, but not sequentially compact.

Exercise 4.14. Let d be a metric on Ω. Show that

$$d'(x, y) := \min\{d(x, y), 1\} \qquad (x, y \in \Omega)$$

defines a metric equivalent to d.

Exercise 4.15. Let (Ω, d) and (Ω', d') be two metric spaces, and let $f, g : \Omega \longrightarrow \Omega'$ be continuous mappings. Show that if $A \subseteq \Omega$ and $f(x) = g(x)$ for all $x \in A$, then $f(x) = g(x)$ even for all $x \in \overline{A}$.

Conclude: if E, F are normed space and $T, S : E \longrightarrow F$ are bounded linear mappings that coincide on the dense linear subspace $E_0 \subseteq E$, then $T = S$.

Exercise 4.16. Let (Ω, d) be a metric space, and let $x_n \to x$ and $y_n \to y$ in Ω. Show that $d(x_n, y_n) \to d(x, y)$. (Hint: Exercise 3.11.)

Exercise 4.17. Let A be a compact subset of a metric space Ω, and let $f : \Omega \longrightarrow \Omega'$ be continuous, where Ω' is another metric space. Prove that $f\big|_A$ is **uniformly continuous**, i.e., satisfies

$$\forall\, \epsilon > 0\ \exists \delta > 0\ \forall\, x, y \in A : d(x, y) < \delta \implies d'(f(x), f(y)) < \epsilon.$$

Exercise 4.18. Let (Ω_j, d_j) are metric space for $j = 1, 2, 3$. Let $f : \Omega_1 \longrightarrow \Omega_2$ be continuous and let $g : \Omega_2 \longrightarrow \Omega_3$ be continuous. Show that $g \circ f : \Omega_1 \longrightarrow \Omega_3$ is continuous.

Exercise 4.19. Show that "equivalence of norms" is indeed an equivalence relation on the set of all norms on a given vector space E.

Exercise 4.20. On $\mathbb{K}^d$ consider the mapping

$$\alpha(x) := \int_0^1 \left| \sum\nolimits_{j=1}^d x_j t^j \right| \mathrm{d}t \qquad (x = (x_1, \dots, x_d) \in \mathbb{K}^d).$$

Show that α is a norm on $\mathbb{K}^d$. (You can use Exercise 2.21. Do you see, how?) Then prove that

$$\inf\{\alpha(x) \mid x_1 + \cdots + x_d = 1\} > 0.$$

Chapter 5

Banach Spaces

In this chapter we shall discuss the important concepts of a Cauchy sequence and the completeness of a metric space. In the final section we see how completeness of a normed space leads to a useful criterium for the convergence of infinite series.

5.1. Cauchy Sequences and Completeness

Look at the interval $(0, 1]$, and forget for the moment that you know about the existence of the surrounding space $\mathbb{R}$. The sequence $(1/n)_{n\in\mathbb{N}}$ does not converge in $(0, 1]$ neither with respect to the standard metric nor to the discrete metric, but — in a sense — for different reasons. In the first case, by looking at the distances $d(x_n, x_m)$ for large $n, m \in \mathbb{N}$ one has the feeling that the sequence "should" converge, however the space $(0, 1]$ lacks a possible limit point. In the second case (discrete metric) one has $d(x_n, x_m) = 1$ for all $n \neq m$, and so one feels that there is no chance to make this sequence convergent by enlarging the space. This leads to the following definition.

Definition 5.1. A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (Ω, d) is called a **Cauchy sequence** if $d(x_n, x_m) \to 0$ as $n, m \to \infty$, i.e., if

$$\forall\, \epsilon > 0 \;\; \exists\, N \in \mathbb{N} \quad \text{such that} \quad d(x_n, x_m) < \epsilon \quad (n, m \geq N).$$

Here are some properties.

Lemma 5.2. *Let (Ω, d) be a metric space. Then the following assertions hold.*

a) *Each convergent sequence in Ω is a Cauchy sequence.*

b) *Each Cauchy sequence is bounded.*

c) *If a Cauchy sequence has a convergent subsequence, then it converges.*

Proof. a) Let $(x_n)_{n\in\mathbb{N}}$ be convergent, with limit $x \in \Omega$. Then by the triangle inequality

$$d(x_n, x_m) \le d(x_n, x) + d(x, x_m) \qquad (n, m \in \mathbb{N}).$$

If $\epsilon > 0$ is fixed, by hypothesis one has $d(x_n, x) < \epsilon/2$ for eventually all $n \in \mathbb{N}$, and so $d(x_n, x_m) < \epsilon$ for eventually all $n, m \in \mathbb{N}$.

b) By definition there is $N \in \mathbb{N}$ such that $d(x_N, x_n) \le 1$ for all $n \ge N$. Define

$$M := \max\{1, d(x_1, x_N), \ldots, d(x_{N-1}, x_N)\} < \infty.$$

If $n, m \in \mathbb{N}$ are arbitrary, then

$$d(x_n, x_m) \le d(x_n, x_N) + d(x_N, x_m) \le M + M = 2M.$$

This proves the claim.

c) Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence and suppose that the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges to $x \in \Omega$. Fix $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon/2$ for $n, n \ge N$. Choose k so large that $n_k \ge N$ and $d(x, x_{n_k}) < \epsilon/2$. Then

$$d(x_n, x) \le d(x_n, x_{n_k}) + d(x_{n_k}, x) < \epsilon/2 + \epsilon/2 = \epsilon$$

if $n \ge N$. □

Ex.5.1
Ex.5.2

Our introductory example shows that there are metric spaces where not every Cauchy sequence converges. So this is a special case, worth an own name.

Definition 5.3. A metric d on a set Ω is called **complete** if every d-Cauchy sequence converges. A metric space (Ω, d) is called **complete** if d is a complete metric on Ω.

Coming back to the introductory example, we may say that $(0, 1]$ with the standard metric is not complete. The space $\mathbb{R}$ with the standard metric is complete. This is almost an axiom about real numbers, see Appendix A.5. Using this fact, we go to higher (but finite) dimensions.

Theorem 5.4. *The Euclidean metric on $\mathbb{K}^d$ is complete.*

Proof. This follows from Corollary A.4 since $\mathbb{K}^d = \mathbb{R}^{2d}$ as metric spaces, when considered with the Euclidean metrics. □

Ex.5.3

The following is a very useful fact when one wants to prove the completeness of a sub-space of a given metric space.

Lemma 5.5. *Let (Ω, d) be a metric space, and let $A \subseteq \Omega$.*

a) *If (Ω, d) is complete and A is closed in Ω, then A with respect to the induced metric is complete.*

b) *If A is (sequentially) compact, then it is complete with respect to the induced metric.*

Proof. We prove a) and leave the proof of b) as an exercise. Suppose that $(x_n)_{n\in\mathbb{N}} \subseteq A$ is a Cauchy sequence with respect to the induced metric. Then it is (trivially) a Cauchy sequence in Ω. By assumption, it has a limit $x \in \Omega$. Since $A \subseteq \Omega$ is closed, it follows that $x \in A$, whence $x_n \to x$ in A (again trivially). □

From the previous lemma we conclude immediately that every closed subset of the Euclidean space $\mathbb{K}^d$ is complete with respect to the induced (=Euclidean) metric.

5.2. Hilbert Spaces

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space. Recall that the inner product induces a natural norm $\|\cdot\|$ by

$$\|f\| = \sqrt{\langle f, f\rangle} \qquad (f \in H)$$

and this norm induces a metric d via

$$d(f,g) := \|f - g\| = \sqrt{\langle f-g, f-g\rangle} \qquad (f, g \in H).$$

We call this the metric *induced* by the inner product. The discussion and the example of the previous section motivate the following definition.

Definition 5.6. An inner product space $(H, \langle\cdot,\cdot\rangle)$ is called a **Hilbert space** if H is complete with respect to the metric induced by the inner product.

From Theorem 5.4 from above we see that $\mathbb{K}^d$ with its standard inner product is a Hilbert space. Here is the infinite-dimensional version of it.

Theorem 5.7. *The space ℓ^2 with its standard inner product is a Hilbert space.*

Proof. For convenience we use function notation, i.e., we write elements from ℓ^2 as functions on $\mathbb{N}$.

Take a Cauchy sequence $f_1, f_2, f_3 \ldots$ in ℓ^2. Note that each f_n is now a function on $\mathbb{N}$. The proof follows a standard procedure: First find the limit function by looking at what the sequence does in each component. Then prove that the alleged limit function is indeed a limit in the given metric.

Fix $j \in \mathbb{N}$. Then obviously

$$|f_n(j) - f_m(j)| \le \|f_n - f_m\|_2 \qquad (n, m \in \mathbb{N}).$$

Hence the sequence $(f_n(j))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$. By the completeness of $\mathbb{K}$, it has a limit, say

$$f(j) := \lim_{n\to\infty} f_n(j).$$

This yields a candidate $f : \mathbb{N} \longrightarrow \mathbb{K}$ for the limit of the sequence f_n. But we still have to prove that $f \in \ell^2$ and $\|f - f_n\|_2 \to 0$.

Fix $\epsilon > 0$ and $M = M(\epsilon) \in \mathbb{N}$ such that $\|f_n - f_m\|_2 < \epsilon$ if $n, m > M$. For fixed $N \in \mathbb{N}$ we obtain

$$\sum_{j=1}^{N} |f_n(j) - f_m(j)|^2 \le \sum_{j=1}^{\infty} |f_n(j) - f_m(j)|^2 = \|f_n - f_m\|_2^2 \le \epsilon^2$$

for all $n, m \ge M$. Letting $m \to \infty$ yields

$$\sum_{j=1}^{N} |f_n(j) - f(j)|^2 \le \epsilon^2$$

for all $n \ge M$ and all $N \in \mathbb{N}$. Letting $N \to \infty$ gives

$$\|f_n - f\|_2^2 = \sum_{j=1}^{\infty} |f_n(j) - f(j)|^2 \le \epsilon^2,$$

i.e., $\|f_n - f\|_2 < \epsilon$ for $n \ge M$. In particular, by the triangle inequality,

$$\|f\|_2 \le \|f_M - f\|_2 + \|f_M\|_2 \le \epsilon + \|f_M\|_2 < \infty,$$

whence $f \in \ell^2$. Moreover, since $\epsilon > 0$ was arbitrary, $\|f_n - f\|_2 \to 0$ as $n \to \infty$, as desired. □

After this positive result, here is a negative one.

Theorem 5.8. *The space* C$[a, b]$*, endowed with the standard inner product, is not complete, i.e., not a Hilbert space.*

Proof. We show this for $[a, b] = [-1, 1]$, the general case being similar. One has to construct a $\|\cdot\|_2$-Cauchy sequence that is not convergent. To this aim consider the functions

$$f_n : [-1,1] \longrightarrow \mathbb{R}, \quad f_n(t) := \begin{cases} 0 & t \in [-1,0] \\ nt & t \in [0,1/n] \\ 1 & t \in [1/n,1]. \end{cases}$$

Then for $m \geq n$ we have $f_n = f_m$ on $[1/n, 1]$ and on $[-1, 0]$, hence

$$\|f_n - f_m\|_2^2 = \int_{-1}^{1} |f_n(t) - f_m(t)|^2 \, \mathrm{d}t = \int_0^{1/n} |f_n(t) - f_m(t)|^2 \, \mathrm{d}t \leq 4/n$$

since $|f_n|, |f_m| \leq 1$. It follows that $(f_n)_{n\in\mathbb{N}}$ is a $\|\cdot\|_2$ Cauchy sequence. We show by contradiction that it does not converge: Suppose that the limit is $f \in \mathrm{C}[-1,1]$. Then

$$\begin{aligned} \int_{-1}^{0} |f(t)|^2 \, \mathrm{d}t &= \int_{-1}^{0} |f(t) - f_n(t)|^2 \, \mathrm{d}t \leq \int_{-1}^{1} |f(t) - f_n(t)|^2 \, \mathrm{d}t \\ &= \|f - f_n\|_2^2 \to 0 \end{aligned}$$

as $n \to \infty$. Hence $\int_{-1}^{0} |f|^2 = 0$, and by Lemma 1.3 $f = 0$ on $[-1, 0]$. On the other hand, for $0 < a < 1$ and $n > 1/a$ we have

$$\begin{aligned} \int_a^1 |f(t) - 1|^2 \, \mathrm{d}t &= \int_a^1 |f(t) - f_n(t)|^2 \, \mathrm{d}t \\ &\leq \int_{-1}^{1} |f(t) - f_n(t)|^2 \, \mathrm{d}t = \|f - f_n\|_2^2 \to 0 \end{aligned}$$

as $n \to \infty$. Hence $\int_a^1 |f - \mathbf{1}|^2 = 0$, and again by Lemma 1.3 $f = \mathbf{1}$ on $[a, 1]$. Since $a \in (0, 1)$ was arbitrary, f is discontinuous at 0, a contradiction. □

Ex.5.4

5.3. Banach spaces

The notion of completeness of an inner product space is actually a property of the norm, not of the inner product. So it makes sense to coin an analogous notion for normed spaces.

Definition 5.9. A normed space $(E, \|\cdot\|)$ is called a **Banach space** if it is complete with respect to its induced metric.

So Hilbert spaces are special cases of Banach spaces. However, we again want to stress that there are many more Banach spaces which are not Hilbert, due to the failing of the parallelogranm law, cf. Remark 2.10.

Example 5.10. *Every finite-dimensional normed space is a Banach space.*

Proof. All norms on a finite-dimensional space are equivalent. It is easy to see (Exercise 5.2) that equivalent norms have the same Cauchy sequences. As we know completeness for the Euclidean norm, we are done. □

Example 5.11. *Let Ω be a non-empty set. Then $(\mathcal{B}(\Omega), \|\cdot\|_\infty)$ is a Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}(\Omega)$ be a Cauchy sequence with respect to $\|\cdot\|_\infty$. We need to find $f \in \mathcal{B}(\Omega)$ such that $\|f_n - f\|_\infty \to 0$. First we try to identify a possible limit function f. Since we know that uniform convergence implies pointwise convergence, we should find f by defining

$$f(x) := \lim_n f_n(x) \qquad (x \in \Omega).$$

This is possible for the following reason. For fixed $x \in \Omega$ we have

$$|f_n(x) - f_m(x)| = |(f_n - f_m)(x)| \le \|f_n - f_m\|_\infty \to 0 \quad \text{as} \quad n, m \to \infty$$

by hypothesis. So $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$, and since $\mathbb{K}$ is complete, the limit (which we call $f(x)$) exists.

Having defined our tentative limit function f we have to show two things: first that $f \in \mathcal{B}(\Omega)$, i.e., f is indeed a bounded function; second that indeed $\|f_n - f\|_\infty \to 0$. To this end we can use no other information than the Cauchy property of the sequence $(f_n)_{n\in\mathbb{N}}$. So fix $\epsilon > 0$. Then there is $N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| \le \|f_m - f_m\|_\infty \le \epsilon$$

for all $x \in X$ and all $n, m \ge N$. Now fix $x \in \Omega$ and $n \ge N$; let $m \to \infty$ we get

$$|f_n(x) - f(x)| = \lim_m |f_n(x) - f_m(x)| \le \epsilon$$

since the function $t \longmapsto |f_n(x) - t|$ is continuous on $\mathbb{K}$. The inequality above holds for all $x \in \Omega$ and all $n \ge N$. Taking the supremum over x we therefore obtain

$$\|f_n - f\|_\infty \le \epsilon$$

for all $n \ge N$. In particular, $\|f_n - f\|_\infty < \infty$, and so $f = f_n - (f_n - f) \in \mathcal{B}(\Omega)$. Summarizing the considerations above, we have shown that to each $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $\|f_n - f\|_\infty \le \epsilon$ for all $n \ge N$; but this is just a reformulation of $\|f_n - f\|_\infty \to 0$, as desired. □

Example 5.12. The space ℓ^∞ of bounded scalar sequences is a Banach space with respect to the sup-norm $\|\cdot\|_\infty$.

Ex.5.5

We have seen above that $\mathrm{C}[a, b]$ is not complete with respect to the 2-norm. (The same is true for the 1-norm.) Things are different for the uniform norm. Ex.5.6

Example 5.13. *The space* $\mathrm{C}[a, b]$ *is a Banach space with respect to the sup-norm* $\|\cdot\|_\infty$.

Proof. We know already that the space $\mathcal{B}[a, b]$ of bounded functions is complete with respect to the uniform norm. By Lemma 5.5 it suffices hence to show that $\mathrm{C}[a, b]$ is *closed* in $(\mathcal{B}[a, b], \|\cdot\|_\infty)$.

To this end, take $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{C}[a, b]$ and $f_n \to f$ uniformly on $[a, b]$, for some bounded function $f \in \mathcal{B}[a, b]$. We fix an arbitrary $x \in [a, b]$ and have to show that f is continuous at x. Using our definition of continuity, we take a sequence $(x_m)_{m\in\mathbb{N}} \subseteq [a, b]$ with $x_m \to x$ and have to show that $f(x_m) \to f(x)$ as $m \to \infty$. By the scalar triangle inequality we may write

$$\begin{aligned}|f(x) - f(x_m)| &\le |f(x) - f_n(x)| + |f_n(x) - f_n(x_m)| + |f_n(x_m) - f(x_m)| \\ &\le 2\|f_n - f\|_\infty + |f_n(x) - f_n(x_m)|\end{aligned}$$

for all $n, m \in \mathbb{N}$. Given $\epsilon > 0$ choose n so large that $\|f_n - f\|_\infty < \epsilon$. For this n, since f_n is continuous at x, we find N such that

$$|f_n(x) - f_n(x_m)| < \epsilon \quad \text{whenever} \quad m \ge N.$$

Then

$$|f(x) - f(x_m)| \le 2\|f_n - f\|_\infty + |f_n(x) - f_n(x_m)| < 3\epsilon$$

for all $m \ge N$. □

Advice/Comment:
The heart of the proof above is called a "3ϵ"-argument. One can shorten it a little, in the following way. As above one derives the inequality

$$|f(x) - f(x_m)| \le 2\|f_n - f\|_\infty + |f_n(x) - f_n(x_m)|$$

for all $n, m \in \mathbb{N}$. We take the lim sup with respect to m and obtain

$$\begin{aligned}\limsup_m |f(x) - f(x_m)| &\le 2\|f_n - f\|_\infty + \limsup_m |f_n(x) - f_n(x_m)| \\ &= 2\|f - f_n\|_\infty .\end{aligned}$$

The right-hand side does not depend on n, so letting $n \to \infty$ shows that it must be zero.

The following example generalizes Example 5.13.

Example 5.14. Let (Ω, d) be a metric space. Denote by

$$\mathrm{C_b}(\Omega) = \mathrm{C_b}(\Omega; \mathbb{K}) := \{f \in \mathcal{B}(\Omega) \mid f \text{ is continuous}\}$$

the space of functions on Ω that are bounded *and* continuous. *Then* $\mathrm{C_b}(\Omega)$ *is a closed subspace of* $(\mathcal{B}(\Omega), \|\cdot\|_\infty)$*, hence is a Banach space with respect to the sup-norm.* Ex.5.7

Ex.5.8
Ex.5.9

5.4. Series in Banach and Hilbert spaces

Let $(E, \|\cdot\|)$ be a normed vector space and let $(f_n)_{n\in\mathbb{N}} \subseteq E$ be a sequence of elements of E. As in the scalar case known from undergraduate courses the formal series

$$\sum_{n=1}^{\infty} f_n \tag{5.1}$$

denotes the *sequence of partial sums* $(s_n)_{n\in\mathbb{N}}$ defined by

$$s_n := \sum_{j=1}^{n} f_j \qquad (n \in \mathbb{N}).$$

If $\lim_{n\to\infty} s_n$ exists in E we call the series (5.1) (simply) **convergent** and use the symbol $\sum_{n=1}^{\infty} f_n$ also to denote its limit.

Definition 5.15. Let $(E, \|\cdot\|)$ be a normed vector space and let $(f_n)_{n\in\mathbb{N}} \subseteq E$ be a sequence of elements of E. The series $\sum_{n=1}^{\infty} f_n$ converges **absolutely** if

$$\sum_{n=1}^{\infty} \|f_n\| < \infty.$$

It is known from undergraduate analysis that if $E = \mathbb{K}$ is the scalar field, then absolute convergence implies (simple) convergence. This is due to completeness, as the following result shows.

Theorem 5.16. *Let* $(E, \|\cdot\|)$ *be a Banach space and let* $(f_n)_{n\in\mathbb{N}} \subseteq E$ *be a sequence in* E *such that* $\sum_{n=1}^{\infty} \|f_n\| < \infty$*. Then* $\sum_{n=1}^{\infty} f_n$ *converges in* E*.*

Proof. The claim is that the sequence $(s_n)_{n\in\mathbb{N}}$ of partial sums converges in E. Since E is a Banach space, i.e., complete, it suffices to show that $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. To this end, take $m > n$ and observe that

$$\|s_m - s_n\| = \Big\| \sum\nolimits_{j=n+1}^{m} f_j \Big\| \le \sum\nolimits_{j=n+1}^{m} \|f_j\| \le \sum\nolimits_{j=n+1}^{\infty} \|f_j\| \to 0$$

as $n \to \infty$. □

Advice/Comment:
Conversely, a normed space in which every absolutely convergent series converges, has to be a Banach space. A proof is given in Appendix B.2.

Example 5.17. The so-called **Weierstrass M-test** says that if $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{C}[a,b]$ such that there is a real number $M > 0$ such that $\sum_{n=1}^{\infty} \|f_n\|_\infty < M$, then the series $\sum_{n=1}^{\infty} f_n$ converges absolutely and uniformly to a continuous function. This is just a special case of Theorem 5.16 since we know that $\mathrm{C}[a,b]$ is complete with respect to the uniform norm.

A typical example for an application is the theory of power series. Indeed, given a power series $\sum_{n=0}^{\infty} a_n z^n$, let R be its radius of convergence and suppose that $R > 0$. Then for $0 < r < R$ the Weierstrass M-test shows that the series converges absolutely and uniformly on $[-r, r]$.

Ex.5.10
Ex.5.11

Already for real numbers one knows series that are convergent but not absolutely convergent, e.g. the *alternating harmonic series* $\sum_{n=1}^{\infty}(-1)^n/n$. Different to the scalar case, there are not many good criteria for the convergence of a series in a general Banach space. For **orthogonal** series in a Hilbert space, however, we have the following inportant result.

Theorem 5.18. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space and let $(f_n)_{n\in\mathbb{N}} \subseteq H$ be a sequence of pairwise orthogonal elements of H. Consider the statements*

(i) *The series $f := \sum_{n=1}^{\infty} f_n$ converges in H.*

(ii) $\sum_{n=1}^{\infty} \|f_n\|^2 < \infty$.

Then (i) *implies* (ii) *and one has* **Parseval's identity**.

$$\|f\|^2 = \sum_{n=1}^{\infty} \|f_n\|^2 . \tag{5.2}$$

If H is a Hilbert space, then (ii) *implies* (i).

Proof. Write $s_n := \sum_{j=1}^{n} f_j$ for the partial sums. If $f = \lim_{m\to\infty} s_n$ exists in H, then by the continuity of the norm and Pythagoras one obtains

$$\begin{aligned}\|f\|^2 &= \left\|\lim_{m\to\infty} s_m\right\|^2 = \lim_{m\to\infty} \|s_m\|^2 = \lim_{m\to\infty} \left\| \sum\nolimits_{j=1}^{m} f_j \right\|^2 \\ &= \lim_{m\to\infty} \sum\nolimits_{j=1}^{m} \|f_j\|^2 = \sum\nolimits_{j=1}^{\infty} \|f_j\|^2 .\end{aligned}$$

Since $\|f\| < \infty$, this implies (ii).

Conversely, suppose that (ii) holds and that H is a Hilbert space. Hence (i) holds if and only if the partial sums $(s_n)_n$ form a Cauchy sequence. If $m > n$ then by Pythagoras' theorem

$$\|s_m - s_n\|^2 = \Big\| \sum\nolimits_{j=n+1}^{m} f_j \Big\|^2 = \sum\nolimits_{j=n+1}^{m} \|f_j\|^2 \le \sum\nolimits_{j=n+1}^{\infty} \|f_j\|^2 \to 0$$

as $n \to \infty$ by (ii), and this concludes the proof. □

Example 5.19. Let $(e_n)_{n\in\mathbb{N}}$ be the sequence of standard unit vectors in ℓ^2. Then

$$\sum_{n=1}^{\infty} \frac{1}{n} e_n$$

converges in ℓ^2 to $(1, 1/2, 1/3, \dots) \in \ell^2$. Note that this series does *not* converge absolutely, since $\sum_{n=1}^{\infty} \|(1/n)e_n\|_2 = \sum_{n=1}^{\infty}(1/n) = \infty$.

Example 5.19 is an instance of an abstract Fourier series. We shall return to this in Chapter 7. Our next goal will be to remedy the non-completeness of $\mathrm{C}[a, b]$ with respect to the 2-norm.

Exercises

Exercise 5.1. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed spaces and let $T : E \longrightarrow F$ be a bounded linear mapping. Show that T maps Cauchy sequences in E to Cauchy sequences in F.

Exercise 5.2. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on a vector space E. Show that a sequence $(x_n)_{n\in\mathbb{N}} \subseteq E$ is $\|\cdot\|_1$-Cauchy if and only if it is $\|\cdot\|_2$-Cauchy.

Exercise 5.3. Show that every discrete metric space is complete.

Exercise 5.4. Prove that the space c_{00} of finite sequences is not a Hilbert space with respect to the standard inner product.

Exercise 5.5. Show that c_{00} (the space of finite sequences) is not a Banach space with respect to the sup-norm.

Exercise 5.6. Show that $\mathrm{C}[a, b]$ is not a Banach space with respect to the 1-norm.

Exercise 5.7. Prove the assertions from Example 5.14.

Exercise 5.8. Show that c_0, the set of scalar null sequences, is a Banach space with respect to the sup-norm.

Exercise 5.9. Show that ℓ^1 is a Banach space with respect to the 1-norm. Show that c_{00} is not a Banach space with respect to the 1-norm.

Exercise 5.10. Let $(E, \|\cdot\|)$ be a normed space, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in E such that $\sum_{n=1}^{\infty} f_n$ converges. Show that $\lim_{n\to\infty} f_n = 0$.

Exercise 5.11. Let $\alpha = (\alpha_n)_{n\in\mathbb{Z}}$ such that $\sum_{n\in\mathbb{Z}} |\alpha_n| < \infty$. Consider the trigonometric double series

$$\sum_{n=-\infty}^{\infty} \alpha_n \mathrm{e}^{2\pi \mathrm{i} n\cdot t}$$

and show that it converges uniformly in $t \in \mathbb{R}$ to a continuous and 1-periodic function on $\mathbb{R}$.

Further Exercises

Exercise 5.12. Prove assertion b) from Lemma 5.5. Prove also the following assertion: *Let (Ω, d) be a metric space and let $A \subseteq \Omega$ be a subset that is complete with respect to the induced metric. Then A is closed in Ω.*

Exercise 5.13. Let $[a, b] \subseteq \mathbb{R}$ be a non-empty interval and let $\alpha \in (0, 1)$. Recall from Exercise 2.20 the space $\mathrm{C}^\alpha[a, b]$ of functions which are Hölder continuous of order α. Consider

$$E := \mathrm{C}_0^\alpha[a, b] := \{f \in \mathrm{C}^\alpha[a, b] \mid f(a) = 0\}$$

Show that with respect to the norm $\|\cdot\|_{(\alpha)}$ introduced in Exercise 2.20, E is a Banach space.

Exercise 5.14. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces and let $T : E \longrightarrow F$ be a bounded linear mapping. Suppose that there is $c \geq 0$ such that

$$\|f\|_E \leq c \|Tf\|$$

for all $f \in E$. Show that $\ker T = \{0\}$ and $\operatorname{ran}(T)$ is closed.

Exercise 5.15. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed spaces. A linear mapping $T : E \longrightarrow F$ is called **invertible** or a **topological isomorphism** if T is bounded and bijective, and T^{-1} is bounded too.

Suppose that $T : E \longrightarrow F$ is invertible and E is a Banach space. Show that F is a Banach space, too.

Exercise 5.16. Let $d_1(x, y) := |x - y|$ and

$$d_2(x, y) := |\arctan(x) - \arctan(y)|$$

for $x, y \in \mathbb{R}$. Show that d_1 and d_2 are equivalent metrics on $\mathbb{R}$. Then show that $(\mathbb{R}; d_2)$ is not complete.

Exercise 5.17 (The Fundamental Principle of Analysis)**.** Let (Ω, d) be a metric space, and let $(x_{n,m})_{n,m\in\mathbb{N}}$ be a *double sequence* in Ω. Suppose that

$$x_{n,m} \to a_n \quad (m \to \infty) \quad \text{and} \quad x_{n,m} \to b_m \quad (n \to \infty)$$

for certain $a_n, b_m \in \Omega$. Suppose further that $a := \lim_{n\to\infty} a_n$ exists, too. We indicate this situation with the diagram

$$\begin{array}{ccc} x_{n,m} & \xrightarrow{\;n\;} & b_m \\ {\scriptstyle m}\downarrow & & \\ a_n & \xrightarrow{\;n\;} & a \end{array}$$

We are interested in the question whether $b_m \to a$ as well. Show (by giving an example) that in general $b_m \not\to a$, even if $b := \lim_{m\to\infty} b_m$ exists. Then show that either of the following hypotheses implies that $b_m \to a$:

(i) $\sup_{m\in\mathbb{N}} d(x_{n,m}, b_m) \to 0$ as $n \to \infty$.

(ii) $\sup_{n\in\mathbb{N}} d(x_{n,m}, a_n) \to 0$ as $m \to \infty$.

(One can rephrase (i) as "$\lim_{n\to\infty} x_{n,m} = b_m$ *uniformly in* m", and (ii) as "$\lim_{m\to\infty} x_{n,m} = a_n$ *uniformly in* n".)

Exercise 5.18 (The Fundamental Principle of Analysis II)**.** Let (Ω, d) be a *complete* metric space, and let $(x_{n,m})_{n,m\in\mathbb{N}}$ be a double sequence in Ω. Suppose that for each n the sequence $(x_{n,m})_{m\in\mathbb{N}}$ is Cauchy and that the sequence $(x_{n,m})_{n\in\mathbb{N}}$ is Cauchy *uniformly in* $m \in \mathbb{N}$. By this we mean that

$$\sup_{m\in\mathbb{N}} d(x_{k,m}, x_{l,m}) \to 0 \quad (k, l \to \infty)$$

Show that under these hypotheses there are elements $a_n, b_m, a \in \Omega$ such that

$$\begin{array}{ccc} x_{n,m} & \xrightarrow{\;n\;} & b_m \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ a_n & \xrightarrow{\;n\;} & a \end{array}$$

Exercise 5.19. A function $f : [a, b] \longrightarrow \mathbb{K}$ is called *of bounded variation* if

$$\|f\|_v := \sup \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})| < \infty$$

where the sup is taken over all decompositions $a = t_0 < t_1 < \cdots < t_n = b$ with $n \in \mathbb{N}$ arbitrary. Denote by

$$\mathrm{BV}_0[a, b] := \{f : [a, b] \longrightarrow \mathbb{K} \mid f(a) = 0, \quad \|f\|_v < \infty\}.$$

Let us believe that $\mathrm{BV}_0[a, b]$ is a linear space and $\|\cdot\|_v$ is a norm on it. Show that $(\mathrm{BV}_0[a, b], \|\cdot\|_v)$ is a Banach space.

(Hint: Show first that $|f(t)| \le \|f\|_v$ for every $f \in \mathrm{BV}_0[a,b]$ and $t \in [a,b]$.

The following exercises presuppose the knowledge of the spaces $\mathrm{L}^2(0,1)$ *and* $\mathrm{L}^1(0,1)$ *from Chapter 6.*

Exercise 5.20. Consider the series of functions

$$\sum_{n=1}^{\infty} \frac{\cos \pi n t}{n^\alpha} \qquad (t \in [0,1]).$$

Determine for which values of $\alpha \ge 0$ the series converges

a) in $\mathrm{C}[0,1]$ with respect to the sup-norm;

b) in $\mathrm{L}^2(0,1)$ with respect to the 2-norm.

Justify your answers.

Exercise 5.21. Define $f_n(t) := \mathrm{e}^{(in^2-n)t}$ for $t \ge 0$. Show that for every $\alpha = (\alpha_n)_n \in \ell^2$ the sum

$$\sum_{n=1}^{\infty} \alpha_n f_n$$

is convergent in $\mathrm{L}^1(\mathbb{R}_+)$. Given that,

$$T : \ell^2 \longrightarrow \mathrm{L}^1(\mathbb{R}_+), \qquad T\alpha := \sum_{n=1}^{\infty} \alpha_n f_n$$

is a well-defined linear mapping. (The proof of linearity is routine, so no need to do this.) Is T bounded? (Justify your answer.)

Exercise 5.22. With a sequence $\alpha = (\alpha_k)_{k\in\mathbb{N}} \in \ell^2$ we associate the power series

$$(T\alpha)(t) := \sum_{k=1}^{\infty} \alpha_k t^{k-1} \qquad (0 < t < 1).$$

Show that this power series converges in $\mathrm{L}^1(0,1)$. Consequently, the mapping

$$T : \ell^2 \longrightarrow \mathrm{L}^1(0,1), \qquad \alpha \mapsto T\alpha$$

is well-defined (and obviously linear). Show that $\|T\| \le \sqrt{\pi^2/6}$.

Intermezzo: Density Principles

In this short Intermezzo we state some easy but very effective results combining the boundedness (of linear operators) with the density (of linear subspaces). The results are mainly for later reference; since there will be many examples and applications later, we do not give any here.

We also omit the proofs here, see Appendix B.3 instead. However, since they are all quite elementary, we recommend to do them as exercises before looking them up.

Theorem DP.1 ("dense in dense is dense"). *Let E be a normed space, let F, G be linear subspaces. If F is dense in E and $F \subseteq \overline{G}$, then G is dense in E.*

Theorem DP.2 ("dense is determining"). *Let E, F be normed spaces and let $T, S : E \longrightarrow F$ be bounded linear mappings. If G is dense in E and $Tf = Sf$ for all $f \in G$, then $T = S$.*

(See also Exercise 4.15.)

Theorem DP.3 ("the image of dense is dense in the image"). *Let E, F be normed spaces and let $T : E \longrightarrow F$ be a bounded linear mapping. If $G \subseteq E$ is dense in E, then $T(G)$ is dense in $T(E)$.*

Theorem DP.4 ("dense implies dense in a weaker norm"). *Let G be a linear subspace of a linear space E, and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on*

E such that there is a constant with $\|f\|_1 \leq c\,\|f\|_2$ for all $f \in G$. If G is $\|\cdot\|_2$-norm dense in E, then it is $\|\cdot\|_1$-norm dense in E, too.

Theorem DP.5 ("dense convergence implies everywhere convergence")**.** *Let E, F be normed spaces and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear mappings $T_n : E \longrightarrow F$ such that*

$$\sup_{n\in\mathbb{N}} \|T_n\| < \infty. \tag{DP.3}$$

If $G \subseteq E$ is dense in E and $T : E \longrightarrow F$ is a bounded linear mapping such that

$$Tf = \lim_{n\to\infty} T_n f \tag{DP.4}$$

for all $f \in G$, then (DP.4) *holds for all $f \in E$.*

A sequence of linear operators that satisfies the conditions (DP.3) is called **uniformly bounded**.

Theorem DP.6 ("densely defined and bounded extends")**.** *Let E be a normed spaces, let F be a Banach space and let $G \subseteq E$ be a dense linear subspace. Furthermore, let $T : G \longrightarrow F$ be a bounded linear operator. Then T extends uniquely to a bounded linear operator $T^{\sim} : E \longrightarrow F$, with the same operator norm.*

By the theorem, there is no danger if we write again T in place of $T^{\sim}$. Note that it is absolutely essential to have the completeness of the space F.

Finally, here is a slight variation of Theorem DP.5, but again: it is essential that F is a Banach space.

Theorem DP.7 ("dense convergence implies everywhere convergence" (2))**.** *Let E be a normed space, let F be a Banach space and let $(T_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of bounded linear mappings $T_n : E \longrightarrow F$. Then if the limit*

$$\lim_{n\to\infty} T_n f \tag{DP.5}$$

exists for each $f \in G$, and if G is dense in E, then the limit (DP.5) *exists for each $f \in E$, and a bounded linear operator $T : E \longrightarrow F$ is defined by $Tf := \lim_{n\to\infty} T_n f$, $f \in E$.*

Chapter 6

The spaces $L^p(X)$

We have seen in Chapter 5 that the space $C[a, b]$ is not complete with respect to the p-norm ($p = 1, 2$). In this chapter we shall remedy this situation by exhibiting a natural "completion".

The major tool to achieve this is to employ the theory of measure and integration, rooting in the fundamental works of H. Lebesgue from the beginning of the twentieth century. A full account of all the needed material requires an own course, hence we shall give only a brief survey.

Evidently, the 1-norm and the 2-norm both use the notion of *integral* of a continuous function, and this integral is to be understood in the Riemann sense. To construct a "completion" of $C[a, b]$ we (at least) must assign to every $\|\cdot\|_1$-Cauchy sequence in $C[a, b]$ a natural limit function. In the example used in the proof of Theorem 5.8 it is not hard to see what this limit would be in this case, namely the function f that is 0 on $[-1, 0]$ and 1 in $(0, 1]$. So we are still in the domain of Riemann integrable functions and we could try to consider the space

$$R[a, b] := \{f : [a, b] \longrightarrow \mathbb{K} \mid f \text{ is Riemann-integrable}\}$$

together with the 1-norm $\|f\|_1 = \int_a^b |f(x)| \, dx$.

A first difficulty arises here: the 1-norm is actually *not a norm*, since there are many non-zero positive Riemann-integrable functions which have zero integral. Below we shall encounter a very elegant mathematical method which amounts to ignoring this fact most of the time.

The more urging problem is that the space $R[a, b]$ is still not complete with respect to $\|\cdot\|_1$, a fact that is by no means obvious. (The proof uses so-called generalized Cantor sets.) So we are forced to go beyond Riemann-integrable functions, and this means that a new notion of integral has to be found that is defined on a larger class of functions and has better properties. This is the so-called *Lebesgue integral*.

To understand the major shift in approach from Riemann to Lebesgue, recall that Riemann's approach uses "Riemann sums"

$$\sum_{j=1}^{n} f(t_{j-1})(t_j - t_{j-1}) \quad \to \quad \int_a^b f(x)\,\mathrm{d}x$$

as approximations of the integral. The quantity $t_j - t_{j-1}$ is simply the *length* $l(A_j)$ of the interval $A_j := (t_{j-1}, t_j]$, and behind the approximation of the integral is an approximation of functions

$$\sum_{j=1}^{n} f(t_{j-1})\mathbf{1}_{A_j}(x) \quad \to \quad f(x) \qquad (x \in [a, b]).$$

(This approximation is not pointwise everywhere, but in the 1-norm.) Here we use the notation $\mathbf{1}_A$ to denote the **characteristic function**

$$\mathbf{1}_A(x) := \begin{cases} 1 & (x \in A) \\ 0 & (x \notin A). \end{cases}$$

of the set A. So in the Riemann approach, the *domain* of the function is partitioned into intervals and this partion is then used for an approximation of f and its integral.

In Lebesgue's approach it is the *range* of the function that is partitioned into intervals. Suppose for simplicity that we are dealing with a positive function f. Then the range of f is contained in $[0, \infty)$. Every partition

$$0 = s_0 < s_1 < \cdots < s_n$$

induces a partition of the domain of f into the sets

$$\{s_0 \le f < s_1\}, \ \ldots, \ \{s_{n-1} \le f < s_n\} \quad \text{and} \quad \{f \ge s_n\}$$

where we use the abbreviation

$$\{c \le f < d\} := \{x \mid c \le f(x) < c\} = f^{-1}[c, d).$$

Now *this* partition is used to give an approximation of f from below:

$$\begin{aligned} g &:= s_0\mathbf{1}_{\{s_0 \le f < s_1\}} + \cdots + s_{n-1}\mathbf{1}_{s_{n-1} \le f < s_n\}} + s_n\mathbf{1}_{\{f \ge s_n\}} \\ &= \sum_{j=1}^{n} s_{j-1}\mathbf{1}_{\{s_{j-1} \le f < s_j\}} + s_n\mathbf{1}_{\{f \ge s_n\}}. \end{aligned}$$

This approximation is a "step function", but with the usual intervals replaced by the more general sets $A_j = \{s_{j-1} \le f < s_j\}$.

Now, *if* all the sets A_j are intervals then we can integrate g and get

$$\int \sum_{j=1}^{n} s_{j-1} \mathbf{1}_{A_j}(x)\,\mathrm{d}x = \sum_{j=1}^{n} s_{j-1} l(A_j)$$

as an approximation of $\int f(x)\,\mathrm{d}x$. In general, however, the sets A_j will not be intervals any more, and so we are led to search for a natural extension of the length (= 1-dimensional volume) of an interval to more general sets. This is the notion of the *Lebesgue measure* of the set.

6.1. Lebesgue measure and integral

We simply state the definition.

Definition 6.1. The **Lebesgue (outer) measure** of a set $A \subseteq \mathbb{R}$ is

$$\lambda^*(A) := \inf \ \textstyle\sum_{n=1}^{\infty} l(Q_n)$$

where the infimum is taken over all sequences of intervals $(Q_n)_{n\in\mathbb{N}}$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} Q_n$.

Advice/Comment:
The new feature here is that *infinite* covers are used. In Riemann's theory and the volume theories based on it (like that of Jordan), the approximations are based on *finite* covers or sums.

We remark that $\lambda^*(Q) = l(Q)$ if Q is an interval. (This is by no means obvious, but we omit a proof.) The set function λ^* is defined on the whole power set of $\mathbb{R}$

$$\lambda : \mathcal{P}(\mathbb{R}) \longrightarrow [0, \infty]$$

and surely can take the value ∞. Unfortunately, its properties are not quite as good as one wishes. For example, in general one can*not* guarantee that

$$A, B \subseteq \mathbb{R}, \quad A \cap B = \emptyset \quad \Longrightarrow \quad \lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B).$$

(This property is called **finite additivity**.) Surprisingly, this is not a defect of the set function λ^* but its chosen domain. If one restricts its domain, things become very nice, and one has even **countable additivity**.

Theorem 6.2. *There is a set $\Sigma \subseteq \mathcal{P}(\mathbb{R})$ of subsets of $\mathbb{R}$ such that the following statements hold*

a) *Every interval is contained in Σ.*

b) Σ *is a* σ-**algebra**, *i.e., it satisfies*

$$\emptyset \in \Sigma; \quad A \in \Sigma \implies A^c \in \Sigma; \quad (A_n)_{n\in\mathbb{N}} \subseteq \Sigma \implies \bigcup_{n\in\mathbb{N}} A_n \in \Sigma.$$

c) $\lambda := \lambda^*\big|_\Sigma$ *is a* **measure**, *i.e., it satisfies*

$$\lambda(\emptyset) = 0; \quad (A_n)_{n\in\mathbb{N}} \subseteq \Sigma,\ p.d. \implies \lambda\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \lambda(A_n).$$

d) *If* $\lambda^*(A) = 0$ *then* $A \in \Sigma$.

Here we have used the notion of a **pairwise disjoint** (p.d.) sequence $(A_n)_{n\in\mathbb{N}}$ of sets, simply meaning that $A_n \cap A_m = \emptyset$ whenever $n \neq m$.

An element A of the σ-algebra Σ from the theorem is called **Lebesgue measurable**. The set function λ defined in Σ is called the **Lebesgue measure**. It is very important to keep in mind that $\Sigma \neq \mathcal{P}(\mathbb{R})$ is *not* the whole power set, i.e., there exist subsets of $\mathbb{R}$ that are not Lebesgue measurable.

Definition 6.3. Let X be any interval. A function $f : X \longrightarrow \mathbb{R}$ is called **(Lebesgue) measurable** if $\{a \le f < b\} \in \Sigma$ for all $a, b \in \mathbb{R}$. A complex-valued function f is measurable if $\operatorname{Re} f, \operatorname{Im} f$ are both measurable.

Example 6.4. If $A \in \Sigma$ then $\mathbf{1}_A$ is measurable. Indeed, $\{a \le f < b\} = A, A^c, \emptyset, \mathbb{R}$, depending whether $0, 1 \in [a, b)$ or not.

In particular, characteristic functions of intervals are measurable, and it turns out that also *all continuous and even all Riemann-integrable functions are measurable.* We denote the class of measurable functions on an interval X by

$$\mathcal{M}(X) = \mathcal{M}(X;\mathbb{K}) := \{f : X \longrightarrow \mathbb{K} \mid f \text{ measurable}\}.$$

This class has nice properties:

Lemma 6.5. *If* $f, g \in \mathcal{M}(X)$ *then* $fg, f+g, |f|, \operatorname{Re} f, \operatorname{Im} f \in \mathcal{M}(X)$*; moreover, if* $f_n \to f$ *pointwise and each* $f_n \in \mathcal{M}(X)$*, then also* $f \in \mathcal{M}(X)$.

Now, by following the original idea of Lebesgue one can construct an integral

$$\mathcal{M}_+(X) \longrightarrow [0,\infty] \qquad f \longmapsto \int_X f \,\mathrm{d}\lambda$$

on the cone $\mathcal{M}_+(X)$ of positive measurable functions in such a way that $\int_X \mathbf{1}_A \,\mathrm{d}\lambda = \lambda(A)$ for every $A \in \Sigma$ and that the integral is additive and positively-homogeneous:

$$\int_X (f + \alpha g)\,\mathrm{d}\lambda = \int_X f\,\mathrm{d}\lambda + \alpha \int_X g\,\mathrm{d}\lambda$$

if $f, g \in \mathcal{M}_+(X), \alpha \geq 0$. Moreover,

$$\int_X f \,\mathrm{d}\lambda = \int_a^b f(x)\,\mathrm{d}x$$

if $0 \leq f \in \mathrm{R}[a,b]$. Ex.6.1

Note that the integral is (up to now) only defined for positive functions and it may take the value infinity. A (not necessarily positive) measurable function $f \in \mathcal{M}(X)$ is called **(Lebesgue) integrable** if

$$\|f\|_1 := \int_X |f| \,\mathrm{d}\lambda < \infty.$$

Let us denote by

$$\mathcal{L}^1(X) := \{f \in \mathcal{M}(X) \mid \|f\|_1 < \infty\}$$

the space of integrable functions.

Theorem 6.6. *The space $\mathcal{L}^1(X)$ is a vector space, and one has*

$$\|f+g\|_1 \leq \|f\|_1 + \|g\|_1, \quad \|\alpha f\|_1 = |\alpha| \, \|f\|_1$$

for all $f, g \in \mathcal{L}^1(X), \alpha \in \mathbb{K}$. Moreover, $\|f\|_1 = 0$ if and only if $\lambda\{f \neq 0\} = 0$.

For a real-valued $f \in \mathcal{L}^1(X)$ we define its integral by

$$\int_X f \,\mathrm{d}\lambda := \int_X f^+ \,\mathrm{d}\lambda - \int_X f^- \,\mathrm{d}\lambda$$

where

$$f^+ := \frac{1}{2}(|f| + f) \qquad \text{and} \qquad f^- := \frac{1}{2}(|f| - f)$$

are the **positive part** and the **negative part** of f, respectively. For a $\mathbb{C}$-valued f we define

$$\int_X f \,\mathrm{d}\lambda := \int_X \operatorname{Re} f \,\mathrm{d}\lambda + \mathrm{i} \int_X \operatorname{Im} f \,\mathrm{d}\lambda.$$

Then we arrive at the following.

Lemma 6.7. *Let $X \subseteq \mathbb{R}$ be an interval. The integral*

$$\mathcal{L}^1(X) \longrightarrow \mathbb{K}, \qquad f \longmapsto \int_X f \,\mathrm{d}\lambda$$

is a linear mapping *satisfying*

$$\left| \int_X f \,\mathrm{d}\lambda \right| \leq \int_X |f| \,\mathrm{d}\lambda = \|f\|_1$$

for every $f \in \mathcal{L}^1(X)$.

Proof. To show linearity is tedious but routine. Let us prove the second statement. If f is real valued, then

$$\left|\int f\right| \le \left|\int f^+\right| + \left|\int f^-\right| = \int f^+ + \int f^- = \int f^+ + f^- = \int |f| .$$

In the case that f is $\mathbb{C}$-valued, find $c \in \mathbb{C}$ with $|c| = 1$ such that $c \int f \in \mathbb{R}$. Then, taking real parts,

$$\left|\int f\right| = \left|c \int f\right| = \operatorname{Re} \int cf = \int \operatorname{Re}(cf) \le \int |cf| = \int |f|$$

as claimed. □

Advice/Comment:
The Lebesgue integral coincides with the Riemann integral for Riemann integrable functions defined on an interval $[a, b]$. Therefore we may write

$$\int_a^b f(x)\,\mathrm{d}x$$

in place of $\int_{[a,b]} f \,\mathrm{d}\lambda$, and we shall usually do this.

6.2. Null sets

We call a set $A \subseteq \mathbb{R}$ a (Lebesgue) **null set** if $\lambda^*(A) = 0$. By Theorem 6.2.d) each null set is Lebesgue measurable.

Example 6.8. If $x \in \mathbb{R}$ is a single point and $\epsilon > 0$, then setting

$$Q_1 := \left(x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}\right), \quad Q_n := \emptyset \quad (n \ge 2)$$

shows that $\{x\}$ is a null set.

To generate more null sets, may use the following lemma.

Lemma 6.9. a) *Every subset of a null set is a null set.*

b) *If $(A_n)_{n\in\mathbb{N}}$ is a sequence of null sets, then $\bigcup_{n\in\mathbb{N}} A_n$ is a null set, too.*

Proof. The first assertion is trivial. To prove the second, fix $\epsilon > 0$ and find for A_k a cover $(Q_{kn})_{n\in\mathbb{N}}$ such that $\sum_{n=1}^\infty |Q_{kn}| < \epsilon/2^k$. Then

$$A = \bigcup_{k\in\mathbb{N}} A_k \subseteq \bigcup_{k,n\in\mathbb{N}} Q_{kn}$$

and $\sum_{k,n=1}^\infty |Q_{kn}| < \sum_{k=1}^\infty \epsilon/2^k = \epsilon$. Note that since $\mathbb{N} \times \mathbb{N}$ is countable, we may arrange the Q_{nk} into a single sequence. □

Example 6.10. *Each countable subset of* $\mathbb{R}$ *is a null set.* Indeed, if $A = \{a_n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ is countable, then $A = \bigcup_{n\in\mathbb{N}}\{a_n\}$ and as each $\{a_n\}$ is a null set (already seen), A is a null set, too.

Advice/Comment:
It is tempting to believe that each null set is countable. However, this is far from true. A nice example of an uncountable null set is the so called "Cantor middle thirds" set. It is constructed by removing from the interval $[0, 1]$ the open middel third interval $(1/3, 2/3)$, then doing the same for the two remaining intervals and proceed iteratively. What remains (i.e., the intersection of all the constructed sets) is clearly a null set. But it contains exactly the real numbers $r \in [0, 1]$ which can be written in a tryadic notation as

$$r = \sum_{j=1}^{\infty} \frac{d_j}{3^j}$$

with $d_j \in \{0, 2\}$ for all $j \in \mathbb{N}$. These are obviously uncountably many.

Definition 6.11. We say that a property P of points of an interval X holds **almost everywhere** (a.e.) if the set

$$\{x \in X \mid P(x) \text{ is not true}\}$$

is a null set. For example, if $f, g : X \longrightarrow \mathbb{K}$ are two functions then we say that

$$f = g \quad \text{almost everywhere}$$

if the set $\{f \neq g\}$ is a null set. In this case we write $f \sim_\lambda g$.

Ex.6.2
Ex.6.3
Ex.6.4

Example 6.12. According to our definition, a sequence of functions $(f_n)_{n\in\mathbb{N}}$ on X converges to a function f almost everywhere, if $f_n(x) \to f(x)$ except for x from a set of measure zero.

For instance, if $f_n(x) := x^n$, $x \in [0, 1]$, then $f_n \to 0$ almost everywhere. (Note that this is false if we consider the f_n as functions on $\mathbb{R}$.)

Lemma 6.13. *The relation* $\sim_\lambda$ *("is equal almost everywhere to") is an equivalence relation on* $\mathcal{F}(X)$*. Moreover, the following statements hold.*

a) *If* $f = \tilde{f}$ *a.e. and* $g = \tilde{g}$ *a.e., then*

$$|f| = |\tilde{f}|, \quad \lambda f = \lambda\tilde{f}, \quad f + g = \tilde{f} + \tilde{g}, \quad fg = \tilde{f}\tilde{g}$$

almost everywhere.

b) *If* $f_n = g_n$ *almost everywhere, for all* $n \in \mathbb{N}$ *and if* $\lim_n f_n = f$ *a.e. and* $\lim_n g_n = g$ *a.e., then* $f = g$ *almost everywhere.*

Proof. Obviously one has $f \sim_\lambda f$ for every f since $\{f \neq f\} = \emptyset$ is a null set. *Symmetry* is trivial, so let us show *transitivity*. Suppose that $f = g$ almost everywhere and $g = h$ almost everywhere. Now $\{f \neq h\} \subseteq \{f \neq g\} \cup \{g \neq h\}$, and so this is a null set by Lemma 6.9.

The proof of a) is left as exercise. We prove b). Set $A_n := \{f_n \neq g_n\}$ and

$$A = \{x \mid f_n(x) \not\to f(x)\}, \quad B = \{x \mid g_n(x) \not\to g(x)\}.$$

Then $\{f \neq g\} \subseteq A \cup B \cup \bigcup_n A_n$, because if $x \notin A_n$ for each n and $x \notin A$ and $x \notin B$, then $f_n(x) = g_n(x)$ and $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$. But that implies that $f(x) = g(x)$. Now, by Lemma 6.9 the set $\{f \neq g\}$ must be a null set. □

Ex.6.5

Recall that $\mathcal{L}^1(X)$ misses a decisive property of a norm: definiteness. Indeed, we know that $\|f - g\|_1 = 0$ if and only if $f = g$ almost everywhere. To remedy this defect, we pass to *equivalence classes* modulo equality almost everywhere and define

$$\mathrm{L}^1(X) := \mathcal{L}^1(X)/\sim_\lambda.$$

Another way to view this is as a factor space of $\mathcal{L}^1(X)$ with respect to the linear subspace(!) $\mathcal{N} := \{f \in \mathcal{L}^1(X) \mid \|f\|_1 = 0\}$. The computations (addition, scalar multiplication, taking 1-norms, taking the integral) are done by using *representatives* for the classes. Of course one must show that
Ex.6.6 these operations are well-defined.

Advice/Comment:
We shall write $f \in \mathrm{L}^1(X)$ and work with f *as if* it was a function. This turns out to be very convenient, and after some time one tends to forget about the fact that these objects *are* not really functions but are only *represented* by functions.

The most annoying consequence of this is that for $f \in \mathrm{L}^1(X)$ the "value" $f(x_0)$ of f at a point $x_0 \in \mathbb{R}$ is meaningless! Indeed, if we alter f on the single point x_0 then we remain still in the same equivalence class, since $\{x_0\}$ is a null set.

Because finite sets are null set, in the definition of $\mathrm{L}^1(X)$ it is inessential whether one starts with closed or open intervals. For instance

$$\mathrm{L}^1[a, b] = \mathrm{L}^1(a, b) \quad \text{and} \quad \mathrm{L}^1[0, \infty) = \mathrm{L}^1(0, \infty).$$

6.3. The Dominated Convergence Theorem

The advantage of the Lebesgue integral lies in its flexibility and especially its convergence results. The most fundamental is the so-called **Monotone Convergence Theorem** B.10. Using this result it is not hard to arrive at the following central fact.

Theorem 6.14 (Dominated Convergence). *Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{L}^1(X)$ and suppose that $f := \lim_{n\to\infty} f_n$ exists pointwise almost everywhere. If there is $0 \le g \in \mathrm{L}^1(X)$ such that $|f_n| \le g$ almost everywhere, for each $n \in \mathbb{N}$, then $f \in \mathrm{L}^1(X)$, $\|f_n - f\|_1 \to 0$ and*

$$\int_X f_n \,\mathrm{d}\lambda \to \int_X f \,\mathrm{d}\lambda.$$

Proof. Note that the *function* f here is defined only almost everywhere. But as such it defines a unique equivalence class modulo equality almost everywhere. It is actually easy to see that $f \in \mathrm{L}^1(X)$: since $f_n \to f$ almost everywhere and $|f_n| \le g$ almost everywhere, for every $n \in \mathbb{N}$, by "throwing away" the countably many null sets we see that $|f| \le g$ almost everywhere. And hence

$$\int_X |f| \,\mathrm{d}\lambda \le \int_X g \,\mathrm{d}\lambda < \infty$$

since $g \in \mathrm{L}^1(X)$. So indeed $f \in \mathrm{L}^1(X)$.

Secondly, if we know already that $\|f_n - f\|_1 \to 0$, then the convergence of the integrals is clear from

$$\left|\int_X f_n \,\mathrm{d}\lambda - \int_X f \,\mathrm{d}\lambda\right| = \left|\int_X f_n - f \,\mathrm{d}\lambda\right| \le \|f_n - f\|_1 \to 0$$

(Lemma 6.7). In other words, the integral is a bounded linear mapping from $\mathrm{L}^1(X)$ to $\mathbb{K}$.

So the real step in the Dominated Convergence Theorem is the assertion that $\|f - f_n\|_1 \to 0$. We shall not give a proof of this here but refer to Appendix B.4. □

Advice/Comment:
In Appendix B.4 you can find a proof of the Dominated Convergence Theorem based on the Monotone Convergence Theorem. On the other hand, Exercise 6.17 shows that the latter is also a consequence of the former.

The dominated convergence theorem has a vast number of applications. Examples are the continuity of the Laplace transform and the Fourier transform of an L^1-function, see Exercise 6.14 and 6.16. Here is a simple model of how this works.

Example 6.15 (Integration Operator). For $f \in \mathrm{L}^1(a,b)$ one defines the function Jf by

$$(Jf)(t) := \int_a^b \mathbf{1}_{[a,t]} f \,\mathrm{d}\lambda = \int_a^t f(x)\,\mathrm{d}x \qquad (t \in [a,b]).$$

Then Jf is continuous: indeed, if $t_n \to t$ in $[a,b]$ then $\mathbf{1}_{[a,t_n]} \to \mathbf{1}_{[a,t]}$ pointwise, except for the point t itself. So $\mathbf{1}_{[a,t_n]} f \to \mathbf{1}_{[a,t]} f$ almost everywhere, and since

$$\left|\mathbf{1}_{[a,t_n]} f\right| \le |f| \in \mathrm{L}^1(a,b)$$

one can apply Dominated Convergence to conclude that $Jf(t_n) \to Jf(t)$.

Hence $J : \mathrm{L}^1(a,b) \longrightarrow \mathrm{C}[a,b]$ is a linear operator. It is also bounded, since

$$|Jf(t)| = \left|\int_a^t f(s)\,\mathrm{d}s\right| \le \int_a^t |f(s)|\ \mathrm{d}s \le \int_a^b |f(s)|\ \mathrm{d}s = \|f\|_1$$

for all $t \in [a,b]$. This yields $\|Jf\|_1 \le \|f\|_\infty$ for all $f \in \mathrm{L}^1(a,b)$.

Using the Dominated Convergence Theorem one can prove the completeness. (See Appendix B.4 for a proof.)

Theorem 6.16 (Completeness of L^1). *The space $\mathrm{L}^1(X)$ is a Banach space. More precisely, let $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{L}^1(X)$ be a $\|\cdot\|_1$-Cauchy sequence. Then there are functions $f, g \in \mathrm{L}^1(X)$ and a subsequence $(f_{n_k})_k$ such that*

$$|f_{n_k}| \le g \quad a.e. \quad and \quad f_{n_k} \to f \quad a.e..$$

Furthermore $\|f_n - f\|_1 \to 0$.

Advice/Comment:
We have discussed the relation between pointwise convergence and convergence in norm already in Section 3.3. We know that usually pointwise convergence does not imply convergence in the 2-norm. This is also true for the 1-norm, see Exercise 6.7. In case of the uniform norm, norm convergence implies pointwise convergence trivially.

However, convergence in the 1-norm is so weak that it does not even imply convergence almost everywhere. Indeed, consider the sequence $(f_k)_{k\in\mathbb{N}}$ given by

$$\mathbf{1}_{[0,1/2]}, \mathbf{1}_{[1/2,1]}, \mathbf{1}_{[0,1/3]}, \mathbf{1}_{[2/3,1]}, \mathbf{1}_{[0,1/4]}, \ldots$$

This sequence converges to 0 in the 1-norm, since $\left\|\mathbf{1}_{[j/n,(j+1)/n]}\right\|_1 = 1/n$. On the other hand for every $x \in [0,1]$, the sequence $(f_k(x))_{k\in\mathbb{N}}$ is a $\{0,1\}$-sequence with both values occuring infinitely often. Hence $(f_k(x))_k$ does not converge at any point in $[0,1]$.

This failure is remedied by Theorem 6.16 which tells us that one must at least have a *subsequence* that converges almost everywhere.

Ex.6.7

6.4. The space $\mathrm{L}^2(X)$

Even more important than the 1-norm is the 2-norm, since this leads to a Hilbert space. For a $f \in \mathcal{M}(X)$ we define

$$\|f\|_2 := \left(\int_X |f|^2 \,\mathrm{d}\lambda\right)^{1/2}$$

let

$$\mathcal{L}^2(X) := \{f \in \mathcal{M}(X) \mid \|f\|_2 < \infty\}.$$

and $\mathrm{L}^2(X) := \mathcal{L}^2(X)/{\sim_\lambda}$. It is not trivial to see that $\mathrm{L}^2(X)$ is a vector space and $\|\cdot\|_2$ is a norm on it. To this aim we need the following variant of the Cauchy–Schwarz inequality.

Lemma 6.17 (Hölder's inequality for $p = 2$)**.** *Let $f, g \in \mathrm{L}^2(X)$ then $fg \in \mathrm{L}^1(X)$ and*

$$\left|\int_X fg \,\mathrm{d}\lambda\right| \le \int_X |fg| \,\mathrm{d}\lambda \le \|f\|_2 \|g\|_2 .$$

Proof. The proof is based on the easy-to-prove identity

$$ab = \inf_{t>0} \frac{t^2}{2}a^2 + \frac{t^{-2}}{2}b^2$$

for real numbers $a, b \ge 0$ (Exercise 6.22). Writing $a = |f(x)|, b = |g(x)|$ we obtain

$$|f(x)g(x)| \le \frac{t^2}{2}|f(x)|^2 + \frac{t^{-2}}{2}|g(x)|^2 .$$

for all $t > 0$ and all $x \in X$. Integrating yields

$$\int_X |fg| \,\mathrm{d}\lambda \le \frac{t^2}{2}\int_X |f|^2 \,\mathrm{d}\lambda + \frac{t^{-2}}{2}\int_X |g|^2 \,\mathrm{d}\lambda$$

for all $t > 0$. Taking the infimum over $t > 0$ again

$$\int_X |fg| \, d\lambda \leq \inf_{t>0} \left[\frac{t^2}{2} \int_X |f|^2 \, d\lambda + \frac{t^{-2}}{2} \int_X |g|^2 \, d\lambda \right]$$
$$= \left(\int_X |f|^2 \, d\lambda \right)^{1/2} \left(\int_X |g|^2 \, d\lambda \right)^{1/2} = \|f\|_2 \, \|g\|_2 .$$

This shows that $fg \in L^1(X)$ and concludes the proof. □

If $f, g \in L^2(X)$ we write

$$|f+g|^2 \leq |f|^2 + 2\,|f|\,|g| + |g|^2 ;$$

if we integrate and use Lemma 6.17, then we see that $\int_X |f+g|^2 \, d\lambda < \infty$. Since $\alpha f \in L^2(X)$ trivially, $L^2(X)$ becomes an inner product space with respect to the inner product

$$\langle f, g \rangle_{L^2} = \int_X f \, \overline{g} \, d\lambda \qquad (f, g \in L^2(X)).$$

One can easily derive an L^2-version of the Dominated Convergence Theorem from Theorem 6.14. Using this, one arrives at completeness of L^2 similar as
Ex.6.8 before.

Theorem 6.18 (Completeness of L^2)**.** *The space $L^2(X)$ is a Hilbert space.*

More precisely, let $(f_n)_{n\in\mathbb{N}} \subseteq L^1(X)$ be a $\|\cdot\|_2$-Cauchy sequence. Then there are functions $f, g \in L^2(X)$ and a subsequence $(f_{n_k})_k$ such that

$$|f_{n_k}| \leq g \quad \text{a.e.} \quad \text{and} \quad f_{n_k} \to f \quad \text{a.e.}.$$

Furthermore $\|f_n - f\|_2 \to 0$.

6.5. Density

Finally, we return to our starting point. Again, we have to quote the result without being able to provide a proof here.

Theorem 6.19. *The space $C[a,b]$ is dense in $L^2(a,b)$ and in $L^1(a,b)$.*

Advice/Comment:
Actually, the theorem is not formulated accurately, since if we speak of density we must always say with respect to which norm/metric this is meant. We make use of a widespread custom: when no norm is explicitly mentioned, we take the one with respect to which the space is known to be a Banach space. In our case, this means:

- the uniform norm $\|\cdot\|_\infty$ in case the space is $C_b(X)$;
- the 1-norm $\|\cdot\|_1$ in case the space is $L^1(X)$;

- the 2-norm $\|\cdot\|_2$ in case the space is $\mathrm{L}^2(X)$.

So the theorem is to be read as: $\mathrm{C}[a,b]$ is dense in $\mathrm{L}^1(a,b)$ w.r.t. the 1-norm, and in $\mathrm{L}^2(a.b)$ w.r.t. the 2-norm.

Based on Theorem 6.19 one can prove various refinements, see the exercises. The theorem tells us that we have reached our goal, i.e., to find "natural completions" of the space $\mathrm{C}[a,b]$ with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$. A consequence of the density theorem is the following very useful characterization of $\mathrm{L}^p(a,b)$. Ex.6.9

Corollary 6.20. *Let $X = [a,b]$ be a finite interval, let $f : [a,b] \longrightarrow \mathbb{R}$ be a function and let $p \in \{1,2\}$. Then $f \in \mathrm{L}^p(a,b)$ if and only if there is a $\|\cdot\|_p$-Cauchy sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{C}[a,b]$ such that $f_n \to f$ almost everywhere.*

Proof. For simplicity we shall confine to the case $p = 1$. Let $f \in \mathrm{L}^1(a,b)$. By Theorem 6.19 there is a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{C}[a,b]$ such that $f_n \to f$ in 1-norm. By Theorem 6.16 one may pass to a subsequence (named $(f_n)_n$ again) that is almost everywhere convergent and in 1-norm to some h. As limits are unique, we must have $h = f$ almost everywhere.

Conversely, let $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{C}[a,b]$ be a 1-norm Cauchy-sequence that converges a.e. to f. By Theorem 6.16 there is $h \in \mathrm{L}^1(a,b)$ such that $f_n \to h$ in 1-norm and a subsequence converges a.e. to h. This implies that $f = h$, and so $f \in \mathrm{L}^1(a,b)$. □

If $X = [a,b]$ is a finite interval, then one has

$$\mathrm{C}[a,b] \subseteq \mathrm{L}^2(a,b) \subseteq \mathrm{L}^1(a,b)$$

with $\|f\|_1 \le \sqrt{b-a}\,\|f\|_2$ for all $f \in \mathrm{L}^2(a,b)$ (proof as exercise). It follows in particular that $\mathrm{L}^2(a,b)$ is dense in $\mathrm{L}^1(a,b)$ (how?). Ex.6.10

Exercises

Exercise 6.1. Prove from the displayed facts about the integral and about measurable functions, that the integral is monotone, i.e., satisfies:

$$f \le g \quad\Longrightarrow\quad \int f\,\mathrm{d}\lambda \le \int g\,\mathrm{d}\lambda$$

for all $f, g \in \mathcal{M}_+(X)$.

Exercise 6.2. Show that if $f : X \longrightarrow \mathbb{K}$ is such that $f = 0$ a.e., then $f \in \mathcal{M}(X)$. Conclude that if $f \in \mathcal{M}(X)$ and $g = f$ a.e., then $g \in \mathcal{M}(X)$, too.

Exercise 6.3. Let $f, g \in \mathrm{C}[a, b]$ and $f = g$ almost everywhere. Show that then $f = g$, i.e., $f(x) = g(x)$ for *all* $x \in X$.

Exercise 6.4. Show that A is a null set if and only if $\mathbf{1}_A = 0$ almost everywhere.

Exercise 6.5. Let X be an interval and let $f, g, \tilde{f}, \tilde{g}$ be functions on X. Show that $f = \tilde{f}$ a.e. and $g = \tilde{g}$ a.e., then

$$|f| = |\tilde{f}|, \quad \lambda f = \lambda \tilde{f}, \quad f + g = \tilde{f} + \tilde{g}, \quad fg = \tilde{f}\tilde{g}, \quad \operatorname{Re} f = \operatorname{Re} \tilde{f}$$

almost everywhere.

Exercise 6.6. Let $f, g \in \mathcal{L}^1(X)$ such that $f = g$ almost everywhere. Show that $\|f\|_1 = \|g\|_1$ and

$$\int_X f \, \mathrm{d}\lambda = \int_X g \, \mathrm{d}\lambda.$$

Exercise 6.7. Let $X = \mathbb{R}$, and let $f_n := \mathbf{1}_{[n,n+1]}$, $n \in \mathbb{N}$. Show that $f_n \to 0$ everywhere, but $(f_n)_n$ is not a $\|\cdot\|_p$-Cauchy sequence for $p = 1, 2$.

Exercise 6.8. Formulate and prove a L^2-version of the Dominated Convergence Theorem, e.g., by using Theorem 6.14.

Exercise 6.9. Show that the space $\mathrm{P}[a, b]$ of polynomials is dense in $\mathrm{L}^2(a, b)$. (Use the Weierstrass theorem and Theorem 6.19.)

Exercise 6.10. Let $X = [a, b]$ a finite interval. Show that

$$\mathrm{L}^2(X) \subseteq \mathrm{L}^1(X)$$

with $\|f\|_1 \le \sqrt{b-a}\, \|f\|_2$ for all $f \in \mathrm{L}^2(X)$. Then show that this inclusion is proper. Show also that

$$\mathrm{L}^1(\mathbb{R}) \not\subseteq \mathrm{L}^2(\mathbb{R}) \quad \text{and} \quad \mathrm{L}^2(\mathbb{R}) \not\subseteq \mathrm{L}^1(\mathbb{R}).$$

Further Exercises

Exercise 6.11. Let $\alpha \in \mathbb{R}$ and consider $f_n := n^\alpha \mathbf{1}_{[0,1/n]}$ for $n \in \mathbb{N}$. Compute $\|f_n\|_p$ for $p = 1, 2$. What is the a.e. behaviour of the sequence $(f_n)_{n\in\mathbb{N}}$?

Exercise 6.12. Let $A \subseteq X$ be measurable and let $p \in \{1, 2\}$. Show that if $f \in \mathrm{L}^p(X)$, then $\mathbf{1}_A \cdot f \in \mathrm{L}^p(X)$, too.

Exercise 6.13. Let $(A_n)_{n\in\mathbb{N}}$ be an *increasing* sequence of measurable subsets of X and let $A := \bigcup_{n\in\mathbb{N}} A_n$. Show that if $f \in \mathrm{L}^p(X)$ then $\mathbf{1}_{A_n} f \to \mathbf{1}_A f$ pointwise and in p-norm.

Exercise 6.14 (Laplace transform)**.** We abbreviate $\mathbb{R}_+ := [0, \infty)$. For $f \in \mathrm{L}^1(\mathbb{R}_+)$ define its *Laplace transform*

$$(\mathcal{L}f)(t) := \int_0^\infty \mathrm{e}^{-ts} f(s)\,\mathrm{d}s \qquad (t \geq 0).$$

Show that $\|\mathcal{L}f\|_\infty \leq \|f\|_1$, that $\mathcal{L}f : \mathbb{R}_+ \longrightarrow \mathbb{K}$ is continuous, and that $\lim_{t\to\infty}(\mathcal{L}f)(t) = 0$.

Exercise 6.15 (Laplace transform (2))**.** Similar to Exercise 6.14 we define the *Laplace transform* of $f \in \mathrm{L}^2(\mathbb{R}_+)$

$$(\mathcal{L}f)(t) := \int_0^\infty \mathrm{e}^{-ts} f(s)\,\mathrm{d}s \qquad (t > 0).$$

(Note that $\mathcal{L}f(0)$ is not defined in general.) Show that $\mathcal{L}f : (0, \infty) \longrightarrow \mathbb{K}$ is continuous with $\lim_{t\to\infty}(\mathcal{L}f)(t) = 0$. (Cf. Example 9.7.)

Exercise 6.16 (Fourier transform)**.** We define the *Fourier transform* of $f \in \mathrm{L}^1(\mathbb{R})$

$$(\mathcal{F}f)(t) := \int_{-\infty}^\infty \mathrm{e}^{-its} f(s)\,\mathrm{d}s \qquad (t \in \mathbb{R}).$$

Show that $\|\mathcal{F}f\|_\infty \leq \|f\|_1$ and that $\mathcal{F}f : \mathbb{R} \longrightarrow \mathbb{K}$ is continuous.

Exercise 6.17 (Monotone Convergence)**.** Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{L}^1(X)$ be an increasing sequence of *positive* integrable functions such that

$$\sup_{n\in\mathbb{N}} \int_X f_n \,\mathrm{d}\lambda < \infty.$$

Use Theorem 6.14 show that there is $f \in \mathrm{L}^1(X)$ such that $f_n \to f$ almost everywhere and in $\|\cdot\|_1$.

Exercise 6.18. Show that the space

$$\mathrm{C_c}(\mathbb{R}) := \{f \in \mathrm{C}(\mathbb{R}) \mid \exists\, a < b : \{f \neq 0\} \subseteq [a, b]\}$$

of **continuous functions with compact support** is dense in $\mathrm{L}^1(\mathbb{R})$ and in $\mathrm{L}^2(\mathbb{R})$.

Exercise 6.19. Let $X = [a, b]$ be a finite interval, and let

$$\mathrm{St}[a, b] := \{\sum_{j=1}^n \alpha_j \mathbf{1}_{t_{j-1},t_j} \mid n \in \mathbb{N},\, a = t_0 < t_1 < \cdots < t_n = b,\, \alpha \in \mathbb{K}^n\}$$

the space of all **(interval) step functions**. Show that $\mathrm{St}[a, b]$ is dense in $\mathrm{L}^2[a, b]$ and in $\mathrm{L}^1[a, b]$.

Exercise 6.20 (The space L^∞)**.** A measurable function $f \in \mathcal{M}(X)$ is called **essentially bounded** if there is a real number $c > 0$ such that $|f| \le c$ almost everywhere. One defines

$$\mathcal{L}^\infty(X) := \{f \in \mathcal{M}(X) \mid f \text{ is essentially bounded}\}$$

and $\|f\|_{\mathrm{L}^\infty} := \inf\{c > 0 \mid \ |f| \le c \text{ a.e.}\}$.

a) Show that $|f| \le \|f\|_{\mathrm{L}^\infty}$ almost everywhere.

b) Show that $\mathcal{L}^\infty(X)$ is a vector space and that

$$\|f + g\|_{\mathrm{L}^\infty} \le \|f\|_{\mathrm{L}^\infty} + \|g\|_{\mathrm{L}^\infty}\,, \quad \|\alpha f\|_{\mathrm{L}^\infty} = |\alpha|\,\|f\|_{\mathrm{L}^\infty}$$

for all $f, g \in \mathcal{L}^\infty(X)$, $\alpha \in \mathbb{K}$.

c) Show that $\|f\|_{\mathrm{L}^\infty} = 0$ if and only if $f = 0$ almost everywhere.

d) Show that $\mathrm{C_b}(X) \subseteq \mathcal{L}^\infty(X)$ with $\|f\|_\infty = \|f\|_{\mathrm{L}^\infty}$.

e) Show that if $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}^\infty(X)$ is a $\|\cdot\|_{\mathrm{L}^\infty}$-Cauchy sequence, then there is a null set $N \subseteq \mathbb{R}$ such that $(f_n)_n$ converges *uniformly* on $X \setminus N$.

f) Show that $\mathrm{L}^\infty(X) := \mathcal{L}^\infty(X)/\!\sim_\lambda$ is a Banach space.

g) Let $p \in \{1, 2, \infty\}$. Show that if $f \in \mathrm{L}^\infty(X)$ and $g \in \mathrm{L}^p(X)$ then $fg \in \mathrm{L}^p(X)$ and

$$\|fg\|_{\mathrm{L}^p} \le \|f\|_{\mathrm{L}^\infty}\,\|g\|_{\mathrm{L}^p}\,.$$

Exercise 6.21 (Affine transformations)**.** For a set $A \subseteq \mathbb{R}$, $c \in \mathbb{R}, \alpha > 0$ define

$$-A := \{-a \mid a \in A\}, \quad c + A := \{c + a \mid a \in A\}, \quad \{\alpha a \mid a \in A\}.$$

Show that $\lambda^*(-A) = \lambda^*(A) = \lambda^*(c + A) = \alpha^{-1}\lambda^*(\alpha A)$.

Remark: It should not come as a surprise that if A is measurable then also $-A, c + A, \alpha A$ are measurable. It can further be shown that if $f \in \mathcal{M}(\mathbb{R})$ then $f(-\cdot), f(c + \cdot), f(\alpha\cdot) \in \mathcal{M}(\mathbb{R})$.

Exercise 6.22. Let $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$, and let $a, b \ge 0$. Prove that

$$ab = \inf_{t>0}\left[\frac{t^p}{p}a^p + \frac{t^{-q}}{q}b^q\right].$$

Use this to show that

$$\int_X fg\,\mathrm{d}\lambda \le \left(\int_X f^p\,\mathrm{d}\lambda\right)^{1/p}\left(\int_X g^q\,\mathrm{d}\lambda\right)^{1/q}$$

for all $f \in \mathcal{M}_+(X)$. (This is the general form of *Hölder's inequality*.)

Chapter 7

Hilbert Space Fundamentals

Hilbert spaces have particularly nice properties that make them the favourite tool in the study of partial differential equations.

7.1. Best Approximations

Let (Ω, d) be a metric space, $A \subseteq \Omega$ be a subset and $x \in \Omega$. We call

$$d(x, A) = \inf\{d(x, y) \mid y \in A\}$$

the **distance** of x to the set A. The function $d(\cdot, A)$ is continuous (Exercise 7.9) but we shall not need this fact. Any element $a \in A$ which *realizes* this distance, i.e., such that

$$d(x, A) = d(x, a),$$

is called a **best approximation** of x in A.

Advice/Comment:
A special case of this concept occurs when A is a closed subset of a normed space and $x = 0$ is the zero vector. Then a best approximation is just an element of A with *minimal norm.* Abstract as it may seem, this concept has a very applied side. The actual state of a physical system is usually that with the smallest energy. In many cases, the energy is a certain norm (adapted to the problem one is considering) and so "minimal energy" becomes "minimal norm", hence a best approximation problem. We shall see a concrete example in the next chapter.

A best approximation a of x in A is a *minimizer* of the function

$$(a \longmapsto d(x,a)) : A \longrightarrow \mathbb{R}_+.$$

In general situations such minimizers do not necessarily exist and when they exist, they need not be unique.

Example 7.1. By Lemma 3.16, $d(x,A) = 0$ if and only if $x \in \overline{A}$. Hence, if A is not closed then for $x \in \overline{A} \setminus A$ there cannot be a best approximation in A: since $d(x,A) = 0$, a best approximation a would satisfy $d(x,a) = 0$ and $a \in A$, and hence $x = a \in A$ which is false by choice of x.

A special case of this is $A = c_{00}$ the space of finite sequences and $\Omega = \ell^2$ and $x = (1/n)_{n\in\mathbb{N}}$.

If the set A is closed and we are in a finite dimensional setting, then a
Ex.7.1 best approximation always exists (Exercise 7.1). This is not true in infinite-dimensional situations.

Example 7.2 (Non-existence)**.** Let $E := \{f \in \mathrm{C}[0,1] \mid f(0) = 0\}$ with the sup-norm. (This is a Banach space!). Let

$$A := \{f \in E \mid \int_0^1 f(t)\,\mathrm{d}t = 0\}.$$

Then A is a closed subspace of E. Let $f(t) := t$, $t \in [0,1]$. Then $f \in E \setminus A$, since $f(0) = 0$ but $\int_0^1 f(t)\,\mathrm{d}t = 1/2 \neq 0$. Hence

$$\frac{1}{2} = \int_0^1 f(t) - g(t)\,\mathrm{d}t \le \int_0^1 |f(t) - g(t)|\,\mathrm{d}t \le \|f - g\|_\infty$$

for every $g \in A$. One can show that $d(f,A) = 1/2$ but there exists no best approximation of f in A. (Exercise 7.14)

So existence can fail. On the other hand, the following example shows that in some cases there are *several different* best approximations.

Example 7.3 (Non-Uniqueness)**.** Let $E = \mathbb{R}^2$ with the $\|\cdot\|_1$-norm, and $A := \mathbb{R}(1,-1)$ the straight line through the points $(-1,1), (0,0), (1,-1)$. If $x := (1,1)$, then $d(x,A) = 2$ and every $a = (\lambda, -\lambda)$ with $\lambda \in [-1,1]$ is a distance minimizer since

$$\|(\lambda,-\lambda) - (1,1)\|_1 = |1-\lambda| + |1+\lambda| = 2$$

for $-1 \le \lambda \le 1$.

We shall show that in Hilbert spaces, unique best approximations exist under a (relatively weak) condition for the set A, namely *convexity.*

Definition 7.4. A subset A of a normed vector space E is called **convex** if

$$f, g \in A,\ t \in [0,1] \quad \Rightarrow \quad tf + (1-t)g \in A.$$

Hence a set A is convex if it contains with any two points also the whole straight line segment joining them.

Theorem 7.5. *Let H be an inner product space, and let $A \neq \emptyset$ be a complete convex subset of H. Furthermore, let $f \in H$. Then there is a unique vector $P_A f := g \in A$ with $\|f - g\| = d(f, A)$.*

Proof. Let us define $d := d(f, A) = \inf\{\|f - g\| \mid g \in A\}$. For $g, h \in A$ we have $(g+h)/2 \in A$ as A is convex, and hence by the parallelogram identity

$$\begin{aligned} \|g - h\|^2 &= \|(g - f) - (h - f)\|^2 \\ &= 2\|g - f\|^2 + 2\|h - f\|^2 - 4\left\|\frac{g+h}{2} - f\right\|^2 \\ &\le 2\|g - f\|^2 + 2\|h - f\|^2 - 4d^2. \end{aligned}$$

If both h, g minimize $\|\cdot - f\|$ then $\|f - g\|^2 = d^2 = \|f - h\|^2$ and we obtain

$$\|g - h\| \le 2d^2 + 2d^2 - 4d^2 = 0,$$

whence $g = h$. This proves uniqueness. To show existence, let $(g_n)_{n\in\mathbb{N}}$ be a minimizing sequence in A, i.e. $g_n \in A$ and $d_n := \|f - g_n\| \searrow d$. For $m \ge n$ we replace g_n, g_m in the above and obtain

$$\|g_n - g_m\|^2 \le 2\|g_n - f\|^2 + 2\|g_m - f\|^2 - 4d^2 \le 4(d_n^2 - d^2).$$

Since $d_n \to d$, also $d_n^2 \to d^2$. Therefore, $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in A, and since A is complete, there is a limit $g := \lim_{n\to\infty} g_n \in A$. But the norm is continuous, and so

$$\|g - f\| = \lim_n \|f - g_n\| = \lim_n d_n = d,$$

and we have found our desired minimizer. □

Note that the proof shows actually that *every minimizing sequence* converges to the best approximation! The conditions of Theorem 7.5 are in particular satisfied if A is a closed convex subset of a Hilbert space. It is in general not easy to compute the best approximation explicitly. (See Exercise 7.2 for an instructing example.) Ex.7.2

7.2. Orthogonal Projections

Let H be a Hilbert space, and let $F \subseteq H$ be a closed linear subspace of H. Then F is in particular convex and complete, so for every $f \in H$ there exists the best approximation $P_F f \in F$ to f in F. Here is a second characterization of the vector $P_F f$.

Lemma 7.6. *Let H be a Hilbert space, let $F \subseteq H$ be a closed linear subspace of H, and let $f, g \in H$. Then the following assertions are equivalent.*

(i) $g = P_F f$.

(ii) $g \in F$ *and* $f - g \perp F$.

Proof. (ii) $\Rightarrow$ (i): Take $h \in F$. Since $f - g \perp F$ and $g \in F$, one has $f - g \perp g - h$, and hence Pythagoras yields

$$\|f - h\|^2 = \|f - g\|^2 + \|g - h\|^2 \geq \|f - g\|^2 .$$

Taking square roots and the infimum over $h \in F$ yields $d(f, F) \geq d(f, g)$, and this shows that g is a best approximation of f in F, i.e., $g = P_F f$.

(i) $\Rightarrow$ (ii): Suppose $g = P_F f$. Then $\|f - g\|^2 \leq \|f - h\|^2$ for all $h \in F$. Since F is a linear subspace, we may replace h by $g - h$ in this inequality, i.e.,

$$\|f - g\|^2 \leq \|(f - g) + h\|^2 = \|f - g\|^2 + 2\,\mathrm{Re}\,\langle f - g, h\rangle + \|h\|^2$$

for all $h \in F$. Now we replace h by th with $t > 0$ and divide by t. We obtain

$$0 \leq 2\,\mathrm{Re}\,\langle f - g, h\rangle + t\,\|h\|^2 \qquad (h \in F).$$

Since this is true for all $t > 0$, we can let $t \searrow 0$ to get

$$0 \leq \mathrm{Re}\,\langle f - g, h\rangle \qquad (h \in F).$$

Finally we can replace h by $-h$ to see that $\mathrm{Re}\,\langle f - g, h\rangle = 0$ for all $h \in F$. So if $\mathbb{K} = \mathbb{R}$ we are done; in the complex case we replace h by ih to finally obtain (ii). □

Ex.7.3 Lemma 7.6 facilitates the computation of best approximations. We now have a closer look at the mapping P_F if F is a closed subspace of a Hilbert space.

Definition 7.7. If F is a closed subspace of a Hilbert space, then the mapping

$$P_F : H \longrightarrow F$$

is called the **orthogonal projection** onto F.

The following lemma summarizes the properties of the orthogonal projection.

Theorem 7.8. *Let F be a closed subspace of a Hilbert space H. Then the orthogonal projection P_F has the following properties:*

a) $P_F f \in F$ *and* $f - P_F f \perp F$ *for all* $f \in H$.

b) $P_F f \in F$ *and* $\|f - P_F f\| = d(f, F)$ *for all* $f \in H$.

c) $P_F : H \longrightarrow H$ *is a bounded linear mapping satisfying* $P_F^2 = P_F$ *and*

$$\|P_F f\| \le \|f\| \qquad (f \in H).$$

d) $\operatorname{ran} P_F = F$ *and* $\ker P_F = F^\perp$.

Proof. b) is true by definition, a) by Lemma 7.6. By a), $\operatorname{ran} P_F \subseteq F$. Furthermore, if $f \in F$ then $d(f, F) = 0$, so f is the best approximation of f in F. This shows that $P_F f = f$ for $f \in F$; in particular, $F \subseteq \operatorname{ran} F$. Again by a), $\ker P_F \subseteq F^\perp$. On the other hand, if $f \perp F$ then $g := 0$ satisfies (ii) of Lemma 7.6, so $P_F f = 0$, i.e., $f \in \ker P_F$. Hence d) is proved.

To prove c), fix $f \in H$ and note first that $P_F f \in F$. But P_F acts as the identity on F, which means that $P_F^2 f = P_F(P_F f) = P_F f$. Since $f - P_F f \perp P_F f$, Pythagoras yields

$$\|f\|^2 = \|f - P_F f\|^2 + \|P_F f\|^2 \ge \|P_F f\|^2$$

To show that P_F is a linear mapping, let $f, g \in H$ and $\alpha \in \mathbb{K}$. Let $h := P_F f + \alpha P_F g$. Then $h \in F$ and

$$(f + \alpha g) - h = (f - P_F f) + \alpha(g - P_F g) \perp F,$$

cf. Lemma 1.8. By Lemma 7.6 we obtain $h = P_F(f + \alpha g)$, and this is linearity. □

Remark 7.9. It follows from Theorem 7.8.a) that our new concept of orthogonal projection coincides with the one for finite-dimensional F, introduced in Chapter 1.

Advice/Comment:
A prominent example of an orthogonal projection appears in probability theory. If $(\Omega, \Sigma, \mathrm{P})$ is a probability space and $\mathcal{F} \subseteq \Sigma$ is a sub-σ-algebra, then $\mathrm{L}^2(\Omega, \mathcal{F}, \mathrm{P})$ is in a natural way a closed subspace of $\mathrm{L}^2(\Omega, \Sigma, \mathrm{P})$. Then the orthogonal projection $P : \mathrm{L}^2(\Omega, \Sigma, \mathrm{P}) \longrightarrow \mathrm{L}^2(\Omega, \mathcal{F}, \mathrm{P})$ is just the *conditional expectation* operator $\mathbb{E}(\cdot \mid \mathcal{F})$.

Corollary 7.10 (Orthogonal Decomposition)**.** *Let* H *be a Hilbert space, and let* $F \subseteq H$ *be a closed linear subspace. Then every vector* $f \in H$ *can be written in a unique way as* $f = u + v$ *where* $u \in F$ *and* $v \in F^\perp$.

Proof. Uniqueness: if $f = u + v = u' + v'$ with $u, u' \in F$ and $v, v' \in F^\perp$, then

$$u - u' = v' - v \in F \cap F^\perp = \{0\}$$

by the definiteness of the scalar product. Hence $u = u', v = v'$ as claimed. Existence: Simply set $u = P_F f$ and $v = f - P_F f$. □

Using terminology from linear algebra (see Appendix A.7), we may say that H is the *direct sum*

$$H = F \oplus F^\perp$$

of the subspaces $F, F^\perp$.

Ex.7.5

Corollary 7.11. *Let F be a subspace of a Hilbert space H. Then $F^{\perp\perp} = \overline{F}$. Moreover,*

$$\overline{F} = H \quad \textit{if and only if} \quad F^\perp = \{0\}.$$

Proof. As $F^\perp = \overline{F}^\perp$ by Corollary 4.11b), we may replace F by $\overline{F}$ in the statement, and suppose without loss of generality that F is closed. The inclusion $F \subseteq F^{\perp\perp}$ is trivial, because it just says that $F \perp F^\perp$. For the converse inclusion, let $g \perp F^\perp$. Then $g - P_F g \in F^{\perp\perp} \cap F^\perp$ and this means that $g - P_F g \perp g - P_F g$ which is equivalent to $g - P_F g = 0$. This shows that $g = P_F g \in F$ as claimed.

The remaining assertions follows easily by taking orthogonals. □

7.3. The Riesz–Fréchet Theorem

Let H be a Hilbert space. If we fix $g \in H$ as the second component in the inner product, we obtain a linear functional

$$\varphi_g : H \longrightarrow \mathbb{K}, \qquad f \longmapsto \varphi_g(f) := \langle f, g \rangle$$

By Cauchy–Schwarz, one has

$$|\varphi_g(f)| = |\langle f, g \rangle| \le \|f\| \, \|g\|$$

for all $f \in H$, hence φ_g is bounded. The Riesz–Fréchet theorem asserts that *every* bounded linear functional on H is of this form.

Theorem 7.12 (Riesz–Fréchet)**.** *Let H be a Hilbert space and let $\varphi : H \longrightarrow \mathbb{K}$ be a bounded linear functional on H. Then there exists a unique $g \in H$ such that $\varphi = \varphi_g$, i.e.,*

$$\varphi(f) = \langle f, g \rangle \qquad (f \in H).$$

Proof. Uniqueness: If $\varphi_g = \varphi_h$ for some $g, h \in H$ then

$$\langle f, g-h \rangle = \langle f, g \rangle - \langle f, h \rangle = \varphi_g(f) - \varphi_h(f) = 0 \qquad (f \in H).$$

Hence $g - h \perp H$ which is only possible if $g = h$. Existence: If $\varphi = 0$, we can take $g := 0$. Otherwise, let

$$F := \ker \varphi = \{f \mid \varphi(f) = 0\}$$

which is a closed subspace of H, since φ is bounded. Since we suppose that $\varphi \neq 0$, we must have $F \neq H$. To get an idea how to find g, we observe that if $\varphi = \varphi_g$ then

$$f \in F \iff 0 = \varphi(f) = \varphi_g(f) = \langle f, g \rangle \iff f \perp g,$$

which means that $F \perp g$. Since $F \neq H$, the orthogonal decomposition of $H = F \oplus F^\perp$ tells us that $F^\perp \neq \{0\}$. So we can find $h \perp F$, $\|h\| = 1$, and it remains to show that we can take g as a multiple of h.

For general $f \in H$ consider

$$u := \varphi(f)h - \varphi(h)f.$$

Then $\varphi(u) = \varphi(f)\varphi(h) - \varphi(f)\varphi(f) = 0$, whence $u \in F$. So

$$0 = \langle u, h \rangle = \langle \varphi(f)h - \varphi(h)f, h \rangle = \varphi(f) \|h\|^2 - \varphi(h) \langle f, h \rangle$$

which yields $\varphi(f) = \varphi(h) \langle f, h \rangle = \langle f, g \rangle = \varphi_g(f)$, with $g := \overline{\varphi(h)}h$. □

We shall see a prototypical application of the Riesz–Fréchet theorem to differential equations in Chapter 8.

7.4. Abstract Fourier Expansions

Let H be a Hilbert space and let $(e_j)_{j\in\mathbb{N}}$ be an ONS in H. Analogous to the finite-dimensional situation considered in Chapter 1 we study now the *infinite* abstract Fourier series

$$Pf := \sum\nolimits_{j=1}^{\infty} \langle f, e_j \rangle e_j$$

for given $f \in H$. Of course, there is an issue of convergence here.

Theorem 7.13. *Let H be a Hilbert space , let $(e_j)_{j\in\mathbb{N}}$ be an ONS in H, and let $f \in H$. Then one has* **Bessel's inequality**

$$\sum\nolimits_{j=1}^{\infty} |\langle f, e_j \rangle|^2 \le \|f\|^2 < \infty. \tag{7.1}$$

Moreover, the series

$$Pf := \sum\nolimits_{j=1}^{\infty} \langle f, e_j \rangle e_j$$

is convergent in H, and $Pf = P_F f$ is the orthogonal projection of f onto the closed subspace

$$F := \overline{\operatorname{span}\{e_j \mid j \in \mathbb{N}\}}.$$

Finally, one has **Parseval's identity** $\quad \|Pf\|^2 = \sum_{j=1}^{\infty} |\langle f, e_j\rangle|^2$.

Proof. For Bessel's inequality it suffices to establish the estimate

$$\sum_{j=1}^{n} |\langle f, e_j\rangle|^2 \le \|f\|^2$$

for arbitrary $n \in \mathbb{N}$. This is immediate from Lemma 1.10. By Bessel's inequality and the fact that H is complete (by assumption) Theorem 5.18 (on the convergence of orthogonal series in Hilbert spaces) yields that the sum $Pf := \sum_{j=1}^{\infty} \langle f, e_j\rangle\, e_j$ is indeed convergent in H with Parseval's identity being true.

To see that $Pf = P_F f$ we only need to show that $Pf \in F$ and $f - Pf \perp F$. Since Pf is a limit of sums of vectors in F, and F is closed, $Pf \in F$. For the second condition, note that

$$\langle f - Pf, e_k\rangle = \langle f, e_k\rangle - \sum_{j=1}^{\infty} \langle f, e_j\rangle \langle e_j, e_k\rangle = \langle f, e_k\rangle - \langle f, e_k\rangle = 0$$

for every k. Hence $f - Pf \perp F$ by Corollary 4.11. □

Corollary 7.14. *Let H be a Hilbert space space, let $(e_j)_{j\in\mathbb{N}}$ be an ONS in H. Then the following assertions are equivalent.*

(i) $\{e_j \mid j \in \mathbb{N}\}^{\perp} = \{0\}$;

(ii) $\operatorname{span}\{e_j \mid j \in \mathbb{N}\}$ *is dense in H;*

(iii) $f = \sum_{j=1}^{\infty} \langle f, e_j\rangle\, e_j \quad$ *for all $f \in H$;*

(iv) $\|f\|^2 = \sum_{j=1}^{\infty} |\langle f, e_j\rangle|^2 \quad$ *for all $f \in H$;*

(v) $\langle f, g\rangle_H = \sum_{j=1}^{\infty} \langle f, e_j\rangle \overline{\langle g, e_j\rangle} \quad$ *for all $f, g \in H$.*

Proof. We use the notation from above. Then (i) just says that $F^{\perp} = \{0\}$ which is (by orthogonal decomposition) equivalent to $F = H$, i.e., (ii). Now, (iii) simply expresses that $f = Pf$ for all $f \in H$, and since $P = P_F$ is the orthogonal projection onto F, this is equivalent to $F = H$. If (iii) holds , the (iv) is also true, by Parseval's identity. On the other hand, by Pythagoras and since P is an orthogonal projection,

$$\|f\|^2 = \|Pf\|^2 + \|f - Pf\|^2$$

which implies that $\|f\|^2 = \|Pf\|^2$ if and only if $f = Pf$. This proves the equivalence (iii)$\Longleftrightarrow$(iv). The equivalence of (iv) and (v) is established in Exercise 7.6. □ Ex.7.6

Definition 7.15. An ONS $(e_j)_{j\in\mathbb{N}}$ in the inner product space H is called **maximal** or **complete** or an **orthonormal basis** of H, if it satisfies the equivalent conditions of Corollary 7.14.

Advice/Comment:
Attention: In algebraic terminology, a *basis* of a vector space is a linearly independent subset such that every vector can be represented as a *finite* linear combination of basis vectors. Hence an *orthonormal basis* in our sense is usually *not* an (algebraic) basis. To distinguish the two notion of bases, in analytic contexts one sometimes says *Hamel basis* for an algebraic basis, to avoid confusion.

We shall not use the term "complete ONS" since it is outdated.

Example 7.16. Let us apply these results to the space $H = \ell^2$ with its standard ONS $(e_n)_{n\in\mathbb{N}}$. We have seen in Example 3.14 that their linear span c_{00} is dense in ℓ^2. That amounts to say that $(e_n)_{n\in\mathbb{N}}$ is indeed an orthonormal basis of ℓ^2.

The following results tells that ℓ^2 is prototypical.

Theorem 7.17. *Let H be a Hilbert space, and let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of H. Then the map*

$$T : H \longrightarrow \ell^2, \qquad f \longmapsto (\langle f, e_j\rangle)_{j\in\mathbb{N}}$$

is an isometric isomorphism.

Proof. It follows directly from Theorem 7.13 and Corollary 7.14 that T is a well-defined linear isometry. The surjectivity is left as (easy) exercise. □ Ex.7.7

Advice/Comment:
Theorem 7.17 says that — in a sense — ℓ^2 is the *only* Hilbert space with a countable orthonormal basis. However, the choice of basis is not canonical in most cases.

Example 7.18. Let us apply these results to the *trigonometric system*

$$e_n = \mathrm{e}^{2\mathrm{i}\pi n \cdot} \qquad (n \in \mathbb{Z})$$

in $\mathrm{L}^2(0,1)$, introduced in Chapter 1. By the trigonometric Weierstrass theorem 3.20, the space of trigonometric polynomials

$$S := \operatorname{span}\{e_n \mid n \in \mathbb{Z}\}$$

is uniformly dense in $\mathrm{C}_{\mathrm{per}}[0,1]$. On the other hand, this space is dense in $\mathrm{L}^2(0,1)$ in the 2-norm (Exercise 7.8). Since sup-norm convergence implies 2-norm convergence, S is 2-norm dense in $\mathrm{L}^2(0,1)$, and hence the trigonometric system forms an orthonormal basis therein. This means that every continuous function $f \in \mathrm{L}^2(0,1)$ can be represented by the Fourier series

$$f = \sum\nolimits_{n=-\infty}^{\infty} c_n e^{2\pi i n t}, \qquad c_n = \langle f, e_n \rangle = \int_0^1 f(t) e^{-2\pi i n t}\, \mathrm{d}t$$

(convergence in the 2-norm). Of course one has to make sense of the doubly infinite summation, see Exercise 7.13 and Appendix C.

Finally and at last, we turn to the question of existence of orthonormal bases. By employing the Gram–Schmidt procedure (Lemma 1.11), we obtain the following.

Lemma 7.19. *A Hilbert space has a countable orthonormal basis if and only if there is a sequence* $(f_n)_{n\in\mathbb{N}} \subseteq H$ *such that*

$$\operatorname{span}\{f_n \mid n \in \mathbb{N}\}$$

is dense in H*.*

Proof. By discarding succesively linearly dependent vectors from the sequence, one ends up with a linearly independent one. Then one can apply literally the Gram–Schmidt Lemma 1.11 to find a countable orthonormal basis. □

A Banach space satisfying the condition of Lemma 7.19 is called **separable**.

Exercises

Exercise 7.1. Let $A \subseteq E$ be closed subset of a normed space E, and let $x \in E$. Show that if A is (sequentially) compact then a best approximation of x exists in A. Conclude that if E is finite-dimensional then there is a best approximation of x in A even if A is not compact. (Hint: Consider the set $A \cap \mathrm{B}_r(x)$ for large enough $r > 0$.)

Exercise 7.2. Let $X \subseteq \mathbb{R}$ be an interval and let $H := \mathrm{L}^2(X;\mathbb{R})$. Let $\mathrm{L}^2_+ := \{f \in \mathrm{L}^2(X;\mathbb{R}) \mid f \geq 0 \,\text{a.e.}\}$ be the *positive cone*. Show that for $f, g \in \mathrm{L}^2(X)$ with $g \geq 0$ one has

$$|f - g| \geq f^- = \left|f - f^+\right| \quad \text{a.e.}.$$

Conclude that f^+ is the best approximation of f in L^2_+.

Exercise 7.3. Let $H = \mathrm{L}^2(0,1)$ and $f(t) = e^t$, $t \in [0,1]$. Find best approximations of f within F in the following cases:

a) F is the space of polynomials of degree at most 1.

b) F is the space $\{at + bt^2 \mid a, b \in \mathbb{C}\}$.

c) F is the space $\{g \in \mathrm{L}^2(0,1) \mid \int_0^1 g = 0\}$.

Exercise 7.4 (Characterization of Orthogonal Projections)**.** Let H be a Hilbert space, and let $P : H \longrightarrow H$ be a linear mapping satisfying $P^2 = P$. Show that $Q := \mathrm{I} - P$ satisfies $Q^2 = Q$, and that $\ker P = \operatorname{ran} Q$. Then show that the following assertions are equivalent:

(i) $\operatorname{ran} P \perp \ker P$.

(ii) $\langle Pf, g\rangle = \langle f, Pg\rangle$ for all $f, g \in H$.

(iii) $\|Pf\| \leq \|f\|$ for all $f \in H$.

(iv) $F := \operatorname{ran} P$ is closed and $P = P_F$.

(Hint for the implication (iii) $\Rightarrow$ (i): if (iii) holds then $\|P(f + cg)\|^2 \leq \|f + cg\|^2$ for all $c \in \mathbb{K}$ and $f, g \in H$; fix $f \in \operatorname{ran} P$, $g \in \ker P$, use Lemma 1.5 and vary c to conclude that $\langle f, g\rangle = 0$.)

Exercise 7.5. For this exercise we assume the results mentioned in Exercise 6.21. Let $E := \mathrm{L}^1(-1,1)$ and consider the mapping

$$(Tf)(t) := f(-t) \quad f \in \mathrm{L}^1(-1,1).$$

A function $f \in \mathrm{L}^1(-1,1)$ is called *even* if $f = Tf$ almost everywhere.

a) Show that if $f \in \mathrm{L}^1(-1,1)$ then $Tf \in \mathrm{L}^1(-1,1)$ as well and $\int_{-1}^1 Tf \,\mathrm{d}\lambda = \int_{-1}^1 f \,\mathrm{d}\lambda$. (Hint: Show first that it is true for $f \in C[-1,1]$. Then use Corollary 6.20.)

b) Show that $Tf \in \mathrm{L}^2(-1,1)$ and $\|Tf\|_2 = \|f\|_2$ for all $f \in \mathrm{L}^2(-1,1)$. (Hint: use a))

c) Let $H = \mathrm{L}^2(-1,1)$. Show that the space

$$F := \{f \in \mathrm{L}^2(-1,1) \mid f \text{ is even}\}$$

is a closed linear subspace of H and show that $P_F = 1/2(\mathrm{I} + T)$ is the orthogonal projection onto F.

d) Describe $F^\perp$ and the orthogonal projection onto $F^\perp$.

Exercise 7.6. Let H be an inner product space with an ONS $(e_n)_n$ and vectors $f, g \in H$. Show that the series

$$\sum_{j=1}^{\infty} \langle f, e_j \rangle \overline{\langle g, e_j \rangle}$$

converges *absolutely*. Then prove the equivalence of (iii) and (iv) in Theorem 7.13. (Hint: see Exercise 1.8.)

Exercise 7.7. Let $(e_j)_{j\in\mathbb{N}}$ an ONS in a Hilbert space H. Show that for every $\alpha \in \ell^2$ there is a (unique) $f \in H$ such that $\langle f, e_j \rangle = \alpha_j$ for all $j \in \mathbb{N}$.

Exercise 7.8. Prove that $\mathrm{C}_{\mathrm{per}}[a,b]$ is $\|\cdot\|_2$-dense in $\mathrm{C}[a,b]$.

Further Exercises

Exercise 7.9. Let (Ω, d) be a metric space and let $A \subseteq \Omega$ be any subset. Show that

$$|d(x,A) - d(y,A)| \le d(x,y)$$

for all $x, y \in \Omega$, and conclude that $d(\cdot, A)$ is a continuous function on Ω.

Exercise 7.10 (Minimal Norm Problems)**.** Let F be a closed subspace of a Hilbert space H and $f_0 \in H$. Show that there is a unique element of minimal norm. in the affine subspace $G := f_0 + F$.

Then, in each of the cases

a) $H = \mathrm{L}^2(0,1)$, $G := \{f \in \mathrm{L}^2(0,1) \mid \int_0^1 t f(t)\,\mathrm{d}t = 1\}$,

b) $H = \mathrm{L}^2(0,1)$, $G := \{f \in \mathrm{L}^2(0,1) \mid \int_0^1 f(t)\,\mathrm{d}t = 1/3, \quad \int_0^1 t^2 f(t)\,\mathrm{d}t = 1/15\}$

find F, f_0 such that $G = F + f_0$, and determine an element of minimal norm in G.

Exercise 7.11. Let $\alpha = (\alpha_j)_{j\in\mathbb{N}}$ be a scalar sequence. Suppose that the series

$$\sum_{j=1}^{\infty} \lambda_j \alpha_j$$

converges for every $\lambda \in \ell^2$ and that there is a constant $c \ge 0$ such that

$$\Big| \sum_{j=1}^{\infty} \lambda_j \alpha_j \Big| \le c$$

for all $\lambda \in \ell^2$ with $\|\lambda\|_2 \le 1$. Show that $\alpha \in \ell^2$ and $\|\alpha\|_2 \le c$.

Exercise 7.12. Let H be a separable Hilbert space and let $F \subseteq H$ be a closed subspace. Show that F is also separable. (Hint: use the orthogonal projection P_F.)

Exercise 7.13. Let $(e_j)_{j\in\mathbb{Z}}$ be an ONS in a Hilbert space H. For $f \in H$ define

$$f_n := \sum\nolimits_{j=-n}^{\infty} \langle f, e_j \rangle\, e_j.$$

This series converges, by Theorem 7.13. Show that $Qf := \lim_{n\to\infty} f_n$ exists in H. Prove that $Qf = P_G f$, where

$$G = \overline{\operatorname{span}\{e_j \mid j \in \mathbb{Z}\}}.$$

Exercise 7.14. Let $E := \{f \in \mathrm{C}[0,1] \mid f(0) = 0\}$ with the ∞-norm. Let

$$A := \{f \in E \mid \int_0^1 f(t)\,\mathrm{d}t = 0\},$$

and let $f(t) := t$, $t \in [0,1]$.

a) Show that E is a closed subspace of $\mathrm{C}[0,1]$ and that A is a closed subspace of E.

b) For given $\epsilon > 0$, find $g \in A$ such that $\|g - f\|_\infty \le 1/2 + \epsilon$. (Hint: modify the function $f - 1/2$ appropriately.)

c) Conclude from this and Example 7.2 that $d(f, A) = 1/2$.

d) Show that for each $g \in A$ one must have $\|f - g\|_\infty > 1/2$.

Exercise 7.15. Show that a Banach space E is separable if there is a sequence $(f_n)_{n\in\mathbb{N}} \subseteq E$ such that $\{f_n \mid n \in \mathbb{N}\}$ is dense in E. (This weakens Lemma 7.19.) (Hint: Use that $\mathbb{Q}$ is countable and dense in $\mathbb{R}$.)

Chapter 8

Sobolev Spaces and the Poisson Problem

In this chapter we shall see a first application of Hilbert space methods to differential equations. More precisely, we shall present a first approach to *Poisson's problem*

$$u'' = -f \quad \text{on} \quad (a, b), \qquad u(a) = u(b) = 0.$$

Classically, f is a continuous function on $[a, b]$, and one can easily solve the equation by successive integration. This approach leads to the concept of *Green's function* and is treated in Chapter 9.

In the present chapter we take a different way, namely we shall apply the Riesz-Fréchet theorem to obtain existence and uniqueness of solutions. To be able to do this, we have to pass from the "classical" $\mathrm{C}[a, b]$-setting to the $\mathrm{L}^2(a, b)$-setting; in particular, we need the notion of a derivative of an $\mathrm{L}^2(a, b)$-"function". Since L^2-elements are not really functions, we cannot do this by using the elementary definition via differential quotients.

8.1. Weak Derivatives

Let $[a, b] \subseteq \mathbb{R}$ be a finite interval. Each function $\psi \in \mathrm{C}^1[a, b]$ such that $\psi(a) = \psi(b) = 0$ is called a **test function**. The space of test functions is denoted by

$$\mathrm{C}_0^1[a, b] = \{\psi \in \mathrm{C}^1[a, b] \mid \psi(a) = \psi(b) = 0\}.$$

Note that if ψ is a test function, then so is $\overline{\psi}$ and $(\overline{\psi})' = \overline{\psi'}$.

Lemma 8.1. *The space of test functions* $C_0^1[a,b]$ *is dense in* $L^2(a,b)$*. If* $g, h \in L^2(a,b)$ *are such that*

$$\int_a^b g(x)\psi(x)\,dx = \int_a^b h(x)\psi(x)\,dx,$$

for all test functions $\psi \in C_0^1[a,b]$*, then* $g = h$ *(almost everywhere).*

Proof. It was shown in Corollary 4.12 that $D := C_0^1[a,b]$ is dense in $C_0[a,b]$ in the sup-norm. By Example 3.13 convergence in sup-norm implies convergence in 2-norm, and so D is 2-norm dense in $C_0[a,b]$ (Theorem DP.4). This space is 2-norm dense in $C[a,b]$, by Exercise 3.10, and $C[a,b]$ is 2-norm dense in $L^2[a,b]$. By Theorem DP.1, D is dense in $L^2(a,b)$, as claimed. If g, h are as in the lemma, then $g - h \perp D$. Hence $g - h \perp \overline{D} = L^2(a,b)$, and so $g - h = 0$. □

Advice/Comment:
At this point it is strongly recommended to look at the Density Principles DP.1–DP.4 and their proofs.

Given a function $f \in L^2(a,b)$ we want to define its derivative function f' without using a differential quotient (since this is the classical approach) but still in such a way that for $f \in C^1[a,b]$ the symbol f' retains its classical meaning. Now, *if* $f \in C^1[a,b]$ then the integration by parts formula gives

$$\int_a^b f'(x)\psi(x)\,dx = -\int_a^b f(x)\psi'(x)\,dx \tag{8.1}$$

for all test functions $\psi \in C_0^1[a,b]$. (The condition $\psi(a) = 0 = \psi(b)$ makes the boundary terms vanish.) This is the key observation behind the following definition.

Definition 8.2. Let $f \in L^2(a,b)$. A function $g \in L^2(a,b)$ is called a **weak derivative** of f if

$$\int_a^b g(x)\psi(x)\,dx = -\int_a^b f(x)\psi'(x)\,dx \tag{8.2}$$

for all functions $\psi \in C_0^1[a,b]$. If f has a weak derivative, we call f **weakly differentiable**. The space

$$H^1(a,b) := \{f \in L^2(a,b) \mid f \text{ has a weak derivative}\}$$

and is called the **(first) Sobolev space**.

Note that the condition for a weak derivative can be equivalently written as

$$\langle g, \psi \rangle = -\langle f, \psi' \rangle \qquad (\psi \in \mathrm{C}_0^1[a,b]).$$

By what we said just before the definition, if $f \in \mathrm{C}^1[a,b]$ then its classical derivative f' is also its weak derivative. We shall see below that *not every* L^2*-function has a weak derivative.* However, if f has a weak derivative, then it can have only one: indeed if $g, h \in \mathrm{L}^2(a,b)$ are weak derivatives then they satisfy (8.2) for every test function ψ, and hence $g = h$ by Lemma 8.1. It is therefore unambiguous to write f' for a (the) weak derivative of f, *provided* such a weak derivative exists in the first place. Ex.8.1 Ex.8.2

The next example shows that there exist weakly differentiable functions that are not (classically) differentiable.

Example 8.3. Let $[a,b] = [-1,1]$ and $f(t) := |t|$, $t \in [-1,1]$. Then f has weak derivative $g := \mathbf{1}_{(0,1)} - \mathbf{1}_{(-1,0)}$; indeed,

$$\begin{aligned} -\int_{-1}^{1} f(t)\psi'(t)\,\mathrm{d}t &= -\int_{-1}^{0} (-t)\psi'(t)\,\mathrm{d}t - \int_{0}^{1} t\psi'(t)\,\mathrm{d}t \\ &= t\psi(t)\Big|_{-1}^{0} - \int_{-1}^{0} \psi(t)\,\mathrm{d}t - t\psi(t)\Big|_{0}^{1} + \int_{0}^{1} \psi(t)\,\mathrm{d}t \\ &= 0 - \int_{-1}^{0} \psi(t)\,\mathrm{d}t - 0 + \int_{0}^{1} \psi(t)\,\mathrm{d}t = \int_{-1}^{1} g(t)\psi(t)\,\mathrm{d}t. \end{aligned}$$

for all $\psi \in \mathrm{C}_0^1[-1,1]$. See also Exercise 8.3. Ex.8.3

8.2. The Fundamental Theorem of Calculus

Before we can work with weak derivatives, we have to establish their basic properties. And of course it is interesting to see to which extent classical results involving derivatives extend to the weak setting. To facilitate computations with weak derivatives we shall employ the **integration operator** J defined by

$$(Jf)(t) := \int_a^t f(x)\,\mathrm{d}x = \langle f, \mathbf{1}_{(a,t)} \rangle \qquad (t \in [a,b], f \in \mathrm{L}^2(a,b)),$$

cf. Example 6.15.

Advice/Comment:
Note that in general this is *not* a Riemann- integral, even if it looks like one.

Classically, J is a *right inverse* to differentiation:

$$(Jf)'(t) = \frac{d}{dt}\int_a^t f(s)\,\mathrm{d}s = f(t) \qquad (t \in [a,b]) \tag{8.3}$$

for $f \in \mathrm{C}[a,b]$. The next result shows that this is still true for weak differentiation.

Lemma 8.4. *The operator $J : \mathrm{L}^2(a,b) \longrightarrow \mathrm{C}[a,b]$ is linear and bounded. Moreover, $(Jf)' = f$ in the weak sense for all $f \in \mathrm{L}^2(a,b)$.*

Proof. If $f \in \mathrm{L}^2(a,b)$, then $f \in \mathrm{L}^1(a,b)$ and

$$\|f\|_2 \le \sqrt{b-a}\,\|f\|_1$$

by Exercise 6.10 (cf. also Example 3.13). By Example 6.15, we have $Jf \in$
Ex.8.4 $\mathrm{C}[a,b]$ and hence

$$\|Jf\|_\infty \le \|f\|_1 \le \sqrt{b-a}\,\|f\|_2 .$$

This is the boundedness of J as an operator from $\mathrm{L}^2(a,b)$ (with the 2-norm) to $\mathrm{C}[a,b]$ (with the sup-norm). The claim $(Jf)' = f$ is, by definition, equivalent to

$$\langle Jf, \psi' \rangle = -\langle f, \psi \rangle \tag{8.4}$$

for all test functions ψ. To establish (8.4) we employ a density argument. Fix a test function ψ and consider the linear mappings

$$T : f \longmapsto \langle Jf, \psi' \rangle, \quad S : f \longmapsto -\langle f, \psi \rangle$$

from $\mathrm{L}^2(a,b)$ to the scalar field. An elementary estimation (do it!) yields that both S, T are bounded. Moreover, $Tf = Sf$ for all $f \in \mathrm{C}^1[a,b]$, by classical integration by parts. Since $\mathrm{C}^1[a,b]$ is dense in $\mathrm{L}^2(a,b)$ (by, e.g., Lemma 8.1) Theorem DP.2 yields that $S = T$, and thus (8.4) holds for all $f \in \mathrm{L}^2(a,b)$. As $\psi \in \mathrm{C}^1_0[a,b]$ was arbitrary, the proof is complete. □

Classically, i.e. for $f \in \mathrm{C}^1[a,b]$ one has that $J(f') - f$ is a constant function. This comes from the fundamental theorem of calculus, and implies that if the (classical) derivative of a function is zero, then the function is constant. Let us see whether we can recover these results in the setting of weak derivatives.

We write $\mathbf{1}$ for the function being constant to one on $[a,b]$. Then a constant function has the form $f = c\mathbf{1}$ for some number $c \in \mathbb{K}$, and hence $\mathbb{C}\mathbf{1}$ is the one-dimensional subspace of constant functions. A spanning unit vector for this space is $e := (b-a)^{-1/2}\mathbf{1}$, and so

$$P_{\mathbb{C}\mathbf{1}} f = \langle f, e \rangle\, e = \frac{\langle f, \mathbf{1} \rangle}{b-a}\mathbf{1} \qquad (f \in \mathrm{L}^2(a,b))$$

is the orthogonal projection onto $\mathbb{C}\mathbf{1}$. In particular, if a function f is constant to c (almost everywhere) then this constant is given by

$$c = \frac{\langle f, \mathbf{1}\rangle}{b-a}.$$

We are interested in the orthogonal complement $\mathbf{1}^\perp$ of $\mathbb{C}\mathbf{1}$.

Lemma 8.5. *The space* $\mathrm{L}^2(a,b)$ *decomposes orthogonally into*

$$\mathrm{L}^2(a,b) = \mathbb{C}\mathbf{1} \oplus \overline{\{\psi' \mid \psi \in \mathrm{C}_0[a,b]\}},$$

the closure on the right-hand side being taken with respect to the 2*-norm.*

Proof. We let $G := \mathbf{1}^\perp$ and $F := \{\psi' \mid \psi \in \mathrm{C}_0^1[a,b]\}$. Then it is to show that $G = \overline{F}$. Now $\mathbf{1} \perp F$, since

$$\int_a^b \mathbf{1} \cdot \psi'(s)\,\mathrm{d}s = \psi(b) - \psi(a) = 0 - 0 = 0$$

for each $\psi \in \mathrm{C}_0^1[a,b]$. This yields $F \subseteq \mathbf{1}^\perp = G$, and hence $\overline{F} \subseteq G$, because G is closed.

For the other inclusion, note that the orthogonal projection onto G is $P_G = \mathrm{I} - P_{\mathbb{C}\mathbf{1}}$, and hence

$$P_G f = f - \frac{\langle f, \mathbf{1}\rangle}{b-a}\mathbf{1} \qquad (f \in \mathrm{L}^2(a,b)).$$

Now, P_G has $\operatorname{ran}(P_G) = G$ and so we have to show that $\operatorname{ran}(P_G) \subseteq \overline{F}$. To this aim, fix $f \in \mathrm{C}[a,b]$. Then $P_G f$ is obviously continuous as well, hence

$$JP_G f = Jf - \frac{\langle f, \mathbf{1}\rangle}{b-a} J\mathbf{1}$$

is in $\mathrm{C}^1[a,b]$. Moreover, $JP_G f(a) = 0$ (by definition of J) and also

$$\begin{aligned}(JP_G f)(b) &= (Jf)(b) - \frac{\langle f, \mathbf{1}\rangle}{b-a}(J\mathbf{1})(b) = \int_a^b f(s)\,\mathrm{d}s - \frac{\langle f, \mathbf{1}\rangle}{b-a}\int_a^b 1\,\mathrm{d}s \\ &= \langle f, \mathbf{1}\rangle - \langle f, \mathbf{1}\rangle = 0.\end{aligned}$$

But this means that $JP_G f$ is a test function, so $P_G f = (JP_G f)' \in F$.

For a general $f \in \mathrm{L}^2(a,b)$ we find a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{C}[a,b]$ with $f_n \to f$ in $\mathrm{L}^2(a,b)$. Then, since orthogonal projections are bounded operators, $P_G f_n \to P_G f$, which shows that $P_G f \in \overline{F}$. (This is an instance of Theorem DP.3.) □

Corollary 8.6. *Let* $f \in \mathrm{L}^2(a,b)$ *such that* f *has weak derivative* 0*. Then* f *is a constant (a.e.).*

Proof. Let $f \in \mathrm{L}^2(a,b)$ and suppose that $f' = 0$ in the weak sense. By definition, this means

$$\int_a^b f(s)\psi'(s)\,\mathrm{d}s = 0 \qquad (\psi \in \mathrm{C}_0^1[a,b]),$$

and this can be rephrased as $f \perp F$, with F being as in the previous proof. Hence $f \perp \overline{F} = (\mathbb{C}\mathbf{1})^\perp$, and thus $f \in \mathbb{C}\mathbf{1}$, by Corollary 7.11. □

Let us formulate an interesting corollary.

Corollary 8.7. *One has* $\mathrm{H}^1(a,b) \subseteq \mathrm{C}[a,b]$. *More precisely,* $f \in \mathrm{H}^1(a,b)$ *if and only if* f *has a representation*

$$f = Jg + c\mathbf{1}$$

with $g \in \mathrm{L}^2(a,b)$ *and* $c \in \mathbb{K}$. *Such a representation is unique, namely*

$$g = f', \quad c = \langle f - Jf', \mathbf{1}\rangle / (b-a).$$

Moreover, the **Fundamental Theorem of Calculus**

$$\int_c^d f'(x)\,\mathrm{d}x = f(d) - f(c)$$

holds for every interval $[c,d] \subseteq [a,b]$.

Proof. If $f = Jg + c\mathbf{1}$ is represented as described, then $f' = (Jg)' + 0 = g$ and $f \in \mathrm{H}^1(a,b)$. Conversely let $f \in \mathrm{H}^1(a,b)$ and set $g := f'$ and $h := Jg$. Then $(f-h)' = f' - (Jg)' = f' - g = 0$, and by Corollary 8.6 there exists $c \in \mathbb{K}$ such that $f - Jg = c\mathbf{1}$. The remaining statement follows from

$$\int_c^d f'(x)\,\mathrm{d}x = (Jg)(d) - (Jg)(c) = f(d) - f(c).$$

□

In particular, a function which does not coincide almost everywhere with a continuous function, cannot have a weak derivative. For example, within $(-1,1)$ the caracteristic function $\mathbf{1}_{(0,1)}$ does not have a weak derivative, see Exercise 8.5.

8.3. Sobolev Spaces

On $\mathrm{H}^1(a,b)$ we define the inner product(!)

$$\langle f, g\rangle_{\mathrm{H}^1} := \langle f, g\rangle_{\mathrm{L}^2} + \langle f', g'\rangle_{\mathrm{L}^2} = \int_a^b f(x)\overline{g(x)}\,\mathrm{d}x + \int_a^b f'(x)\overline{g'(x)}\,\mathrm{d}x$$

for $f, g \in \mathrm{H}^1(a,b)$.

> **Advice/Comment:**
> If one writes out the norm of $f \in \mathrm{H}^1(a,b)$ one obtains:
> $$\|f\|_{\mathrm{H}^1} = \Big(\|f\|_2^2 + \|f'\|_2^2\Big)^{1/2}.$$
> This shows that convergence of a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{H}^1(a,b)$ is the same as the L^2-convergence of $(f_n)_{n\in\mathbb{N}}$ and of $(f_n')_{n\in\mathbb{N}}$.

Theorem 8.8. *The first Sobolev space* $\mathrm{H}^1(a,b)$ *is a Hilbert space, and the mapping*
$$\mathrm{H}^1(a,b) \longrightarrow \mathrm{L}^2(a,b), \qquad f \longmapsto f'$$
is linear and bounded. Moreover, the inclusion $\mathrm{H}^1(a,b) \subseteq \mathrm{C}[a,b]$ *is continuous, i.e., there is a constant* $c > 0$ *such that*
$$\|f\|_\infty \le c\,\|f\|_{\mathrm{H}^1} \qquad (f \in \mathrm{H}^1(a,b)).$$

Proof. By Exercise 8.2, $\mathrm{H}^1(a,b)$ is a vector space, and the derivative is a linear mapping. By definition of the norm
$$\left\|f'\right\|_2^2 \le \|f\|_2^2 + \left\|f'\right\|_2^2 = \|f\|_{\mathrm{H}^1}^2,$$
and so $(f \longmapsto f')$ is a bounded mapping.

Suppose that $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{H}^1(a,b)$ is a Cauchy sequence, i.e. $\|f_n - f_m\|_{\mathrm{H}^1} \to 0$ as $n,m\to\infty$. This means that both sequences $(f_n)_{n\in\mathbb{N}}$ and $(f_n')_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathrm{L}^2(a,b)$. By the completeness of $\mathrm{L}^2(a,b)$, there are functions $f, g \in \mathrm{L}^2(a,b)$ such that
$$\|f_n - f\|_2 \to 0, \quad \text{and} \quad \left\|f_n' - g\right\|_2 \to 0$$
as $n \to \infty$. It suffices to show that g is a weak derivative of f. In order to establish this, let $\psi \in \mathrm{C}_0^1[a,b]$. Then
$$\langle g, \psi\rangle_{\mathrm{L}^2} = \lim_n \left\langle f_n', \psi\right\rangle_{\mathrm{L}^2} = \lim_n -\left\langle f_n, \psi'\right\rangle = -\langle f, \psi\rangle,$$
where we used that f_n' is a weak derivative of f_n. The remaining statement is an exercise. □ Ex.8.6

Corollary 8.9. *The space* $\mathrm{C}^1[a,b]$ *is dense in* $\mathrm{H}^1(a,b)$ *(with respect to its proper norm).*

Proof. Let $f \in \mathrm{H}^1(a,b)$ and $g := f'$. Then $f = Jg + c\mathbf{1}$ for some $c \in \mathbb{K}$, by Corollary 8.7. Since $\mathrm{C}[a,b]$ is dense in $\mathrm{L}^2(a,b)$, there is a sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathrm{C}[a,b]$ such that $\|g_n - g\|_2 \to 0$. By Exercise 8.8, $J : \mathrm{L}^2(a,b) \longrightarrow \mathrm{H}^1(a,b)$ is bounded, and hence $Jg_n \to Jg$ in $\mathrm{H}^1(a,b)$. But then $f_n := Jg_n + c\mathbf{1} \to$

$Jg + c\mathbf{1} = f$ in $\mathrm{H}^1(a,b)$ as well. Evidently $f_n \in \mathrm{C}^1[a,b]$ and the proof is complete. □

When one has digested the definition of H^1 and the fact that some classically non-differentiable functions are weakly differentiable, it is then only a small step towards higher (weak) derivatives. One defines recursively

$$\mathrm{H}^{n+1}(a,b) := \{f \in \mathrm{H}^1(a,b) \mid f' \in \mathrm{H}^n(a,b)\} \qquad (n \in \mathbb{N})$$

with norm

$$\|f\|_{\mathrm{H}^n}^2 := \|f\|_2^2 + \|f'\|_2^2 + \cdots + \|f^{(n)}\|_2^2 \qquad (f \in \mathrm{H}^n(a,b)).$$

Ex.8.7 It can be shown by induction on n that $\mathrm{H}^n(a,b)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_{\mathrm{H}^1} := \sum\nolimits_{k=0}^{n} \left\langle f^{(k)}, g^{(k)}\right\rangle_{\mathrm{L}^2} \qquad (f, g \in \mathrm{H}^n(a,b))$$

The space $\mathrm{H}^n(a,b)$ are called (higher) **Sobolev spaces**.

8.4. The Poisson Problem

We shall give a typical application in the field of (partial) differential equations. (Actually, no *partial* derivatives here, but that's a contingent fact, because we are working in dimension one.) For simplicity we work with $\mathbb{K} = \mathbb{R}$ here. Consider again for given $f \in \mathrm{L}^2(a,b)$ the boundary value problem ("Poisson problem")

$$u'' = -f, \qquad u(a) = u(b) = 0$$

to be solved within $\mathrm{H}^2(0,1)$. We shall see in a later chapter how to do this with the help of an integral operator; here we present the so-called *variational method*. Since the differential equation above is to be understood in the weak sense, we can equivalently write

$$\left\langle u', \psi'\right\rangle_{\mathrm{L}^2} = \langle f, \psi\rangle_{\mathrm{L}^2} \qquad (\psi \in \mathrm{C}_0^1[a,b]).$$

The idea is now readily sketched. Define

$$\mathrm{H}_0^1(a,b) := \{u \in \mathrm{H}^1(a,b) \mid u(a) = u(b) = 0\}.$$

This is a closed subspace of $\mathrm{H}^1(a,b)$ since $\mathrm{H}^1(a,b) \subseteq \mathrm{C}[a,b]$ continuously (Theorem 8.8) and the point evaluations are bounded on $\mathrm{C}[a,b]$. Thus $H := \mathrm{H}_0^1(a,b)$ is a Hilbert space with respect to the (induced) scalar product

$$\langle f, g\rangle_H := \left\langle f', g'\right\rangle_{\mathrm{L}^2} + \langle f, g\rangle_{\mathrm{L}^2} = \int_a^b f'(x)g'(x)\,\mathrm{d}x + \int_a^b f(x)g(x)\,\mathrm{d}x.$$

It is an important fact that we may leave out the second summand here.

Lemma 8.10 (Poincaré inequality). *There is a constant $C \geq 0$ depending on $b-a$ such that*

$$\|u\|_{L^2} \leq C \left\|u'\right\|_{L^2}$$

for all $u \in H_0^1(a,b)$.

Note that a similar estimate cannot hold for $H^1(a,b)$, since for a constant function f one has $f' = 0$. It is not at all easy to determine the *optimal* constant c, see Chapter 11.

Proof. Let $u \in H_0^1(a,b)$. We claim that $u = Ju'$. Indeed, if $(Ju')' = u'$ and by Corollary 8.6 $Ju' - u = c$ is a constant. But $Ju' - u$ vanishes at a and so the constant is zero. Using $Ju' = u$ we finally obtain

$$\|u\|_{L^2} = \left\|Ju'\right\|_{L^2} \leq C \left\|u'\right\|_{L^2}.$$

for some constant C, since by Lemma 8.4 the integration operator $J : L^2(a,b) \longrightarrow L^2(a,b)$ is bounded. □

As a consequence, we see that $\langle u,v\rangle_H := \langle u',v'\rangle_{L^2}$ is an inner product on H, and the norm induced by it is equivalent to the orginal norm. Indeed, for $u \in H = H_0^1(a,b)$

$$\|u\|_H^2 = \left\|u'\right\|_{L^2}^2 \leq \|u\|_{L^2}^2 + \left\|u'\right\|_{L^2}^2 = \|u\|_{H^1}^2$$

and

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \left\|u'\right\|_{L^2}^2 \leq (C^2+1) \left\|u'\right\|_{L^2}^2 = (C^2+1) \|u\|_H^2$$

by Poincaré's inequality.

In particular, $(H, \langle\cdot,\cdot\rangle_H)$ is a Hilbert space. The Poincaré inequality shows also that the inclusion mapping

$$H = H_0^1(a,b) \longrightarrow L^2(a,b), \qquad v \longmapsto v$$

is continuous. Hence the linear functional

$$\varphi : H \longrightarrow \mathbb{K}, \qquad v \longmapsto \langle v,f\rangle_{L^2}$$

is bounded. By the Riesz–Fréchet theorem 7.12 there exists a unique $u \in H$ such that

$$\langle v',u'\rangle_{L^2} = \langle v,u\rangle_H = \varphi(v) = \langle v,f\rangle_{L^2}$$

for all $v \in H_0^1(a,b)$. In particular, this is true for all $v \in C_0^1[a,b]$, whence u is a solution of our original problem. Ex.8.9

8.5. Further Reading: The Dirichlet Principle

Our treatment of the Poisson problem is just a one-dimensional version of the so-called **Dirichlet principle** in arbitrary dimensions. We shall only give a rough sketch without any proofs. One starts with a *bounded, open* set $\Omega \subseteq \mathbb{R}^d$. On Ω one considers the d-dimensional Lebesgue measure and the Hilbert space $\mathrm{L}^2(\Omega)$. Then one looks at the *Poisson problem*

$$\Delta u = -f, \quad u\big|_{\partial\Omega} = 0.$$

Here, Δ is the *Laplace operator* defined by

$$\Delta u = \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2}.$$

In the *classical* case, $f \in \mathrm{C}(\Omega)$, and one wants a solution $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}(\overline{\Omega})$ satisfying literally the PDE above. The functional analytic way to treat this is a two-step procedure: first find a solution within $\mathrm{L}^2(\Omega)$, where the derivatives are interpreted in a "weak" manner, then try to find conditions on f such that this solution is a classical solution. The second step belongs properly to the realm of PDE, but the first step can be done by our abstract functional analysis methods.

As the space of *test functions* one takes

$$\mathrm{C}_0^1(\Omega) := \{\psi \in \mathrm{C}^1(\overline{\Omega}) \mid \psi\big|_{\partial\Omega} = 0\}.$$

A *weak gradient* of a function $f \in \mathrm{L}^2(\Omega)$ is a d-tuple $g = (g_1, \dots, g_d) \in \mathrm{L}^2(\Omega)^d$ such that

$$\int_\Omega f(x) \frac{\partial \psi}{\partial x_j}(x)\,\mathrm{d}x = -\int_\Omega g_j(x)\psi(x)\,\mathrm{d}x \qquad (j = 1, \dots, d)$$

for all $\psi \in C_0^1(\Omega)$. One proves that such a weak gradient is unique, and writes $\nabla f = g$ and $\partial f/\partial x_j := g_j$. One defines

$$\mathrm{H}^1(\Omega) := \{u \in \mathrm{L}^2(\Omega) \mid u \text{ has a weak gradient}\}$$

which is a Hilbert space for the scalar product

$$\begin{aligned}\langle u, v\rangle_{\mathrm{H}^1} &= \langle u, v\rangle_{\mathrm{L}^2} + \int_\Omega \langle \nabla u(x), \nabla v(x)\rangle_{\mathbb{R}^d}\,\mathrm{d}x \\ &= \int_\Omega u(x)v(x)\,\mathrm{d}x + \sum_{j=1}^{d} \int_\Omega \frac{\partial u}{\partial x_j}(x) \frac{\partial v}{\partial x_j}(x)\,\mathrm{d}x.\end{aligned}$$

The boundary condition is incorporated into a closed subspace $\mathrm{H}_0^1(\Omega)$ of $\mathrm{H}^1(\Omega)$:

$$\mathrm{H}_0^1(\Omega) = \overline{\mathrm{C}_0^1(\Omega)}$$

(closure within $\mathrm{H}^1(\Omega)$.) One then shows Poincaré's inequality:

$$\int_\Omega |u|^2 \,\mathrm{d}x \le c \int_\Omega |\nabla u|^2 \,\mathrm{d}x \qquad (u \in \mathrm{H}^1_0(\Omega))$$

for some constant c depending on Ω. In the end, Riesz–Fréchet is applied to obtain a unique $u \in \mathrm{H}^1_0(\Omega)$ such that

$$\int_\Omega \langle \nabla u(x), \nabla \psi(x) \rangle \,\mathrm{d}x = -\int_\Omega f(x)\psi(x)\,\mathrm{d}x \qquad (\psi \in \mathrm{C}^1_0(\Omega)),$$

that is $\Delta u = -f$ in the weak sense. (Of course one would like to have $u \in \mathrm{H}^2(\Omega)$, but this will be true only if Ω is sufficiently regular, e.g., if $\partial\Omega$ is smooth.)

Exercises

Exercise 8.1. Let $f \in \mathrm{H}^1(a,b)$ with weak derivative $g \in \mathrm{L}^2(a,b)$. Show that $\overline{g}$ is a weak derivative of $\overline{f}$.

Exercise 8.2. Show that $\mathrm{H}^1(a,b)$ is a vector space and that

$$\mathrm{H}^1(a,b) \longrightarrow \mathrm{L}^2(a,b), \qquad f \longmapsto f'$$

is a linear mapping.

Exercise 8.3. Consider $[a,b] = [-1,1]$ and $f(t) = |t|^q$ for some $1/2 < q$. Show that $f \in \mathrm{H}^1(-1,1)$ and compute its weak derivative. Is $q = 1/2$ also possible?

Exercise 8.4. Show that for $f \in \mathrm{L}^2(a,b)$ one has

$$|Jf(t) - J(s)| \le \sqrt{t-s}\,\|f\|_2 \qquad (s,t \in [a,b]).$$

I.e., Jf is *Hölder* continuous with exponent $1/2$. (This improves on the mere continuity of Jf established in Example 6.15.)

Exercise 8.5. Show that there is no $f \in \mathrm{C}[-1,1]$ such that $f = \mathbf{1}_{(0,1)}$ almost everywhere. (This is somehow similar to the proof of Theorem 5.8.) Conclude that $\mathbf{1}_{(0,1)}$ does not have a weak derivative in $\mathrm{L}^2(-1,1)$.

Exercise 8.6. Find a constant $c \ge 0$ such that

$$\|f\|_\infty \le c\,\|f\|_{\mathrm{H}^1} \qquad (f \in \mathrm{H}^1(a,b)).$$

(Hint: Corollary 8.7).

Exercise 8.7. Show that, for every $n \in \mathbb{N}$, $\mathrm{H}^n(a,b)$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle_{\mathrm{H}^n} := \sum\nolimits_{k=0}^n \left\langle f^{(k)}, g^{(k)} \right\rangle_{\mathrm{L}^2} \qquad (f,g \in \mathrm{H}^n(a,b)).$$

Exercise 8.8. Consider the integration operator J on the interval $[a, b]$. In the proof of Lemma 8.4 it was shown that

$$\|Jf\|_\infty \leq \sqrt{b-a}\,\|f\|_2 \qquad (f \in \mathrm{L}^2(a,b)).$$

Show that

$$\|Jf\|_2 \leq \frac{b-a}{\sqrt{2}}\,\|f\|_2 \qquad (f \in \mathrm{L}^2(a,b)).$$

Then determine $c \geq 0$ such that

$$\|Jf\|_{\mathrm{H}^1} \leq c\,\|f\|_2 \qquad (f \in \mathrm{L}^2(a,b)).$$

Exercise 8.9. Show that $\mathrm{C}_0^1[a,b]$ is dense in $\mathrm{H}_0^1(a,b)$. Conclude that the solution in $\mathrm{H}_0^1(a,b)$ of Poisson's equation is *unique*.

Further Exercises

Exercise 8.10. Show that on $\mathrm{H}^2(a,b)$,

$$|\!|\!| f |\!|\!| := \left(\|f\|_2 + \left\|f''\right\|_2\right)^{1/2}$$

is an equivalent Hilbert space norm, i.e., is a norm which comes from an inner product, and this norm is equivalent to the norm given in the main text.

Exercise 8.11 (Product Rule). For $f, g \in \mathrm{C}^1[a,b]$ we know that

$$(fg)' = f'g + fg'. \tag{8.5}$$

The aim of this exercise is to show that this is still true in the weak sense, for $f, g \in \mathrm{H}^1(a,b)$. It will be done in two steps.

a) For fixed $g \in \mathrm{H}^1(a,b)$ define

$$T_g f := fg \quad \text{and} \quad S_g f := f'g + fg' \qquad (f \in \mathrm{H}^1(a,b)).$$

Show that $T, S : \mathrm{H}^1(a,b) \longrightarrow \mathrm{L}^2(a,b)$ are well-defined bounded oprators.

b) Let $g \in \mathrm{H}^1(a,b)$ and $f \in \mathrm{C}^1[a,b]$. Show that $f'g + fg'$ is a weak derivative of fg. (Hint: if $\psi \in \mathrm{C}^1[a,b]$ then $\psi g \in \mathrm{C}_0^1[a,b]$ as well.)

c) Fix $\psi \in \mathrm{C}_0^1[a,b]$ and $g \in \mathrm{H}^1(a,b)$, and consider

$$q(f) := \langle fg, \psi' \rangle + \langle f'g + gf', \psi \rangle = \langle T_g f, \psi' \rangle + \langle S_g f, \psi \rangle$$

for $f \in \mathrm{H}^1(a,b)$). Show that $q(f) = 0$ for all $f \in \mathrm{H}^1(a,b)$. (Hint: use b) and Exercise 8.9.)

d) Conclude from c) that for all $f, g \in \mathrm{H}^1(a,b)$ one has $fg \in \mathrm{H}^1(a,b)$ and $(fg)' = f'g + fg'$ and the **integration by parts** formula

$$\int_a^b f(s)g'(s)\,\mathrm{d}s = f(b)g(b) - f(a)g(a) - \int_a^b f'(s)g(s)\,\mathrm{d}s.$$

Exercise 8.12 (The general Poincaré inequality). a) Determine a constant c such that

$$\|Jf - \langle Jf, \mathbf{1}\rangle_{\mathrm{L}^2} \mathbf{1}\|_2 \le c \, \|f\|_2$$

for all $f \in \mathrm{L}^2(0,1)$.

b) Use a) to establish the general Poincaré inequality

$$\|u - \langle u, \mathbf{1}\rangle_{\mathrm{L}^2} \mathbf{1}\|_2 \le c \left\|u'\right\|_2 \qquad (u \in \mathrm{H}^1(0,1)). \tag{8.6}$$

(Hint: Corollary 8.7.)

c) How would (8.6) have to be modified if the interval $(0,1)$ is replaced by a general interval (a,b)?

Exercise 8.13 (Neumann boundary conditions). Fix real numbers $\lambda > 0$ and $a < b$.

a) Show that

$$\|u\|_\lambda := \left(\lambda \|u\|_2^2 + \left\|u'\right\|_2^2\right)^{1/2} \qquad (u \in \mathrm{H}^1(a,b))$$

is a norm on $\mathrm{H}^1(a,b)$ which is induced by an inner product $\langle \cdot, \cdot \rangle_\lambda$. Then show that this norm is equivalent to the standard norm $\|\cdot\|_{\mathrm{H}^1}$ on $\mathrm{H}^1(a,b)$.

b) Let $f \in \mathrm{L}^2(a,b)$. Show that there is a unique $u \in \mathrm{H}^1(a,b)$ satisfying

$$\lambda \int_a^b u(s)v(s)\,\mathrm{d}s + \int_a^b u'(s)v'(s)\,\mathrm{d}s = \int_a^b f(s)v(s)\,\mathrm{d}s$$

for all $v \in \mathrm{H}^1(a,b)$.

c) Let f and u as in b). Show that u satisfies

$$u \in \mathrm{H}^2(a,b), \quad \lambda u - u'' = f, \quad u'(b) = u'(a) = 0$$

(Hint for c): Take first $v \in \mathrm{C}_0^1[a,b]$ in b), then $v \in \mathrm{C}^1[a,b]$ and use the product rule from Exercise 8.11.)

Chapter 9

Bounded Linear Operators

Bounded linear mappings (=operators) have been introduced in Chapter 2, and we have seen already a few examples, for instance the orthogonal projections in Hilbert spaces. In this chapter we have a closer look at the abstract concept and encounter more examples.

9.1. Integral Operators

In the previous section we introduced the integration operator J, which maps $\mathrm{L}^1(a,b)$ into $\mathrm{C}[a,b]$ and $\mathrm{L}^2(a,b)$ into $\mathrm{H}^1(a,b)$. We have

$$Jf(t) = \int_a^t f(s)\,\mathrm{d}s = \int_a^b \mathbf{1}_{[a,t]}(s) f(s)\,\mathrm{d}s \qquad (t \in [a,b]).$$

Let us introduce the function $k : [a,b] \times [a,b] \longrightarrow \mathbb{K}$

$$k(t,s) = \mathbf{1}_{[a,t]}(s) = \begin{cases} 1 & \text{if} \quad a \le s \le t \le b, \\ 0 & \text{if} \quad a \le t < s \le b. \end{cases}$$

Then we have

$$Jf(t) = \int_a^b k(t,s) f(s)\,\mathrm{d}s$$

for every function $f \in \mathrm{L}^1(a,b)$. Hence J is an integral operator in the following sense.

Definition 9.1. Let X, Y be intervals of $\mathbb{R}$. An operator A is called an **integral operator** if there is a function

$$k : X \times Y \longrightarrow \mathbb{K}$$

such that $A = A_k$ is given by

$$(Af)(t) = \int_Y k(t,s) f(s)\,\mathrm{d}s \qquad (t \in X) \tag{9.1}$$

for functions f where this is meaningful. The function k is called the **integral kernel** or the **kernel function** of A_k.

> **Advice/Comment:**
> Attention: the integral kernel k of the integral operator A_k has *nothing* to do with the kernel (= null space) $\ker(A_k)$.

Remark 9.2 (Lebesgue measure in $\mathbb{R}^2$ and Fubini's theorem). To work with integral operators properly, one has to know something about the theory of product measure spaces and in particular the Fubini-Tonelli theorem. Roughly speaking, one defines Lebesgue outer measure on $\mathbb{R}^2$ as in Definition 6.1, replacing intervals by *rectangles*, i.e., cartesian products of intervals. Then Theorem 6.2 holds if $\mathbb{R}$ is replaced by $\mathbb{R}^2$ and one obtains the **two-dimensional Lebesgue measure** λ^2. The notion of measurable functions (Definition 6.3) carries over, with the interval X being replaced by the rectangle $X \times Y$. (A function $f \in \mathcal{M}(X \times Y)$ is then called **product measurable**.) Then the whole theory of null sets, $\mathcal{L}^p$ and L^p-spaces carries over to the 2-dimensional setting. In particular, we can form the Banach spaces

$$\mathrm{L}^1(X \times Y) \quad \text{and} \quad \mathrm{L}^2(X \times Y)$$

and one has a Dominated Convergence and a Completeness Theorem.

The integral of an integrable function $f \in \mathrm{L}^1(X \times Y)$ with respect to two-dimensional Lebesgue measure is computed via iterated integration in either order:

$$\int_{X \times Y} f(\cdot,\cdot)\,\mathrm{d}\lambda^2 = \int_X \int_Y f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

This is called **Fubini's theorem** and it includes the statement that if one integrates out just one variable, the function

$$x \longmapsto \int_Y f(x,y)\,\mathrm{d}y$$

is again measurable.

Advice/Comment:
Actually, it is not that simple, and there are quite some measure-theoretical subtleties here. However, we shall boldly ignore them and refer to a book on measure theory instead.

For measurable functions $f \in \mathcal{M}(X)$ and $g \in \mathcal{M}(Y)$ we define the function

$$f \otimes g : X \times Y \longrightarrow \mathbb{K}, \qquad (f \otimes g)(x, y) := f(x)g(y). \tag{9.2}$$

Then $f \otimes g$ is product measurable and the space

$$\operatorname{span}\{f \otimes g \mid f \in \mathrm{L}^2(X),\, g \in \mathrm{L}^2(Y)\}$$

is dense in $\mathrm{L}^2(X \times Y)$. By Density Principle DP.1 one can then conclude that

$$\operatorname{span}\{f \otimes g \mid f \in \mathrm{C}[a, b],\, g \in \mathrm{C}[c, d]\}$$

is dense in $\mathrm{L}^2([a, b] \times [c, d])$.

For future reference we note the following simple fact.

Lemma 9.3. *Let $f \in \mathrm{L}^1(a, b)$ and $n \in \mathbb{N}$. Then*

$$(J^n f)(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)\, \mathrm{d}s$$

for all $t \in [a, b]$. In particular, J^n is also an integral operator with kernel function

$$G_n(t, s) = \frac{1}{(n-1)!} \mathbf{1}_{[a,t]}(s)(t-s)^{n-1} \qquad (s, t \in [a, b]).$$

Proof. This is induction and Fubini's theorem. □ Ex.9.1

Example 9.4 (Green's function for the Poisson problem)**.** Consider again the *Poisson problem*

$$u'' = -f, \qquad u(a) = u(b) = 0.$$

We have seen in the previous chapter that for each $f \in \mathrm{L}^2(a, b)$ there is a *unique* solution $u \in \mathrm{H}^2(a, b)$. Now we want to derive a more explicit formula how to *compute* the solution u from the given data f. For simplicity we shall work on $[a, b] = [0, 1]$, but see also Exercise 9.2. Ex.9.2

Applying the integration operator J we find successively

$$u'(t) = -(Jf)(t) + c, \quad u(t) = -(J^2 f)(t) + tc + d \qquad (t \in [0, 1])$$

for certain constants $c, d \in \mathbb{K}$. Using the boundary conditions we obtain $d = 0$ and

$$c = (J^2 f)(1) = \int_0^1 (1-s)f(s)\,\mathrm{d}s.$$

Inserting c, d back into the formula for u one obtains

$$\begin{aligned} u(t) &= -\int_0^t (t-s)f(s)\,\mathrm{d}s + t\int_0^1 (1-s)f(s)\,\mathrm{d}s \\ &= \int_0^1 [(s-t)\mathbf{1}_{\{s\le t\}}(t,s) + t(1-s)]f(s)\,\mathrm{d}s = \int_0^1 g_0(t,s)f(s)\,\mathrm{d}s \end{aligned}$$

with

$$g_0(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1 \\ t(1-s), & 0 \le t \le s \le 1. \end{cases} \tag{9.3}$$

Ex.9.3 This is the so-called **Green's function** for the Poisson problem.

The Green's function for the Poisson problem is **product square integrable**, i.e.,

$$\int_a^b \int_a^b |g_0(t,s)|^2 \,\mathrm{d}s\,\mathrm{d}t < \infty,$$

since $|g_0(t,s)| \le 1$ for all s, t. This property has a special name.

Definition 9.5. Let $X, Y \subseteq \mathbb{R}$ be intervals and let $k : X \times Y \longrightarrow \mathbb{K}$ be product measurable. If $k \in \mathrm{L}^2(X \times Y)$, i.e., if

$$\int_X \int_Y |k(x,y)|^2 \,\mathrm{d}y\,\mathrm{d}x < \infty.$$

then k is called a **Hilbert-Schmidt kernel function**. The associated integral operator A_k is called a **Hilbert–Schmidt integral operator**.

The next result shows that a Hilbert–Schmidt kernel function induces indeed a bounded operator on the L^2-spaces.

Theorem 9.6. *Let $k \in \mathrm{L}^2(X \times Y)$ be a Hilbert-Schmidt kernel function. Then the associated Hilbert–Schmidt integral operator A_k (given by (9.1)) satisfies*

$$\|A_k f\|_{\mathrm{L}^2(X)} \le \|k\|_{\mathrm{L}^2(X\times Y)} \|f\|_{\mathrm{L}^2(Y)}$$

for all $f \in \mathrm{L}^2(Y)$, hence A_k is a bounded operator

$$A_k : \mathrm{L}^2(Y) \longrightarrow \mathrm{L}^2(X).$$

Moreover, the kernel function k is uniquely determined (λ^2-a.e.) by the associated operator A_k.

Proof. Let $f \in \mathrm{L}^2(Y)$. By Cauchy–Schwarz

$$\left|\int_Y k(x,y)f(y)\,\mathrm{d}y\right| \le \int_Y |k(x,y)f(y)|\,\mathrm{d}y \le \left[\int_Y |k(x,y)|^2\,\mathrm{d}y\right]^{1/2} \|f\|_{\mathrm{L}^2(Y)}$$

for all $x \in [a,b]$. Hence

$$\begin{aligned}\|A_k f\|^2_{\mathrm{L}^2(X)} &= \int_X \left|\int_Y k(x,y)f(y)\,\mathrm{d}y\right|^2 \mathrm{d}x \\ &\le \left(\int_X \int_Y |k(x,y)|^2\,\mathrm{d}y\,\mathrm{d}x\right) \|f\|^2_{\mathrm{L}^2(Y)}.\end{aligned}$$

Taking square-roots we arrive at

$$\|A_k f\|_{\mathrm{L}^2(X)} \le \left(\int_X \int_Y |k(x,y)|^2\,\mathrm{d}y\,\mathrm{d}x\right)^{1/2} \|f\|_{\mathrm{L}^2(Y)}$$

and this was to prove.

That k is determined by A_k amounts to saying that if $A_k = 0$ then $k = 0$ in $\mathrm{L}^2(X \times Y)$. This can be proved by using the density result mentioned in Remark 9.2, see Exercise 9.4. □ Ex.9.4

We shall deal mostly with Hilbert–Schmidt integral operators. However, there are other ones.

Example 9.7 (The Laplace Transform)**.** Consider the case $X = Y = \mathbb{R}_+$ and the kernel function

$$k(x,y) := \mathrm{e}^{-xy} \qquad (x, y > 0).$$

The associated integral operator is the **Laplace transform**

$$(\mathcal{L}f)(x) = \int_0^\infty \mathrm{e}^{-xy} f(y)\,\mathrm{d}y \qquad (x > 0).$$

In Exercise 6.14 it is shown that $\mathcal{L}$ is a bounded operator from $\mathrm{L}^1(\mathbb{R}_+)$ to $\mathrm{C_b}(\mathbb{R}_+)$. Here we are interested in its behaviour on $\mathrm{L}^2(\mathbb{R}_+)$. Letting aside (as usual) the measurability questions, we estimate

$$|\mathcal{L}f(x)| \le \int_0^\infty \mathrm{e}^{-xy/2} y^{-1/4} \cdot \mathrm{e}^{-xy/2} y^{1/4} \,|f(y)|\,\mathrm{d}y$$

and hence by Cauchy–Schwarz

$$|\mathcal{L}f(x)|^2 \le \int_0^\infty \mathrm{e}^{-xy} y^{-1/2}\,\mathrm{d}y \cdot \int_0^\infty \mathrm{e}^{-xy} y^{1/2}\,|f(y)|^2\,\mathrm{d}y.$$

We evaluate the first integral by change of variables ($y \mapsto y/x, y \mapsto y^2$)

$$\int_0^\infty e^{-xy} y^{-1/2}\,dy = x^{-1/2}\int_0^\infty e^{-y} y^{-1/2}\,dy = x^{-1/2} 2\int_0^\infty e^{-y^2}\,dy$$
$$= \sqrt{\pi} x^{-1/2}.$$

If we use this information and integrate, we obtain

$$\|\mathcal{L}f\|_2^2 = \int_0^\infty |\mathcal{L}f(x)|^2 \,dx \le \sqrt{\pi}\int_0^\infty \int_0^\infty x^{-1/2} e^{-xy} y^{1/2} |f(y)|^2 \,dy\,dx$$
$$= \sqrt{\pi}\int_0^\infty \int_0^\infty x^{-1/2} e^{-xy}\,dx\, y^{1/2} |f(y)|^2 \,dy = \pi \|f\|_2^2 .$$

This shows that $\mathcal{L} : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+)$ is a bounded operator, with $\|\mathcal{L}f\|_2 \le \sqrt{\pi}\,\|f\|_2$ for each $f \in L^2(\mathbb{R}_+)$.

Ex.9.5 However, the Laplace transform is *not* a Hilbert–Schmidt operator.

9.2. The Space of Operators

Recall from Section 2.3 that a linear operator $T : E \longrightarrow F$ is **bounded** if there is a constant $c \ge 0$ such that

$$\|Tf\| \le c\,\|f\|$$

for all $f \in E$, which is precisely the case if its **operator norm**

$$\|T\| := \sup\{\|Tf\| \mid f \in E,\ \|f\| \le 1\} < \infty.$$

is a finite number. And in this case we have the important formula

$$\|Tf\| \le \|T\| \cdot \|f\| \qquad (f \in E). \tag{9.4}$$

If there is $\|f\| = 1$ such that $\|T\| = \|Tf\|$, then we say that the norm **is attained**. (This is not always the case, cf. Example 9.8.6 below.)

In Theorem 2.13 we have proved that the operator norm turns the space $\mathcal{L}(E;F)$ of bounded linear operators from E to F into a normed vector space. However, we have not yet dealt so far neither with computing operator norms nor with convergent sequences in the space of operators.

Examples 9.8. 1) (Zero and identity) The zero operator 0 maps the whole space E to 0, so $\|0\|_{\mathcal{L}(E;F)} = 0$. The **identity** mapping $I \in \mathcal{L}(E)$ satisfies $If = f$ and so $\|I\| = 1$.

2) (Isometries) More generally, if $T : E \longrightarrow F$ is an isometry, then $\|Tf\| = \|f\|$, and so $\|T\| = 1$ (unless $E = \{0\}$).

3) (Orthogonal Projections) If $P \ne 0$ is an orthogonal projection on a Hilbert space H, then $\|P\| = 1$.

4) (Point Evaluations) If $[a,b]$ is an interval and $x_0 \in [a,b]$ then the **point-evaluation** or **Dirac functional** in x_0

$$\delta_{x_0} := (f \longmapsto f(x_0)) : \mathrm{C}[a,b] \longrightarrow \mathbb{K}$$

is a bounded linear functional on $\mathrm{C}[a,b]$ with norm $\|\delta_{x_0}\| = 1$. Analogously, point evaluations are bounded linear functionals on the spaces $\mathcal{B}(\Omega)$ (Ω an arbitrary set) and ℓ^p, $p \in \{1,2,\infty\}$.

However, point evaluations on $E = (\mathrm{C}[a,b], \|\cdot\|_p)$ (with $p \in \{1,2\}$) are *not* bounded, see Exercise 2.8. And point evaluations cannot even be defined in a reasonable way on $\mathrm{L}^2(a,b)$, since singletons are null-sets and 'functions' are determined only up to equality almost everywhere.

5) (Inner Products) If H is an inner product space, and $g \in H$ then

$$\psi_g : H \longrightarrow \mathbb{K}, \qquad \psi_g(f) := \langle f, g \rangle$$

is a linear functional with norm $\|\psi_g\| = \|g\|$, i.e.,

$$\sup_{\|f\| \le 1} |\langle f, g \rangle| = \|g\|$$

Hence, if $A : H \longrightarrow K$ is a bounded linear operator, then

$$\|A\| = \sup_{\|g\| \le 1} \|Ag\| = \sup_{\|g\| \le 1, \|f\| \le 1} |\langle f, Ag \rangle| \tag{9.5}$$

6) (Multiplication Operators) A bounded sequence $\lambda = (\lambda_n)_{n \in \mathbb{N}} \in \ell^\infty$ induces a **multiplication operator** $A_\lambda : \ell^2 \longrightarrow \ell^2$ by

$$(A_\lambda f)(n) := \lambda_n f(n) \qquad (n \in \mathbb{N}, f \in \ell^2).$$

The sequence λ is called is called the **multiplier**. Since it is bounded, we obtain

$$\|A_\lambda f\|_2^2 = \sum\nolimits_{n=1}^\infty |\lambda_n f(n)|^2 \le \sum\nolimits_{n=1}^\infty \|\lambda\|_\infty^2 |f(n)|^2 = \|\lambda\|_\infty^2 \|f\|_2^2 \tag{9.6}$$

for every $f \in \ell^2$. Therefore A_λ is bounded and $\|A_\lambda\| \le \|\lambda\|_\infty$. On the other hand,

$$\|A_\lambda e_n\|_2 = \|\lambda_n e_n\|_2 = |\lambda_n| \|e_n\|_2$$

where e_n is the n-th unit vector. Hence $\|A_\lambda\| \ge |\lambda_n|$ for every $n \in \mathbb{N}$, and thus $\|A_\lambda\| \ge \sup_n |\lambda| = \|\lambda\|_\infty$. Combining both estimates yields $\|A_m\| = \|m\|_\infty$.

Similarly, one can define multiplication operators on ℓ^1 and ℓ^∞ and on the spaces $\mathrm{C}[a,b]$, $\mathrm{L}^p(a,b)$ with $p \in \{1,2,\infty\}$. Ex.9.6 Ex.9.7

Advice/Comment:
In order that the norm of the multiplication operator A_m from above is attained at f, say, one must have equality in the estimate (9.6). But this requires $|\lambda_n| = \|\lambda\|_\infty$ wherever $f(n) \neq 0$. *So the norm is attained if and only if $|\lambda|$ attains its supremum.* More examples of operators which do not attain their norms are in Exercises 9.6 and 9.7.

7) (Shifts) On $E = \ell^2$ the **left shift** L and the **right shift** R are defined by

$$L : (x_1, x_2, x_3 \dots) \longmapsto (x_2, x_3, \dots)$$
$$R : (x_1, x_2, x_3 \dots) \longmapsto (0, x_1, x_2, x_3, \dots).$$

Thus, in function notation,

$$(Lf)(n) := f(n+1), \quad (Rf)(n) = \begin{cases} 0 & \text{if } \; n = 1 \\ f(n-1) & \text{if } \; n \geq 2. \end{cases}$$

It is easy to see that

$$\|Rf\|_2 = \|f\|_2 \qquad (f \in \ell^2)$$

so R is an *isometry* and hence $\|R\| = 1$. Turning to L we obtain for $f \in \ell^2$

$$\begin{aligned} \|Lf\|_2^2 &= \sum\nolimits_{n=1}^{\infty} |(Lf)(n)|^2 = \sum\nolimits_{n=1}^{\infty} |f(n+1)|^2 \\ &= \sum\nolimits_{n=2}^{\infty} |f(n)|^2 \leq \sum\nolimits_{n=1}^{\infty} |f(n)|^2 = \|x\|_2^2 \end{aligned}$$

which implies that $\|L\| \leq 1$. Inserting e_2, the second unit vector, we have $Le_2 = e_1$ and so $\|Le_2\|_2 = \|e_1\|_2 = 1 = \|e_2\|_2$, which implies that $\|L\| = 1$.

Note that $LR = \mathrm{I}$, R is injective and L is surjective. But R is not surjective, and L is not injective. (Such a situation cannot occur in finite dimensions!)

Shift operators can be defined also on the sequence spaces ℓ^1, ℓ^∞. Moreover there are continuous analogues, for instance on the spaces $\mathrm{L}^1(\mathbb{R}_+)$ and $\mathrm{L}^2(\mathbb{R}_+)$.

8) (Hilbert–Schmidt operators) Let $X, Y \subseteq \mathbb{R}$ be intervals and let $k \in \mathrm{L}^2(X \times Y)$ be a Hilbert–Schmidt kernel. This determines a Hilbert–Schmidt operator A_k by (9.1). We call

$$\|A_k\|_{HS} := \|k\|_{\mathrm{L}^2(X\times Y)} = \left(\int_X \int_Y |k(x,y)|^2 \,\mathrm{d}y\,\mathrm{d}x \right)^{1/2}$$

the **Hilbert–Schmidt norm** of the operator A_k. By Theorem 9.6 one has $\|A_k\|_{\mathcal{L}} \le \|A_k\|_{HS}$, i.e., the operator norm is always smaller than the Hilbert–Schmidt norm. Ex.9.8

Advice/Comment:
In general one has $\|A_k\|_{\mathcal{L}} < \|A_k\|_{HS}$, i.e., the operator norm is usually *strictly* smaller than the Hilbert–Schmidt norm. One example is the integration operator J on $\mathrm{L}^2(a,b)$. Its HS-norm is $\|J\|_{HS} = (b-a)/\sqrt{2}$, but its operator norm is $\|J\| = 2(b-a)/\pi$, cf. Chapter 11.

Recall from Lemma 2.14 that the multiplication (=composition) of bounded linear operators is again a bounded linear operator, and one has

$$\|ST\| \le \|S\| \, \|T\|$$

whenever $S \in \mathcal{L}(F;G)$ and $T \in \mathcal{L}(E;F)$. If $E = F$ then we can iterate T, and by induction obtain

$$\|T^n\| \le \|T\|^n \qquad (n \in \mathbb{N}_0).$$

Attention: In general one has $\|T^n\| \ne \|T\|^n$!

Example 9.9 (Integration Operator)**.** The n-th power of the integration operator J on $E = \mathrm{C}[a,b]$ is induced by the integral kernel

$$k_n(t,s) = \mathbf{1}_{\{s \le t\}}(t,s)\frac{(t-s)^{n-1}}{(n-1)!}.$$

From this it follows that $\|J^n\|_{\mathcal{L}(E)} = 1/n! \ne 1^n = \|J\|^n$. (See Exercise 9.9.) Ex.9.9

9.3. Operator Norm Convergence

In the next section we shall see convergence in operator norm at work. But before, we have to establish some formal properties.

Theorem 9.10. *If F is complete, i.e., a Banach space, then $\mathcal{L}(E;F)$ is also a Banach space.*

Proof. Let $(T_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}(E;F)$. For each $f \in E$ by (9.4) we have

$$\|T_n f - T_m f\| = \|(T_n - T_m)f\| \le \|T_n - T_m\| \, \|f\| \tag{9.7}$$

for all $n, m \in \mathbb{N}$. This shows that $(T_n f)_{n\in\mathbb{N}}$ is a Cauchy sequence in F, and since F is complete there exists the limit

$$Tf := \lim_{n\to\infty} T_n f.$$

The linearity of T follows by letting $n \to \infty$ in the equation

$$T_n(\alpha f + \beta g) = \alpha T_n f + \beta T_n g$$

(where $\alpha, \beta \in \mathbb{K}$ and $f, g \in E$). Since every Cauchy sequence is bounded (Lemma 5.2), there is $M \geq 0$ such that $\|T_n\| \leq M$ for all $n \in \mathbb{N}$. If we let $n \to \infty$ in the inequality

$$\|T_n f\| \leq M \|f\| \qquad (f \in E)$$

we see that T is also bounded, with $\|T\| \leq M$.

Now fix $\epsilon > 0$ and find $N \in \mathbb{N}$ so that $\|T_n - T_m\| \leq \epsilon$ for all $n, m \geq N$. This means that

$$\|T_n f - T_m f\| \leq \epsilon$$

for all f with $\|f\| \leq 1$, and $n, m \geq N$. Letting $m \to \infty$ yields

$$\|T_n f - Tf\| \leq \epsilon$$

for all $n \geq N$ and unit vectors f, and taking the supremum over the f we arrive at

$$\|T_n - T\| \leq \epsilon$$

for $n \geq N$. Since $\epsilon > 0$ was arbitrary, we find that $T_n \to T$ in $\mathcal{L}(E; F)$. □

The following lemma tells us that both operator multiplication and application of operators to vectors are continuous operations.

Lemma 9.11. *Let E, F, G be normed space, $T, T_n \in \mathcal{L}(E; F)$, $S, S_n \in \mathcal{L}(F; G)$ and $f, f_n \in E$ for $n \in \mathbb{N}$. Then*

$$T_n \to T,\ S_n \to S \quad \Longrightarrow \quad S_n T_n \to ST \qquad \textit{and}$$
$$T_n \to T,\ f_n \to f \quad \Longrightarrow \quad T_n f \to Tf.$$

Proof. This is proved analogously to Theorem 4.10. Write

$$S_n T_n - ST = (S_n - S)(T_n - T) + S(T_n - T) + (S_n - S)T,$$

then take norms and estimate

$$\begin{aligned} \|S_n T_n - ST\| &\leq \|(S_n - S)(T_n - T)\| + \|S(T_n - T)\| + \|(S_n - S)T\| \\ &\leq \|S_n - S\| \, \|T_n - T\| + \|S\| \, \|T_n - T\| + \|S_n - S\| \, \|T\| \to 0. \end{aligned}$$

The proof of the second assertion is analogous, see also Exercise 9.35. □

9.4. The Neumann Series

Let us begin with an example. Consider on $[0, 1]$ the initial-value problem

$$(9.10) \qquad u'' - pu = g \qquad u \in \mathrm{C}^2[0, 1],\ u(0) = 0,\ u'(0) = 1$$

for the unknown function u, where $g \in \mathrm{C}[0, 1]$ and $p \in \mathrm{C}[0, 1]$ are given data. If we integrate twice and respect the initial conditions, we see that (9.10) is equivalent to

$$u \in \mathrm{C}[0, 1], \quad u(t) = J^2(g + pu)(t) + t \qquad t \in [0, 1].$$

If we write $f(t) := t + (J^2 g)(t)$ then the problem becomes the abstract fixed-point problem

$$u = f + Au, \qquad u \in E,$$

where $E := \mathrm{C}[0, 1]$ and $A : E \longrightarrow E$ is given by $Au := J^2(pu)$.

To solve the fixed-point problem, one has the reflex to find a solution by applying the iteration

$$u_{n+1} := f + Au_n.$$

starting from $u_0 := 0$, say. By continuity of A, if the sequence u_n converges to some u, then this u must solve the problem. The first iterations here are

$$u_0 = 0, \quad u_1 = f, \quad u_2 = f + Af, \quad u_3 = f + Af + A^2 f \dots$$

so

$$u_n = \sum_{j=0}^{n-1} A^j f \qquad (n \in \mathbb{N}).$$

In effect, we have proved the following lemma.

Lemma 9.12. *Let E be a normed space and $A \in \mathcal{L}(E)$. If $f \in E$ is such that the series $u := \sum_{n=0}^{\infty} A^n f$ converges in E, then $u - Au = f$.*

Let us call a bounded operator $T \in \mathcal{L}(E)$ **invertible** if T is bijective and T^{-1} is again bounded. If we want to have a *unique* solution u to the problem

$$u - Au = f$$

for *each* $f \in E$ and in such a way that the solution f depends continuously on y, then this amounts to the invertibility of the operator $\mathrm{I} - A$. Here is a useful criterion.

Theorem 9.13. *Let E be a Banach space and let $A \in \mathcal{L}(E)$ be such that*

$$\sum_{n=0}^{\infty} \|A^n\| < \infty.$$

Then the operator $\mathrm{I} - A$ is invertible and its inverse is given by the so-called **Neumann series**

$$(\mathrm{I} - A)^{-1} = \sum_{n=0}^{\infty} A^n.$$

Proof. The space $\mathcal{L}(E)$ is a Banach space, since E is a Banach space (Theorem 9.10). Therefore $S := \sum_{n=0}^{\infty} A^n$ exists in $\mathcal{L}(E)$, cf. Theorem 5.16. Now Lemma 9.12 shows that $(\mathrm{I} - A)S = \mathrm{I}$, whence $\mathrm{I} - A$ is surjective and S is injective. But $S(\mathrm{I} - A) = (\mathrm{I} - A)S$ since $S = \lim_n S_n$ where S_n is the n-th partial sum, which satisfies $(\mathrm{I} - A)S_n = S_n(\mathrm{I} - A)$. Altogether we have shown that S is the inverse of $\mathrm{I} - A$. □

The Neumann series is absolutely convergent for instance in the case that A is a **strict contraction**, i.e., $\|A\| < 1$; indeed,

$$\sum_{n=0}^{\infty} \|A^n\| \le \sum_{n=0}^{\infty} \|A\|^n < \infty.$$

However this is not a necessary condition: the integration operator J on
Ex.9.10 $\mathrm{C}[a, b]$ is a counterexample, see Example 9.9.

Volterra Operators. Recall the example we started with: $E = \mathrm{C}[0, 1]$ and $Au = J^2(pu)$ where $p \in \mathrm{C}[0, 1]$ is given. If we write this out, we obtain

$$(Au)(t) = \int_0^t (t - s)p(s)u(s)\,\mathrm{d}s.$$

In general, we call an operator $V : \mathrm{C}[a, b] \longrightarrow \mathrm{C}[a, b]$ an abstract **Volterra operator** if it has the form

$$(Vf)(t) = \int_a^t k(t, s)f(s)\,\mathrm{d}s \qquad (t \in [a, b],\ f \in \mathrm{C}[a, b]) \tag{9.11}$$

where $k : [a, b] \times [a, b] \longrightarrow \mathbb{K}$ is a *continuous* function. One can show that Vf is indeed a continuous function, see Exercise 9.33. The following lemma is fundamental.

Lemma 9.14. *Let $k : [a, b] \times [a, b] \longrightarrow \mathbb{K}$ is continuous and let V be the associated Volterra operator, given by* (9.11)*. Then*

$$|V^n f(t)| \le \frac{\|k\|_\infty^n (t - a)^n}{n!} \|f\|_\infty$$

for all $f \in \mathrm{C}[a, b], t \in [a, b], n \in \mathbb{N}$. Consequently

$$\|V^n\|_{\mathcal{L}(\mathrm{C}[a,b])} \le \|k\|_\infty \frac{(b - a)^n}{n!}$$

for every $n \in \mathbb{N}$.

Ex.9.11 **Proof.** The proof is an easy induction and left as an exercise. □

Advice/Comment:
Note that an abstract Volterra operator V as above is an integral operator with kernel function
$$\mathbf{1}_{\{s\le t\}}(t,s)k(t,s),$$
which is in general *not* continuous on $[a,b]\times[a,b]$!

If V is an abstract Volterra operator, then by the previous lemma
$$\sum\nolimits_{n=0}^{\infty}\|V^n\| \le \sum\nolimits_{n=1}^{\infty}\frac{\|k\|_\infty (b-a)^n}{n!} = \mathrm{e}^{\|k\|_\infty (b-a)} < \infty.$$
Hence by Theorem 9.13 the operator $\mathrm{I}-V$ is invertible. This leads to the following corollary.

Corollary 9.15. *If $k : [a,b]\times[a,b] \longrightarrow \mathbb{K}$ is continuous, then for every $f\in \mathrm{C}[a,b]$ the equation*
$$u(t) - \int_a^t k(t,s)u(s)\,\mathrm{d}s = f(t) \qquad (t\in[a,b])$$
has a unique solution $u\in \mathrm{C}[a,b]$.

Hence for given $p, f \in \mathrm{C}[0,1]$ our initial-value problem (9.10) has a unique solution. Ex.9.12
Ex.9.13

9.5. Adjoints of Hilbert Space Operators

Suppose $X, Y \subseteq \mathbb{R}$ are intervals, $k\in \mathrm{L}^2(X\times Y)$ is square-integrable and A_k is the associated Hilbert–Schmidt operator. Then if $f\in \mathrm{L}^2(Y)$, $g\in\mathrm{L}^2(X)$,
$$\begin{aligned}\int_{X\times Y}\left|k(x,y)f(y)\overline{g(x)}\right|\,\mathrm{d}\lambda^2(x,y) &= \int_{X\times Y}|k\cdot(\overline{g}\otimes f)|\,\mathrm{d}\lambda^2\\ &\le \|k\|_{\mathrm{L}^2(X\times Y)}\,\|f\|_{\mathrm{L}^2(Y)}\,\|g\|_{\mathrm{L}^2(X)} < \infty.\end{aligned}$$
by Cauchy–Schwarz. So one can apply Fubini's theorem:
$$\begin{aligned}\int_X\int_Y k(x,y)f(y)\,\mathrm{d}y\,\overline{g(x)}\,\mathrm{d}x &= \int_Y f(y)\int_X k(x,y)\overline{g(x)}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_Y f(y)\overline{\left(\int_X \overline{k(x,y)}g(x)\,\mathrm{d}x\right)}\,\mathrm{d}y\\ &= \int_Y f(y)\overline{\left(\int_X k^*(y,x)g(x)\,\mathrm{d}x\right)}\,\mathrm{d}y\end{aligned}$$
where $k^*\in\mathrm{L}^2(Y\times X)$ is defined as
$$k^*(y,x) := \overline{k(x,y)} \qquad (x\in X, y\in Y).$$

The function k^* is called the **adjoint kernel function** and our computations above amount to the formula

$$\langle A_k f, g\rangle = \langle f, A_{k^*} g\rangle .$$

Actually, there is an abstract concept behind this.

Theorem 9.16. *Let H, K be Hilbert spaces, and let $A : H \longrightarrow K$ be a bounded linear operator. Then there is a unique operator $A^* : K \longrightarrow H$ such that*

$$\langle Af, g\rangle_K = \langle f, A^* g\rangle_H \quad \textit{for all} \quad f \in H,\ g \in K.$$

Furthermore, one has $\|A^*\| = \|A\|$.

Proof. Fix $g \in K$. Then the mapping

$$H \longrightarrow \mathbb{K}, \qquad f \longmapsto \langle Af, g\rangle$$

is a bounded linear functional, because

$$|\langle Af, g\rangle| \le \|Af\|\,\|g\| \le \|A\|\,\|f\|\,\|g\|$$

by Cauchy–Schwarz. By the Riesz–Fréchet theorem, there is a *unique* vector $h \in H$ such that $\langle Af, g\rangle = \langle f, h\rangle$ for all $f \in H$. We write $A^*g := h$, so that

$$\langle Af, g\rangle = \langle f, A^* g\rangle \qquad (f \in H,\ g \in K)$$

It is routine to show that $A^* : K \longrightarrow H$ is linear. Moreover, A^* is again bounded and one has

$$\|A^*\| = \sup_{\|f\|\le 1, \|g\|\le 1} |\langle f, A^* g\rangle| = \sup_{\|f\|\le 1, \|g\|\le 1} |\langle Af, g\rangle| = \|A\|$$

by (9.5). □

The new operator A^* is called the (Hilbert space) **adjoint** of A. The formal properties of adjoints are as follows:

$$(A+B)^* = A^* + B^*, \quad (\alpha A)^* = \overline{\alpha} A^*, \quad (A^*)^* = A, \quad (AB)^* = B^* A^*$$

(where we suppose that $H = K$ in the last identity in order to render the
Ex.9.14 composition meaningful). The proof these identities is left as Exercise 9.14.

Lemma 9.17. *If $A \in \mathcal{L}(H; K)$ then*

$$H = \overline{\operatorname{ran}}(A^*) \oplus \ker(A)$$

is an orthogonal decomposition.

Proof. Observe that

$$\begin{aligned} f \perp \overline{\operatorname{ran}}(A^*) &\iff f \perp \operatorname{ran}(A^*) \iff \langle f, A^* g\rangle = 0 \text{ for all } g \in K \\ &\iff \langle Af, g\rangle = 0 \text{ for all } g \in K \iff Af = 0 \iff f \in \ker(A). \end{aligned}$$

Hence $\ker(A)$ is the orthogonal complement of $\overline{\operatorname{ran}}(A^*)$. □

Examples 9.18. 1) If $H = \mathbb{K}^d$ is finite-dimensional with the canonical scalar product, then $A \in \mathcal{L}(H)$ is given by a $d \times d$-matrix $M = (a_{ij})_{i,j}$. Writing the elements of $\mathbb{K}^d$ as column vectors we have

$$(Mx)^t\overline{y} = x^t M^t \overline{y} = x^t \overline{(\overline{M}^t y)} \qquad (x, y \in \mathbb{K}^d).$$

This shows that A^* corresponds to the conjugate-transposed matrix $M^* = \overline{M}^t = (\overline{a_{ji}})_{i,j}$.

2) Consider the shifts L, R on $H = \ell^2(\mathbb{N})$. Then

$$\langle Re_n, e_k\rangle = \langle e_{n+1}, e_k\rangle = \delta_{n+1,k} = \delta_{n,k-1} = \langle e_n, Le_k\rangle$$

for all $n, k \in \mathbb{N}$. Since $\operatorname{span}\{e_m \mid m \in \mathbb{N}\}$ is dense in H, by sesquilinearity and continuity we conclude that $\langle Rx, y\rangle = \langle x, Ly\rangle$ for all $x, y \in H$, whence $L^* = R$ and $R^* = L$.

3) The adjoint of a Hilbert–Schmidt operator with kernel function $k \in \mathrm{L}^2(X \times Y)$ is given by the Hilbert-Schmidt operator induced by the adjoint kernel function k^*. This has been shown above. Ex.9.15 Ex.9.16

9.6. Compact Operators on Hilbert Spaces

Look (again) at the Poisson problem

$$u'' = -f \qquad u \in \mathrm{H}^2(0,1), \quad u(0) = u(1) = 0$$

for given $f \in \mathrm{L}^2(0,1)$. We know that there is a unique solution given by

$$u(t) = (G_0 f)(t) := \int_0^1 g_0(t,s) f(s)\, \mathrm{d}s$$

where g_0 is the Green's function (see (9.3)). Since numerically one can handle only finite data sets, in actual computations one wants to replace the original problem by a finite-dimensional one, with controllable error. This amounts to approximating the operator L by operators L_n that have finite-dimensional ranges.

Definition 9.19. An operator $T : E \longrightarrow F$ is called of **finite rank** or a **finite-dimensional operator**, if $\operatorname{ran} T$ is of finite dimension. Ex.9.17

That $(L_n)_{n\in\mathbb{N}}$ is an approximation of L can mean at least two things:

1) **strong convergence:** $L_n f \to Lf$ in the norm of $\mathrm{L}^2(0,1)$, for each input data $f \in \mathrm{L}^2(0,1)$;

2) **norm convergence**: $L_n \to L$ in the operator norm.

In the second sense, the error $\|L_n f - Lf\| \le \|L_n - L\| \, \|f\|$ is controlled by $\|f\|$ only. In the first sense, no such control is implied, and the speed of convergence $L_n f \to Lf$ might be arbitrarily low for unit vectors f.

Advice/Comment:
Strong convergence is "pointwise convergence" and norm convergence is "uniform convergence on the unit ball". So clearly (2) is the stronger notion. It should not surprise us that the two notions really differ.

Example 9.20. Suppose that $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of the Hilbert space H. The orthogonal projection onto $F_n := \operatorname{span}\{e_1, \dots, e_n\}$ is

$$P_n := \sum_{j=1}^{n} \langle \cdot, e_j \rangle \, e_j$$

and we know from Chapter 7 that $P_n f \to f$ for each $f \in H$. So $(P_n)_{n\in\mathbb{N}}$ approximates the identity operator I strongly. However, $\mathrm{I} - P_n$ is the projection onto $F_n^\perp$ and so $\|\mathrm{I} - P_n\| = 1$. In particular, $P_n \not\to \mathrm{I}$ in operator norm.

Now, let $A : H \longrightarrow H$ be a bounded operator. Then $AP_n f \to Af$ for each $f \in H$ by continuity, and hence $(AP_n)_{n\in\mathbb{N}}$ approximates A strongly. Clearly, each AP_n is a finite-dimensional operator.

The previous example shows that at least on separable Hilbert spaces strong approximation of an operator by finite-dimensional ones is always possible by choosing any orthonormal basis. The norm approximability deserves an own name.

Definition 9.21. A bounded operator $A : H \longrightarrow K$ between two Hilbert spaces is called **compact** if there is a sequence $(A_n)_{n\in\mathbb{N}}$ of finite dimensional operators in $\mathcal{L}(H; K)$ such that $\|A_n - A\| \to 0$.

We denote by

$$\mathcal{C}(H, K) := \{A \in \mathcal{L}(H, K) \mid A \text{ is compact}\}$$

the space of compact operators. To show that an operator is compact, one uses the definition or the following useful theorem.

Theorem 9.22. *Let H, K, L be Hilbert spaces. Then $\mathcal{C}(H; K)$ is a closed linear subspace of $\mathcal{L}(H; K)$. If $A : H \longrightarrow K$ is compact, $C \in \mathcal{L}(K; L)$ and $D \in \mathcal{L}(L; H)$, then CA and AD are compact. Moreover, $A^* \in \mathcal{L}(K; H)$ is*
Ex.9.18 *also compact.*

Proof. The set $\mathcal{C}(H, K)$ is the operator-norm closure of the linear subspace(!) of $\mathcal{L}(H; K)$ consisting of all finite-rank operators. Hence it is a closed subspace, by Example 4.2.1. Let $A : H \longrightarrow K$ be compact. Then by definition there is a sequence of finite rank operators $(A_n)_{n\in\mathbb{N}}$ such that $A_n \to A$ in operator norm. Hence

$$A_n D \longrightarrow AD \quad \text{and} \quad CA_n \to CA$$

in norm, by (9.8). However, CA_n and A_nD clearly are of finite rank, so AD and CA are compact. To see that the adjoint A^* is compact as well, note that $\|A^* - A_n^*\| = \|(A - A_n)^*\| = \|A - A_n\| \to 0$ as $n \to \infty$. Also, by Exercise 9.17, A_n^* is of finite rank for each $n \in \mathbb{N}$. □

Example 9.23 (Hilbert–Schmidt operators)**.** *Let $X, Y \subseteq \mathbb{R}$ be intervals and $k \in \mathrm{L}^2(X \times Y)$. Then the associated Hilbert–Schmidt integral operator*

$$A_k : \mathrm{L}^2(Y) \longrightarrow \mathrm{L}^2(X), \qquad (A_k h)(x) = \int_Y k(x, y)h(y)\,\mathrm{d}y$$

is compact.

Proof. We use that the space

$$E := \mathrm{span}\{f \otimes g \mid f \in \mathrm{L}^2(X),\ g \in \mathrm{L}^2(Y)\}$$

is dense in $\mathrm{L}^2(X \times Y)$. (Another proof is given in Exercise 9.39.) If $k = f \otimes g$ then $A_k h = \langle h, \overline{g}\rangle \cdot f$ and so $\mathrm{ran}\, A_k$ is one-dimensional. Hence if $k \in E$ then A_k is of finite rank. If $k \in \mathrm{L}^2(X \times Y)$ we can find $k_n \in E$ with $\|k - k_n\|_{\mathrm{L}^2} \to 0$. Then

$$\|A_k - A_{k_n}\|_{\mathcal{L}} \le \|A_k - A_{k_n}\|_{HS} = \|A_{k-k_n}\|_{HS} = \|k - k_n\|_{\mathrm{L}^2} \to 0$$

and so A_k is compact. □

Theorem 9.24. *Let H, K be Hilbert spaces, let $A : H \longrightarrow K$ be a compact operator, and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in H. Then the sequence $(Ax_n)_{n\in\mathbb{N}} \subseteq K$ has a convergent subsequence.*

To put it differently, the theorem says that if an operator A is compact, the image of the unit ball $\{Ax \mid \|x\| \le 1\}$ is relatively (sequentially) compact in F.

> **Advice/Comment:**
> Recall that a subsequence of a sequence $(x_n)_{n\in\mathbb{N}}$ is determined by a strictly increasing map $\pi : \mathbb{N} \longrightarrow \mathbb{N}$, the subsequence then being $(x_{\pi(n)})_{n\in\mathbb{N}}$. See Appendix A.1.

Proof. First of all we note that the theorem is true if A is of finite rank. Indeed, by the boundedness of A, the sequence $(Ax_n)_{n\in\mathbb{N}}$ is a bounded sequence in the finite-dimensional space $\operatorname{ran} A$. By the Bolzano-Weierstrass theorem, it must have a convergent subsequence.

In the general case we shall employ a so-called **diagonal argument**. We can find a sequence $(A_m)_{m\in\mathbb{N}}$ of finite-rank operators such that $\|A_m - A\| \to 0$ as $m \to \infty$. Now take the original sequence $(x_n)_{n\in\mathbb{N}}$ and pick a subsequence $(x_n^1)_{n\in\mathbb{N}}$ of it such that $(A_1 x_n^1)_{n\in\mathbb{N}}$ converges. Then pick a subsequence $(x_n^2)_{n\in\mathbb{N}}$ of the first subsequence such that $(A_2 x_n^2)_{n\in\mathbb{N}}$ converges. Continuing in this way, we get a nested sequence of subsequences $(x_n^k)_{n\in\mathbb{N}}$, each a subsequence of all its predecessors.

The "diagonal sequence" $(x_n^n)_{n\in\mathbb{N}}$ is therefore (eventually) a subsequence
Ex.9.19 of *every* subsequence constructed before. In particular: $(A_m x_n^n)_{n\in\mathbb{N}}$ converges for *every* $m \in \mathbb{N}$.

Finally, we show that $(Ax_n^n)_{n\in\mathbb{N}}$ converges. Since K is a Hilbert space, it suffices to show that the sequence is Cauchy. The usual estimate yields

$$\begin{aligned}\|Ax_n^n - Ax_m^m\| &\le \|Ax_n^n - A_l x_n^n\| + \|A_l x_n^n - A_l x_m^m\| + \|A_l x_m^m - Ax_m^m\| \\ &\le 2M\,\|A - A_l\| + \|A_l x_n^n - A_l x_m^m\|\end{aligned}$$

where $M := \sup_{j\in\mathbb{N}} \|x_j\|$. Given $\epsilon > 0$ we can find an index l so large that $2M\,\|A - A_l\| < \epsilon$ and for that l we can find $N \in \mathbb{N}$ so large that $\|A_l x_n^n - A_l x_m^m\| < \epsilon$ for $m, n \ge N$. Hence

$$\|Ax_n^n - Ax_m^m\| \le \epsilon + \epsilon = 2\epsilon$$

for $m, n \ge N$. □

Example 9.25. Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis for the Hilbert space H, and let $A \in \mathcal{L}(H)$ be given by

$$Af = \sum_{j=1}^{\infty} \lambda_j \langle f, e_j \rangle e_j \qquad (f \in H)$$

for some sequence $\lambda = (\lambda_j)_{j\in\mathbb{N}} \in \ell^\infty$. *Then A is compact if and only if $\lim_{j\to\infty} \lambda_j = 0$.*

Proof. If $\lambda_j \not\to 0$, there is $\epsilon > 0$ and a subsequence $(\lambda_{j_n})_{n\in\mathbb{N}}$ such that $|\lambda_{j_n}| \ge \epsilon$. Define $x_n := \lambda_{j_n}^{-1} e_{j_n}$. Then $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence and $Ax_n = e_{j_n}$ for each $n \in \mathbb{N}$. Since this has no convergent subsequence (see Example 4.18), the operator A cannot be compact, by Theorem 9.24.

Now suppose that $\lim_{j\to\infty} \lambda_j = 0$. Define the "truncation"

$$A_n f := \sum_{j=1}^{n} \lambda_j \langle f, e_j \rangle e_j \qquad (f \in H).$$

Then by Parseval and Bessel

$$\|Af - A_n f\|^2 = \left\| \sum\nolimits_{j=n+1}^{\infty} \lambda_j \langle f, e_j \rangle e_j \right\|^2 = \sum\nolimits_{j=n+1}^{\infty} |\lambda_j|^2 |\langle f, e_j \rangle|^2$$
$$\le \left(\sup_{j>n} |\lambda_j|^2 \right) \sum\nolimits_{j=n+1}^{\infty} |\langle f, e_j \rangle|^2 \le \left(\sup_{j>n} |\lambda_j|^2 \right) \|f\|^2 .$$

Taking the supremum over all f from the unit ball of H yields

$$\|A - A_n\|^2 \le \left(\sup_{j>n} |\lambda_j|^2 \right) \to 0$$

as $n \to \infty$, since $\lambda_j \to 0$. So A is a norm-limit of finite rank operators, hence compact. □ Ex.9.20

Exercises

Exercise 9.1. Prove Lemma 9.3.

Exercise 9.2. Determine the Green's function for the general Poisson problem

$$u'' = -f, \quad (u \in \mathrm{H}^2(a,b),\, u(a) = u(b) = 0).$$

Exercise 9.3. Show that for each $f \in \mathrm{L}^2(a,b)$ there is exactly one solution $u \in \mathrm{H}^2(a,b)$ of the problem

$$u'' = -f, \quad u(a) = 0 = u'(b).$$

Determine the corresponding Green's function.

Exercise 9.4. Use the fact that

$$\mathrm{span}\{f \otimes g \mid f \in \mathrm{C}[a,b],\, g \in \mathrm{C}[c,d]\}$$

is dense in $\mathrm{L}^2([a,b] \times [c,d])$ to show that each kernel function

$$k \in \mathrm{L}^2([a,b] \times [c,d])$$

is uniquely determined by its associated integral operator $A_k : \mathrm{L}^2[c,d] \longrightarrow \mathrm{L}^2[a,b]$.

Exercise 9.5. Show that the Laplace transform (Example 9.7) is not a Hilbert–Schmidt operator.

Exercise 9.6. Let $E := \{f \in \mathrm{C}[0,1] \mid f(1) = 0\}$, with supremum norm. This is a closed subspace of $\mathrm{C}[0,1]$ (why?). Consider the functional φ on E defined by

$$\varphi(f) := \int_0^1 f(x)\,\mathrm{d}x \qquad (f \in E).$$

Show that φ is bounded with norm $\|\varphi\| = 1$. Then show that for every $0 \ne f \in E$ one has $|\varphi(f)| < \|f\|_\infty$.

Exercise 9.7. Let, as in Exercise 9.6, $E := \{f \in \mathrm{C}[0,1] \mid f(1) = 0\}$ with supremum norm. Consider the multiplication operator A defined by $(Af)x = xf(x)$, $x \in [0,1]$. Show that $\|Af\|_\infty < 1$ for every $f \in E$ such that $\|f\|_\infty \le 1$, but nevertheless $\|A\| = 1$.

Exercise 9.8. Let $H = \mathrm{L}^2(-1,1)$, $e_1 = \mathbf{1}_{(-1,0)}$ and $e_2 = \mathbf{1}_{(0,1)}$, and let $P : H \longrightarrow \operatorname{span}\{e_1, e_2\}$ be the orthogonal projection. Show that P is a Hilbert–Schmidt operator, determine its integral kernel and its Hilbert–Schmidt norm $\|P\|_{HS}$. Compare it to the operator norm $\|P\|_{\mathcal{L}(H)}$ of P.

Exercise 9.9. Determine the operator norm $\|J^n\|$ of J^n, $n \in \mathbb{N}$, acting on $\mathrm{C}[a,b]$ with the sup-norm.

Exercise 9.10. Then determine the Hilbert–Schmidt norm $\|J^n\|_{HS}$ of the operator J^n, $n \in \mathbb{N}$ acting on $\mathrm{L}^2(a,b)$. Show that $\mathrm{I} - J$ is invertible and show that $(\mathrm{I} - J)^{-1} - \mathrm{I}$ is again a Hilbert–Schmidt operator. Determine its integral kernel.

Exercise 9.11. Prove Lemma 9.14.

Exercise 9.12. Let $\alpha, \beta \in \mathbb{K}$ and $p \in \mathrm{C}[a,b]$ be given. Show that for each $f \in \mathrm{C}[a,b]$ there is a unique solution u of the initial-value problem

$$u'' - pu = f, \qquad u \in \mathrm{C}^2[a,b],\ u(0) = \alpha,\ u'(0) = \beta.$$

Exercise 9.13. Let $p \in \mathrm{C}[a,b]$ be a **positive** function, and let $u \in \mathrm{C}[a,b]$ be the solution of $u'' = pu$ with $u(0) = 0$ and $u'(0) = 1$. Show that $u(t) \ge 0$ and $u'(t) \ge 1$ for all $t \in [a,b]$.

Exercise 9.14. Let H, K be Hilbert spaces, and let $A, B \in \mathcal{L}(H;K)$. Show that

$$(A + B)^* = A^* + B^* \quad \text{and} \quad (\alpha A)^* = \overline{\alpha} A^*$$

where $\alpha \in \mathbb{K}$. If $K = H$, show that $(AB)^* = B^* A^*$.

Exercise 9.15. Let J be the integration operator on $H = \mathrm{L}^2(a,b)$. Determine J^*. Show that

$$(J + J^*)f = (Jf)(b)\mathbf{1} = \langle f, \mathbf{1} \rangle \cdot \mathbf{1}$$

for $f \in \mathrm{L}^2(a,b)$. (See also Exercise 9.28.)

Exercise 9.16. For the following operators A on ℓ^2 determine the adjoint A^* and decide whether A is compact or not. (Justify your answer.)

1) $A : (x_1, x_2, \dots) \longmapsto (x_2, x_1 + x_3, x_2 + x_4, \dots)$.
2) $A : (x_1, x_2, \dots) \longmapsto (x_1, \frac{x_1+x_2}{2}, \frac{x_2+x_3}{3}, \dots)$.
3) $A : (x_1, x_2, \dots) \longmapsto (x_1, x_3, x_5, \dots)$.

Exercise 9.17. Let H, K be Hilbert spaces and let $A : H \longrightarrow K$ be of finite rank. Show that there are vectors $g_1, \dots, g_n \in H$ and an ONS $e_1, \dots, e_n \in K$ such that

$$Af = \sum\nolimits_{j=1}^{n} \langle f, g_j \rangle \, e_j$$

for all $f \in H$. Show that $A^* : K \longrightarrow H$ is of finite rank, too.

Exercise 9.18. Let A be a compact operator such that $\mathrm{I} - A$ is invertible. Show that $\mathrm{I} - (\mathrm{I} - A)^{-1}$ is compact, too. Is this true if we replace "compact" by "of finite rank" here?

Exercise 9.19. Show that in the proof of Theorem 9.24, the sequence $(x_n^n)_{n \ge k}$ is a subsequence of $(x_n^k)_{n \ge k}$, for each $k \in \mathbb{N}$.

Exercise 9.20. Show that if H is an infinite-dimensional and $A \in \mathcal{L}(H)$ is invertible, then A cannob be a compact operator. Show that the closed unit ball of an infinite-dimensional Hilbert space is not sequentially compact.

Further Exercises

Exercise 9.21. Determine a Green's function for the problem

$$u'' = -f, \qquad u \in \mathrm{H}^2(0,1),\ u(0) = u(1),\ \int_0^1 u(s)\,\mathrm{d}s = 0.$$

Exercise 9.22. Determine a Green's function for the problem

$$u'' = -f, \qquad u \in \mathrm{H}^2(0,1),\ u(0) = u'(0),\ u(1) = u'(1).$$

Exercise 9.23. Determine a Green's function for the problem

$$u'' = -f, \qquad u \in \mathrm{H}^2(0,1),\ u(0) = u'(1),\ u(1) = u'(0).$$

Exercise 9.24. Let $P : \mathrm{L}^2(0,1) \longrightarrow \{\mathbf{1}\}^\perp$ be the orthogonal projection. Determine the integral kernel of the Hilbert–Schmidt operator

$$A := PJ^2P.$$

Exercise 9.25. Show that JJ^* is a Hilbert–Schmidt integral operator on $\mathrm{L}^2(a,b)$, and determine its kernel function.

Exercise 9.26. Show that

$$\langle J^2 f, g \rangle = \langle f, J^2 g \rangle$$

for all $f, g \in \mathrm{L}^2(a,b)$ such that $f, g \in \{\mathbf{1}\}^\perp$.

Exercise 9.27. Determine the integral kernel of the Hilbert–Schmidt operator

$$A := \frac{1}{2}(J^2 + J^{*2}).$$

on $\mathrm{L}^2(0,1)$.

Exercise 9.28. Let J be the integration operator, considered as an operator $J : \mathrm{L}^2(a, b) \longrightarrow \mathrm{H}^1(a, b)$. Determine a formula for $J^* : \mathrm{H}^1(a, b) \longrightarrow \mathrm{L}^2(a, b)$. (Attention: this J^* is different from the J^* computed in Exercise 9.15! Do you understand, why?)

Exercise 9.29. Let $F := \{u \in \mathrm{H}^1(0, 1) \mid u(0) = 0\}$. Determine $F^\perp$, the orthogonal complement of F in the Hilbert space $\mathrm{H}^1(0, 1)$. (Hint: $F = \operatorname{ran} J$.)

Exercise 9.30. Consider on ℓ^1 the multiplication operator A_λ induced by the sequence $\lambda = (1 - 1/n)_{n\in\mathbb{N}}$. What is its norm? Is it attained?

Exercise 9.31. Let $g \in \mathrm{C}[a, b]$. Consider on $\mathrm{C}[a, b]$ the multiplication operator

$$A : \mathrm{C}[a, b] \longrightarrow \mathrm{C}[a, b] \qquad Af = gf.$$

Prove that $\|A\| = \|g\|_\infty$.

Exercise 9.32. Let $g \in \mathrm{C}[a, b]$. Prove that

$$f \in \mathrm{L}^2(a, b) \quad \rightarrow \quad gf \in \mathrm{L}^2(a, b) \text{ and } \|gf\|_2 \le \|g\|_\infty \|f\|_2 .$$

Then consider on $\mathrm{L}^2(a, b)$ the multiplication operator

$$A : \mathrm{L}^2(a, b) \longrightarrow \mathrm{L}^2(a, b) \qquad Af = gf.$$

Prove that $\|A\|_{\mathcal{L}} = \|g\|_\infty$. (Hint: fix $\delta < \|g\|_\infty$ and consider the set $B := \{t \in [a, b] \mid |g(t)| \ge \delta\}$; show that $\|A\mathbf{1}_B\|_2 \ge \delta \|\mathbf{1}_B\|_2$.)

Exercise 9.33. Let $k : [a, b]^2 \longrightarrow \mathbb{K}$ be continuous.

a) Show that if $t_n \to t_0$ in $[a, b]$ then $k(t_n, s) \to k(t_0, s)$ *uniformly* in $s \in [a, b]$. (Hint: k is uniformly continuous (why?).)

b) Use a) to show that if $f \in \mathrm{L}^1(a, b)$ then with

$$(Vf)(t) = \int_a^t k(t, s)f(s)\,\mathrm{d}s \qquad (t \in [a, b])$$

the function Vf is continuous.

c) Use a) to show that if $f \in \mathrm{L}^1(a, b)$ then with

$$(Af)(t) = \int_a^b k(t, s)f(s)\,\mathrm{d}s \qquad (t \in [a, b])$$

the function Af is continuous.

Exercise 9.34 (Volterra Operators on L^2)**.** Let $k : [a, b]^2 \longrightarrow \mathbb{R}$ be continuous and let $V = V_k$ be defined as

$$(Vf)(t) := \int_a^t k(t, s)f(s)\,\mathrm{d}s \qquad (t \in [a, b]).$$

a) Use Exercise 9.33 to show that $V : \mathrm{L}^2(a,b) \longrightarrow \mathrm{C}[a,b]$ is a bounded operator with

$$\|V\|_{\mathrm{L}^2\to\mathrm{C}} \le \|k\|_\infty \sqrt{b-a}.$$

b) Use Lemma 9.14 to show that

$$\|V^n\|_{\mathrm{L}^2\to\mathrm{C}} \le \sqrt{b-a}\frac{\|k\|_\infty (b-a)^{n-1}}{(n-1)!}$$

for all $n \ge 1$. (This is just a rough estimate.)

c) Show that the Volterra integral equation

$$u(t) - \int_a^t k(t,s)u(s)\,\mathrm{d}s = f$$

has a unique solution $u \in \mathrm{L}^2[a,b]$, for each $f \in \mathrm{L}^2[a,b]$.

Exercise 9.35 (Continuity of multiplications)**.** Let X, Y, Z be normed spaces. Suppose one has defined a "multiplication" $X \times Y \longrightarrow Z$, i.e., a mapping $(x,y) \longmapsto x \cdot y = xy$ satisfying $(x+x')y = xy + x'y$ and $x(y+y') = xy + xy'$. Suppose further that there is a constant $c > 0$ such that

$$\|xy\|_Z \le c\,\|x\|_X\,\|y\|_Y \qquad (x \in X, y \in Y).$$

Show that if $x_n \to x$ in X and $y_n \to y$ in Y, then $x_n y_n \to xy$ in Z.

Show that this applies in the following cases:

a) scalar multiplication $\mathbb{K} \times E \longrightarrow E, \qquad (\lambda, f) \longmapsto \lambda f;$

b) inner product $\langle \cdot,\cdot\rangle : H \times H \longrightarrow \mathbb{K}, \qquad (f,g) \longmapsto \langle f,g\rangle;$

c) operator evaluation $\mathcal{L}(E;F) \times E \longrightarrow F, \qquad (T,f) \longmapsto Tf;$

d) operator composition
$\mathcal{L}(F;G) \times \mathcal{L}(E;F) \longrightarrow \mathcal{L}(E;G), \qquad (S,T) \longmapsto ST := S \circ T;$

e) multiplying functions

$$\mathrm{L}^\infty(X) \times \mathrm{L}^2(X) \longrightarrow \mathrm{L}^2(X), \qquad (f,g) \longmapsto f \cdot g$$

f) "tensoring" functions

$$\mathrm{L}^2(X) \times \mathrm{L}^2(Y) \longrightarrow \mathrm{L}^2(X \times Y), \quad (f,g) \longmapsto f \otimes g.$$

(Hint: copy the proof of the cases a) and b) from Theorem 4.10.)

Exercise 9.36. Let E be a Banach space and let $A \in \mathcal{L}(E)$. Show that

$$\exp(A) := \sum\nolimits_{n=0}^\infty \frac{1}{n!}\,A^n$$

exists in $\mathcal{L}(E)$. Show that if $AB = BA$ then $\exp(A+B) = \exp(A)\exp(B)$. Then show that $\exp(A)$ is invertible for each $A \in \mathcal{L}(E)$.

Exercise 9.37. Let E be a Banach space and let $T \in \mathcal{L}(E)$ be an invertible operator. Show that if $S \in \mathcal{L}(E)$ is such that $\|T - S\| < \|T^{-1}\|^{-1}$, then S is invertible, too. (Hint: show first that $S = (\mathrm{I} - (T - S)T^{-1})T$).

Exercise 9.38 (Hilbert–Schmidt operators I)**.** Let H, K be two Hilbert spaces, let $(e_j)_{j\in\mathbb{N}}$ and $(f_m)_{m\in\mathbb{N}}$ be orthonormal bases in H and K, respectively. Show that for a bounded operator $A : H \longrightarrow K$ the following assertions are equivalent:

(i) $\sum_{j=1}^{\infty} \|Ae_j\|_K^2 < \infty$;

(ii) $\sum_{j,m} |\langle Ae_j, f_m\rangle|^2 < \infty$;

(iii) $\sum_{m=1}^{\infty} \|A^* f_m\|_H^2 < \infty$.

An operator satisfying conditions (i)–(iii) is called an **(abstract) Hilbert–Schmidt operator**. Show that if A is such, then it can be written as

$$Af = \sum_{n=1}^{\infty} \langle f, e_j\rangle \, Ae_j \qquad (f \in H)$$

with the series being *absolutely convergent* in K. Show also that for the finite-dimensional truncation

$$A_n f := \sum_{j=1}^{n} \langle f, e_j\rangle \, Ae_j \qquad (f \in H),$$

one has $\|A - A_n\|_{\mathcal{L}} \to 0$. Conclude that A is compact.

Exercise 9.39 (Hilbert–Schmidt operators II)**.** Let $X, Y \subseteq \mathbb{R}$ be intervals and let

$$H := \mathrm{L}^2(Y), \quad K := \mathrm{L}^2(X) \quad \text{and} \quad E := \mathrm{L}^2(X \times Y).$$

Furthermore, let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathrm{L}^2(Y)$ and let $(f_m)_{m\in\mathbb{N}}$ be an orthonormal basis of $\mathrm{L}^2(X)$.

a) Show that $(f_m \otimes \overline{e_j})_{j,m}$ is an ONS in E.

b) Let $k \in \mathrm{L}^2(X \times Y)$ and let $A = A_k$ be the associated HS-integral operator. Show that

$$\langle Ae_j, f_m\rangle_K = \langle k, f_m \otimes \overline{e_j}\rangle_E$$

for all m, j.

c) Show that $\sum_{j=1}^{\infty} \|Ae_j\|_K^2 < \infty$. Conclude (with the help of Exercise 9.38) that A is a compact operator.

Exercise 9.40 (Hilbert–Schmidt operators III)**.** Let $X, Y \subseteq \mathbb{R}$ be intervals and let

$$H := \mathrm{L}^2(Y), \quad K := \mathrm{L}^2(X) \quad \text{and} \quad E := \mathrm{L}^2(X \times Y).$$

Furthermore, let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of H and let $A : H \longrightarrow K$ be a bounded operator satisfying

$$\sum_{j=1}^{\infty} \|Ae_j\|_K^2 < \infty.$$

(I.e., A is a Hilbert–Schmidt operator as defined in Exercise 9.38).

a) Show that
$$k := \sum\nolimits_{j=1}^{\infty} Ae_j \otimes \overline{e_j}$$
converges in E.

b) Let $k_n := \sum_{j=1}^{n} Ae_j \otimes \overline{e_j}$ for $n \in \mathbb{N}$, and let A_k, A_{k_n} are the Hilbert–Schmidt operators associated with the kernel functions k and k_n. Show that $A_{k_n} \to A_k$ in *operator norm* as $n \to \infty$.

c) Show (e.g., by using d)) that $A = A_k$. (Hint: prove first that $A_{k_n} f = A\left(\sum_{j=1}^{n} \langle f, e_j\rangle\, e_j\right)$ for all $f \in H$.)

Exercise 9.41 (Hilbert–Schmidt operators IV)**.** Let
$$A : \mathrm{L}^2(c,d) \longrightarrow \mathrm{C}[a,b]$$
be a bounded operator, and let $(e_j)_{j\in\mathbb{N}}$ be any orthonormal basis of $H := \mathrm{L}^2(c,d)$.

a) Let $x \in [a,b]$. Show that there is a unique $g_x \in \mathrm{L}^2(c,d)$ such that
$$\langle f, g_x\rangle = (Af)(x)$$
for all $f \in \mathrm{L}^2(c,d)$.

b) Show that $\|g_x\| \le \|A\|_{\mathrm{L}^2\to\mathrm{C}}$.

c) Use a) and b) to show that for every $x \in [a,b]$
$$\sum\nolimits_{j=1}^{\infty} |(Ae_j)(x)|^2 \le \|A\|_{\mathrm{L}^2\to\mathrm{C}}^2 .$$

d) Show that $\sum_{j=1}^{\infty} \|Ae_j\|_{\mathrm{L}^2}^2 < \infty$.

e) Show that for $f \in H$,
$$(Af)(x) = \sum_{j=1}^{\infty} \langle f, e_j\rangle\, (Ae_j)(x) \qquad (x \in [a,b])$$
the series being uniformly convergent on $[a,b]$, and absolutely convergent for each $x \in [a,b]$.

(By d), A, considered as an operator from $\mathrm{L}^2(c,d)$ to $\mathrm{L}^2(a,b)$, is an (abstract) Hilbert–Schmidt operator. By Exercise 9.40, A is induced by the kernel

$k = \sum_{j=1}^{\infty} Ae_j \otimes \overline{e_j}$. One has $k(x, \cdot) = g_x$ almost everywhere, for almost all $x \in [a, b]$.)

Exercise 9.42 (Discrete Hilbert–Schmidt Operators I)**.** Here is the sequence space analogue of the HS-integral operators: Let $a = (a_{ij})_{i,j\in\mathbb{N}}$ be an infinite matrix such that

$$\|a\|_2 := \Bigl(\sum\nolimits_{i,j\in\mathbb{N}} |a_{ij}|^2\Bigr)^{1/2} < \infty$$

Show that a induces a linear operator A on ℓ^2 by

$$(Af)(n) := \sum\nolimits_{j=1}^{\infty} a_{nj} f(j) \qquad (n \in \mathbb{N}),$$

and $\|A\|_{\ell^2\to\ell^2} \le \|a\|_2$.

Exercise 9.43 (Discrete Hilbert–Schmidt Operators II)**.** Let $a = (a_{ij})_{i,j}$ be an infinite matrix such that $\sum_{i,j} |a_{ij}|^2 < \infty$. Let $A : \ell^2 \longrightarrow \ell^2$ be the discrete Hilbert–Schmidt operator associated with the matrix a, i.e., A is given by

$$(Af)(n) = \sum_{j=1}^{\infty} a_{nj} f(j) \qquad (f \in \ell^2,\, n \in \mathbb{N}).$$

Show that A is compact.

Exercise 9.44. Consider $X = Y = \mathbb{R}_+$ and the kernel function

$$k(x, y) = \frac{1}{x + y} \qquad (x, y > 0).$$

The associated integral operator $\mathcal{H}$ is called the **Hilbert–Hankel** operator. Formally, it is given by

$$(\mathcal{H}f(x) = \int_0^{\infty} \frac{f(y)}{x + y}\, \mathrm{d}y \qquad (x > 0).$$

Show that $\mathcal{H}$ is a bounded operator on $\mathrm{L}^2(\mathbb{R}_+)$. (Hint: use the same trick as in Example 9.7.) Then show that $\mathcal{H} = \mathcal{L}^2$.

Exercise 9.45. (a little tricky) Let E be a normed space, $A \in \mathcal{L}(E)$. Suppose that A is "approximately right invertible", i.e. there is $B \in \mathrm{Lin}(E)$ such that $\|\mathrm{I} - AB\| < 1$. Show that for each $f \in E$ the equation

$$Au = f$$

has a solution $u \in E$.

Exercise 9.46. Let $A : H \longrightarrow H$ be a bounded linear operator, and $F \subseteq H$ a closed subspace. Suppose that

$$\langle Af, g\rangle = \langle f, Ag\rangle \qquad (f, g \in F).$$

Find an operator $B \in \mathcal{L}(H)$ such that $B^* = B$ and $A - B$ maps F into $F^{\perp}$. (There are several possibilities.)

Exercise 9.47. Let $a < t < b$ and define $f_n := 2n\mathbf{1}_{t-1/n,t+1/n}$ for $n \in \mathbb{N}$ large enough. The let

$$A_n(f) := \int_a^b f_n(s)f(s)\,\mathrm{d}s \qquad (f \in \mathrm{C}[a,b])$$

a) Show that each A_n is a bounded linear functional on $\mathrm{C}[a,b]$ (with respect to the sup-norm, of course) and

$$A_n(f) \to \delta_t(f) = f(t)$$

for each $f \in \mathrm{C}[a,b]$.

b) (more involved) Show that $(A_n)_{n\in\mathbb{N}}$ does not converge in the operator norm of $\mathcal{L}(\mathrm{C}[a,b];\mathbb{C})$.

Exercise 9.48. Let $m \in \mathrm{C}[a,b]$, and consider the functional T_m given by

$$T_m f := \int_a^b m(s)f(s)\,\mathrm{d}s.$$

We can consider this functional on different spaces.

a) Show that $\|T_m\|_{\mathrm{L}^2(a,b)\to\mathbb{C}} = \|m\|_2$.

b) Show that $\|T_m\|_{\mathrm{C}[a,b]\to\mathbb{C}} = \|m\|_1$.
(Hint: consider $T_m(g_\epsilon)$ with $g_\epsilon = \frac{\overline{m}}{|m|^2+\epsilon}$ for $\epsilon > 0$.)

c) Show that $\|T_m\|_{\mathrm{L}^1(a,b)\to\mathbb{C}} = \|m\|_\infty$.
(Hint: find $t \in (a,b)$ such that $|m(t)|$ is very close to $\|m\|_\infty$, then consider $f_n = 2n\mathbf{1}_{(t-1/n,t+1/n)}$ for $n \in \mathbb{N}$ large and apply Exercise 9.47.a).)

Chapter 10

Spectral Theory of Compact Self-adjoint Operators

One of the most important results of finite-dimensional linear algebra says that a symmetric real square matrix A is orthogononally diagonalizable. Equivalently, each such matrix has an orthonormal basis consisting of eigenvectors of A. In this chapter we shall derive an infinite-dimensional version of this result.

10.1. Eigenvalues

Recall that if $A : E \longrightarrow F$ is a linear operator between vector spaces E, F, then an **eigenvalue** of A is each scalar $\lambda \in \mathbb{K}$ such that the **eigenspace**

$$\ker(\lambda \mathrm{I} - A) \neq \{0\},$$

and every $0 \neq f$ such that $Af = \lambda f$ is called an associated **eigenvector**.

In finite dimensions the eigenvalues tell a great deal about the operator. Indeed, the theory of the so-called "Jordan canonical form" says that in the case $\mathbb{K} = \mathbb{C}$ a square matrix is determined up to similarity (change of basis) by the dimensions of the **generalized eigenspaces**

$$\ker(\lambda \mathrm{I} - A)^k$$

where λ runs through the eigenvalues and k the natural numbers.

There is no analogue of such a result for operators on infinite-dimensional spaces. In fact, there are relatively simple operators having no eigenvalues at all.

Example 10.1. Let $H := \mathrm{L}^2(0,1)$ and $A : H \longrightarrow H$ is the multiplication
Ex.10.1 operator $(Af)(t) := t \cdot f(t)$, $t \in (0,1)$. Then A has no eigenvalues.

So let us generalize the notion of eigenvalue a little.

Definition 10.2. Let E be a normed spaces and $A : E \longrightarrow E$ a bounded operator. A scalar $\lambda \in \mathbb{K}$ is called **approximate eigenvalue** of A if there is a sequence $(f_n)_{n\in\mathbb{N}}$ in E such that $\|f_n\| = 1$ for all $n \in \mathbb{N}$ and $\|Af_n - \lambda f_n\| \to 0$.

Note that an eigenvalue is also an approximate eigenvalue. The next example shows that the converse does not hold.

Example 10.3. In the Example 10.1, every $\lambda \in [0,1]$ is an approximate eigenvalue. Indeed, for $n \in \mathbb{N}$ we can find f_n such that

$$\|f_n\|_2 = 1 \quad \text{and} \quad f_n(t) = 0 \quad (|t - \lambda_0| \geq 1/n).$$

(Choose $f_n := c_n \mathbf{1}_{[\lambda_0 - 1/n, \lambda_0 + 1/n]}$ with a suitable constant c_n.) Then

$$\|Af_n - \lambda_0 f_n\|_2^2 = \int_a^b |t - \lambda_0|^2 \, |f_n(t)|^2 \, \mathrm{d}t \leq \frac{4}{n^2} \int_a^b |f_n(t)|^2 \, \mathrm{d}t = \frac{4}{n^2} \to 0.$$

Ex.10.2
Ex.10.3
Ex.10.4

Advice/Comment:
For a matrix A, the collection of its eigenvalues is called the **spectrum** of A. Example 10.1 shows that this notion of spectrum is not very reasonable beyond finite dimensions.

For a general operator A on a Banach space E, its **spectrum** is defined as

$$\sigma(A) := \{\lambda \in \mathbb{K} \mid \lambda \mathrm{I} - A \text{ is not invertible}\}.$$

For matrices, this coincides with the collection of eigenvalues, but in general, the spectrum can be much more complicated. In particular, it turns out that even the notion of approximate eigenvalue is still too restricted to account for a full "spectral theory" of bounded linear operators. However, it suffices for our purposes, and we refer to the standard books on functional analysis for further information.

Lemma 10.4. *Let A be a bounded operator on the Banach space E. If $\lambda\mathrm{I} - A$ is invertible, then λ cannot be an approximate eigenvalue. If $|\lambda| > \|A\|$ then $\lambda\mathrm{I} - A$ is invertible.*

Proof. If $\|Af_n - \lambda f_n\| \to 0$ and $\lambda\mathrm{I} - A$ is invertible, then

$$f_n = (\lambda\mathrm{I} - A)^{-1}(\lambda f_n - Af_n) \to 0$$

which contradicts the requirement that $\|f_n\| = 1$ for all $n \in \mathbb{N}$. Take $|\lambda| > \|A\|$. Then $\|\lambda^{-1}A\| < 1$ and hence

$$\lambda\mathrm{I} - A = \lambda(\mathrm{I} - \lambda^{-1}A)$$

is invertible with

$$(\lambda\mathrm{I} - A)^{-1} = \lambda^{-1}\sum\nolimits_{n=0}^{\infty}(\lambda^{-1}A)^n = \sum\nolimits_{n=1}^{\infty}\lambda^{-(n+1)}A^n$$

(Theorem 9.13). □ Ex.10.5

Example 10.5. In Example 10.1 (the multiplication operator $(Af)(t) = t \cdot f(t)$ on $\mathrm{L}^2(0,1)$), for every $\lambda \in \mathbb{C} \setminus [0,1]$ the operator $\lambda\mathrm{I} - A$ is invertible and the inverse is given by

$$\big[(\lambda\mathrm{I} - A)^{-1}f\big](t) = \frac{1}{\lambda - t}f(t) \qquad (t \in (0,1)).$$

Ex.10.6

The following result gives a hint why we can expect good results for compact operators.

Theorem 10.6. *Let A be a compact operator on a Hilbert space and let $\lambda \neq 0$ be an approximate eigenvalue of A. Then λ is an eigenvalue and* $\ker(A - \lambda\mathrm{I})$ *is finite-dimensional.*

Proof. By definition, there is a sequence $(f_n)_{n\in\mathbb{N}} \subseteq H$ such that $\|f_n\| = 1$ for all n and $\|Af_n - \lambda f_n\| \to 0$. As A is compact, by passing to a subsequence we may suppose that $g := \lim_n Af_n$ exists. Consequently

$$\|\lambda f_n - g\| \le \|\lambda f_n - Af_n\| + \|Af_n - g\| \to 0.$$

Thus $\|g\| = \lim_n \|\lambda g_n\| = \lim_n |\lambda|\,\|g_n\| = |\lambda| \neq 0$. Moreover,

$$Ag = A(\lim_n \lambda f_n) = \lambda \lim_n Af_n = \lambda g$$

which shows that λ is an eigenvalue with eigenvector g. Suppose that $F := \ker(A - \lambda\mathrm{I})$ is infinite dimensional. Then there is an infinite ONS $(e_n)_{n\in\mathbb{N}}$ in F. For $n \neq m$,

$$\|Ae_n - Ae_m\| = \|\lambda e_n - \lambda e_m\| = |\lambda|\,\|e_n - e_m\| = |\lambda|\sqrt{2}.$$

Since $\lambda \neq 0$, the sequence $(Ae_n)_{n\in\mathbb{N}}$ does not have a convergent subsequence, which is in contradiction to the compactness of A, by Theorem 9.24. □

10.2. Self-adjoint Operators

A bounded operator A on a Hilbert space H is called **self-adjoint** or **Hermitian**, if $A^* = A$. By definition of the adjoint, A is self-adjoint if and only if

$$\langle Af, g\rangle = \langle f, Ag\rangle$$

for all $f, g \in H$.

Examples 10.7. a) Each orthogonal projection is self-adjoint, see Exercise 7.4.

b) Let $\lambda \in \ell^\infty$ then the multiplication operator A_λ on ℓ^2 (Example 9.8.6) is self-adjoint if and only if λ is a real sequence.

c) A Hilbert–Schmidt kernel $A = A_k$ on $\mathrm{L}^2(a, b)$ is self-adjoint if $\overline{k(x, y))} = k(y, x)$ for almost all $x, y \in (a, b)$. This is true for instance for the Green's function for the Poisson problem (see Example 9.4).

We shall need the following (technical) result.

Theorem 10.8. *Let A be a bounded self-adjoint operator on a Hilbert space A. Then $\langle Af, f\rangle \in \mathbb{R}$ for all $f \in H$ and*

$$\|A\| = |\!|\!| A |\!|\!| := \sup\{|\langle Af, f\rangle| \mid f \in H,\ \|f\| = 1\}.$$

The quantity $|\!|\!| A |\!|\!|$ is called the **numerical radius** of A.

Proof. One has $\langle Af, f\rangle = \langle f, Af\rangle = \overline{\langle Af, f\rangle}$ so $\langle Af, f\rangle$ is real. By Cauchy–Schwarz

$$|\langle Af, f\rangle| \le \|Af\|\,\|f\| \le \|A\|\,\|f\|^2 = \|A\|$$

if $\|f\| = 1$. This proves that $|\!|\!| A |\!|\!| \le \|A\|$.

For the converse, we write $T \ge 0$ if $T = T^*$ and $\langle Tg, g\rangle \ge 0$ for all $g \in H$. Fix $\alpha > |\!|\!| A |\!|\!|$ and define $B := \alpha\mathrm{I} - A$ and $C = 2\alpha\mathrm{I} - B = \alpha\mathrm{I} + A$. Then
Ex.10.7 $B, C \ge 0$. Let $\beta := 2\alpha$. Then by simple algebra

$$\begin{aligned}\beta^2 B - \beta B^2 &= B(\beta\mathrm{I} - B)B + (\beta\mathrm{I} - B)B(\beta\mathrm{I} - B) \\ &= BCB + CBC.\end{aligned} \tag{10.1}$$

Hence

$$\begin{aligned}\big\langle(\beta^2 B - \beta B^2)f, f\big\rangle &= \langle BCBf, f\rangle + \langle CBCf, f\rangle \\ &= \langle CBf, Bf\rangle + \langle BCf, Cf\rangle \ge 0.\end{aligned}$$

Dividing by β (which is > 0!) yields

$$\begin{aligned}0 &\le \beta B - B^2 = 2\alpha(\alpha\mathrm{I} - A) - (\alpha\mathrm{I} - A)^2 \\ &= 2\alpha^2\mathrm{I} - 2\alpha A - \alpha^2\mathrm{I} + 2\alpha A - A^2 = \alpha^2\mathrm{I} - A^2.\end{aligned}$$

If we write this out, it means that

$$\|Af\|^2 = \langle Af, Af\rangle = \left\langle A^2 f, f\right\rangle \le \left\langle \alpha^2 f, f\right\rangle = \alpha^2 \|f\|^2$$

for all $f \in H$. Taking square roots and the supremum over $\|f\| \le 1$ yields $\|A\| \le \alpha$. Letting $\alpha \searrow \|\|A\|\|$ we arrive at $\|A\| \le \|\|A\|\|$. □

Finally, we collect the spectral-theoretic facts of self-adjoint operators. Recall that a subspace F is called A-**invariant** if $A(F) \subseteq F$.

Lemma 10.9. *Let A be a self-adjoint operator on a Hilbert space. Then the following assertions hold.*

a) *Every eigenvalue of A is real.*

b) *Eigenvectors with respect to different eigenvalues are orthogonal to each other.*

c) *If F is an A-invariant subspace of H, then also $F^\perp$ is A-invariant.*

Proof. a) If $Af = \lambda f$ and $\|f\| = 1$ then

$$\lambda = \lambda \|f\|^2 = \langle \lambda f, f\rangle = \langle Af, f\rangle \in \mathbb{R}$$

by Theorem 10.8. Ex.10.8

b) Suppose that $\lambda, \mu \in \mathbb{R}$ and $f, g \in H$ such that $Af = \lambda f$ and $Ag = \mu g$. Then

$$(\lambda - \mu)\langle f, g\rangle = \langle \lambda f, g\rangle - \langle f, \mu g\rangle = \langle Ax, y\rangle - \langle x, Ay\rangle = 0$$

since $A = A^*$. Hence $\lambda \ne \mu$ implies that $f \perp g$.

c) Finally, let $f \in F, g \in F^\perp$. Then $Af \in F$ and hence $\langle f, Ag\rangle = \langle Af, g\rangle = 0$. As $f \in G$ was arbitrary, $Af \in G^\perp$. □

Example 10.1 shows that even self-adjoint operators need not have eigenvalues. If A is compact, however, this is different.

Lemma 10.10. *Let A be a compact self-adjoint operator on a Hilbert space. Then A has an eigenvalue λ such that $|\lambda| = \|A\|$.*

Proof. By definition, we can find a sequence $(f_n)_n$ in H such that $\|f_n\| = 1$ and $|\langle Af_n, f_n\rangle| \to \|\|A\|\|$. By passing to a subsequence we may suppose that $\langle Af_n, f_n\rangle \to \lambda$, and passing from A to $-A$ if necessary, we may suppose that $\lambda = \|\|A\|\| = \|A\|$. By Theorem 10.6 it suffices to show that λ is an approximate eigenvalue.

We use the terminology of the proof of Theorem 10.8, i.e., we fix $\alpha > \|\|A\|\|$ and let $B := \alpha \mathrm{I} - A$. We have seen in the mentioned proof that $2\alpha B - B^2 \ge 0$.

This means

$$\|(\alpha - A)f_n\|^2 = \langle B^2 f_n, f_n \rangle \le 2\alpha \langle B f_n, f_n \rangle = 2\alpha \langle \alpha f_n - A f_n, f_n \rangle$$

for all $n \in \mathbb{N}$. Letting $\alpha \searrow \|A\| = \lambda$ yields

$$\|(\lambda - A)f_n\|^2 \le 2\lambda \big(\lambda - \langle A f_n, f_n \rangle \big) \to 0$$

which concludes the proof. □

10.3. The Spectral Theorem

We are now in the position to state and prove the main result.

Theorem 10.11 (The Spectral Theorem)**.** *Let A be a compact self-adjoint operator on a Hilbert spacs H. Then A is of the form*

$$Af = \sum_j \lambda_j \langle f, e_j \rangle e_j \qquad (f \in H) \tag{10.2}$$

for some finite or countably infinite ONS $(e_j)_j$ and real numbers $\lambda_j \neq 0$ satisfying $\lim_{j\to\infty} \lambda_j = 0$*. Moreover, $Ae_j = \lambda e_j$ for each j.*

More precisely, the ONS is either $(e_j)_{j=1}^N$ for some $N \in \mathbb{N}$ or $(e_j)_{j\in\mathbb{N}}$. Of course, the condition $\lim_{j\to\infty} \lambda_j = 0$ is only meaningful in the second case.

Proof. We shall find the e_j, λ_j step by step. If $A = 0$ then there is nothing to show. So let us assume that $\|A\| > 0$.

Write $H_1 = H$. By Lemma 10.10, A has an eigenvalue λ_1 such that $|\lambda_1| = \|A\|$. Let $e_1 \in H$ be such that $\|e_1\| = 1$ and $Ae_1 = \lambda_1 e_1$.

Now $F_1 := \operatorname{span}\{e_1\}$ is clearly an A-invariant linear subspace of H_1. By Lemma 10.9.c, $H_2 := F_1^\perp$ is also A-invariant. Hence we can consider the restriction $A|H_2$ of A on H_2 and iterate. If $A|H_2 = 0$, the process stops. If
Ex.10.9 not, since $A|H_2$ is a compact self-adjoint operator of H_2, we can find a unit vector e_2 and a scalar λ_2 such that $Ae_2 = \lambda_2 e_2$ and

$$|\lambda_2| = \|A|H_2\|_{\mathcal{L}(H_2)} \le \|A|H_1\|_{\mathcal{L}(H_1)}.$$

After n steps we have constructed an ONS $e_1, \dots, e_n$ and a sequence $\lambda_1, \dots, \lambda_n$ such that

$$Ae_j = \lambda_j e_j, \quad |\lambda_j| = \|A|H_j\|_{\mathcal{L}(H_j)} \quad \text{where} \quad H_j = \{e_1, \dots, e_{j-1}\}^\perp$$

for all $j = 1, \dots, n$. In the next step define $H_{n+1} := \{e_1, \dots, e_n\}^\perp$, note that it is A-invariant and consider the restriction $A|H_{n+1}$ thereon. This is again a compact self-adjoint operator. If $A|H_{n+1} = 0$, the process stops, otherwise one can find a unit eigenvector associated with an eigenvalue λ_{n+1} such that $|\lambda_{n+1}| = \|A|H_{n+1}\|_{\mathcal{L}(H_{n+1})}$.

Suppose that process stops after the n-th step. Then $A|H_{n+1} = 0$. If $f \in H$ then

$$f - \sum_{j=1}^{n} \langle f, e_j \rangle e_j \in \{e_1, \dots, e_n\}^\perp = H_{n+1}$$

and so A maps it to 0; this means that

$$Af = A \sum_{j=1}^{n} \langle f, e_j \rangle e_j = \sum_{j=1}^{n} \langle f, e_j \rangle Ae_j = \sum_{j=1}^{n} \lambda_j \langle f, e_j \rangle e_j,$$

i.e., (10.2). Now suppose that the process does not stop, and so we have $|\lambda_n| = \|A|H_n\| > 0$ for all $n \in \mathbb{N}$. We claim that $|\lambda_n| \to 0$, and suppose towards a contradiction that this is not the case. Then there is $\epsilon > 0$ such that $|\lambda_n| \geq \epsilon$ for all $n \in \mathbb{N}$. But then

$$\|Ae_j - Ae_k\|^2 = \|\lambda_j e_j - \lambda_k e_k\|^2 = |\lambda_j|^2 + |\lambda_k|^2 \geq 2\epsilon^2$$

for all $j \neq k$. So $(Ae_j)_{j \in \mathbb{N}}$ cannot have a convergent subsequence, contradicting the compactness of A.

Now let $f \in H$ and define

$$y_n := f - \sum_{j=1}^{n-1} \langle f, e_j \rangle e_j \ \in \ \{e_1, \dots, e_{n-1}\}^\perp = H_n.$$

Note that y_n is the orthogonal projection of f onto H_n, and so $\|y_n\| \leq \|f\|$. Hence

$$\|Ay_n\| \leq \|A|H_n\|_{\mathcal{L}(H_n)} \|y_n\| \leq |\lambda_n| \|f\| \to 0;$$

This implies

$$Af - \sum_{j=1}^{n-1} \lambda_j \langle f, e_j \rangle e_j = Ay_n \to 0,$$

which proves (10.2). □

The Spectral Theorem 10.11 contains additional information. Let us denote the index set for the ONS in the Spectral Theorem by J. So $J = \{1, \dots, N\}$ or $J = \mathbb{N}$. Moreover, let

$$P_0 : H \longrightarrow \ker A$$

be the orthogonal projection onto the kernel of A and $P_r := \mathrm{I} - P_0$ its complementary projection. Then we can write

$$Af = 0 \cdot P_0 f + \sum_{j \in J} \lambda_j \langle f, e_j \rangle e_j$$

for all $f \in H$. This formula is called the **spectral decomposition** of A

Corollary 10.12. *Let A be as in the Spectral Theorem 10.11. Then the following assertions hold.*

a) $\overline{\mathrm{ran}}(A) = \overline{\mathrm{span}\{e_j \mid j \in J\}}$ *and* $\ker(A) = \{e_j \mid j \in J\}^\perp$.

b) $P_r f = \sum_{j \in J} \langle f, e_j \rangle e_j$ *for all* $f \in H$.

c) *Every nonzero eigenvalue of A occurs in the sequence $(\lambda_j)_{j\in J}$, and its geometric multiplicity is*

$$\dim\ker(\lambda \mathrm{I} - A) = \operatorname{card}\{j \in J \mid \lambda = \lambda_j\} < \infty.$$

Proof. a) By the formula (10.2) it is obvious that $\operatorname{ran}(A) \subseteq \overline{\operatorname{span}}\{e_j \mid j \in J\}$. For the converse, simply note that since each $\lambda_j \neq 0$, each $e_j = A(\lambda_j^{-1} e_j) \in \operatorname{ran}(A)$.

Since $A = A^*$, Lemma 9.17 yields the orthogonal decomposition $H = \ker(A) \oplus \overline{\operatorname{ran}}(A)$. Hence $P_r = \mathrm{I} - P_0$ is the orthogonal projection onto $\overline{\operatorname{ran}}(A)$, and so the formula in b) follows from the abstract theory in Chapter 7. Moreover, by a)

$$\ker(A) = \overline{\operatorname{ran}}(A)^\perp = \{e_j \mid j \in J\}^\perp,$$

and this is the second identity in a).

c) Suppose that $Af = \lambda f$. If $\lambda \neq \lambda_j$ for every j, then by Lemma 10.9.b) $f \perp e_j$ for every j and so $Af = 0$, by a). This proves the first assertion of
Ex.10.10 c). The remaining statement is left as an exercise. □

Finally, we discuss the abstract eigenvalue equation

$$Au - \lambda u = f \tag{10.3}$$

where $f \in H$ and $\lambda \in \mathbb{K}$ are given, and A is a compact self-adjoint operator on H. By virtue of the Spectral Theorem we shall have complete overview about existence and uniqueness of solutions u. Take $(e_j)_{j\in J}$ and $(\lambda_j)_{j\in J}$ as in the Spectral Theorem.

Theorem 10.13. *In the situation above, precisely one of the following cases holds*

1) *If $0 \neq \lambda$ is different from every λ_j, then $(\lambda \mathrm{I} - A)$ is invertible and*

$$u := (A - \lambda \mathrm{I})^{-1} f = -\frac{1}{\lambda} P_0 f + \sum\nolimits_{j\in J} \frac{1}{\lambda_j - \lambda} \langle f, e_j\rangle\, e_j$$

is the unique solution to (10.3).

2) *If $0 \neq \lambda$ is an eigenvalue of A then* (10.3) *has a solution if and only if $f \perp \ker(\lambda \mathrm{I} - A)$. In this case a particular solution is*

$$u := -\frac{1}{\lambda} P_0 f + \sum\nolimits_{j\in J_\lambda} \frac{1}{\lambda_j - \lambda} \langle f, e_j\rangle\, e_j,$$

where $J_\lambda := \{j \in J \mid \lambda_j \neq \lambda\}$.

3) *If* $\lambda = 0$ *then* (10.3) *is solvable if and only if* $f \in \operatorname{ran}(A)$*; in this case one particular solution is*

$$u := \sum_{j \in J} \frac{1}{\lambda_j} \langle f, e_j \rangle \, e_j,$$

this series being indeed convergent.

Proof. We first show uniqueness in 1). Suppose that u_1, u_2 satisfy (10.3). Then $u := u_1 - u_2$ satisfies $Au = \lambda u$, and since every eigenvalue of A appears in the sequence $(\lambda_j)_j$ we have $u = 0$.

Now we show that in 2) the condition $f \perp \ker(A - \lambda \mathrm{I})$ is necessary for $f \in \operatorname{ran}(A - \lambda \mathrm{I})$. Indeed, if $f = Au - \lambda u$ and $Ag = \lambda g$, then

$$\langle f, g \rangle = \langle Au - \lambda u, g \rangle = \langle u, Ag - \lambda g \rangle = 0$$

since λ is an eigenvalue of $A = A^*$, hence a real number.

To prove existence in 1) and 2) simultaneously, take $0 \neq \lambda$ and define $J_\lambda := \{ j \in J \mid \lambda \neq \lambda_j \}$. (In the situation 1), $J_\lambda = J$.) Take $f \in H$ such that $f \perp \ker(A - \lambda \mathrm{I})$. (In the situation 1), this is always satisfied.) Then we can write

$$f = P_0 f + P_r f = P_0 f + \sum_{j \in J} \langle f, e_j \rangle \, e_j = P_0 f + \sum_{j \in J_\lambda} \langle f, e_j \rangle \, e_j$$

because for $j \notin J_\lambda$ we have $f \perp e_j$. Now note that

$$c := \sup_{j \in J_\lambda} \left| \frac{1}{\lambda_j - \lambda} \right| < \infty,$$

because $\lambda_j \to 0 \neq \lambda$. Hence

$$\sum_{j \in J_\lambda} \left| \frac{\langle f, e_j \rangle}{\lambda_j - \lambda} \right|^2 \le c^2 \sum_{j \in J} \left| \langle f, e_j \rangle \right|^2 \le c^2 \, \|f\|^2 < \infty,$$

by Bessel's inequality. So by Theorem 5.18 and since we are in a Hilbert space, the series

$$v := \sum_{j \in J_\lambda} \frac{1}{\lambda_j - \lambda} \langle f, e_j \rangle \, e_j$$

converges. Define $u := (-1/\lambda) P_0 f + v$. Then

$$\begin{aligned} Au - \lambda u &= Av - \lambda v + P_0 f \\ &= P_0 f + \sum_{j \in J_\lambda} \frac{\lambda_j}{\lambda_j - \lambda} \langle f, e_j \rangle \, e_j - \sum_{j \in J_\lambda} \frac{\lambda}{\lambda_j - \lambda} \langle f, e_j \rangle \, e_j \\ &= P_0 f + \sum_{j \in J_\lambda} \langle f, e_j \rangle \, e_j = f \end{aligned}$$

as was to prove.

In 3), $f \in \operatorname{ran}(A)$ is certainly necessary for the solvability of $Au = f$. Now suppose that $f = Av$ for some $v \in H$. Then by (10.2)

$$\langle f, e_j \rangle = \lambda_j \langle v, e_j \rangle$$

for all $j \in J$. In particular, $\sum_{j \in J} \left| \lambda_j^{-1} \langle f, e_j \rangle \right|^2 < \infty$; since H is a Hilbert space, the series

$$u := \sum_{j \in J} \frac{\langle f, e_j \rangle}{\lambda_j} e_j$$

is convergent. Note that actually $u = P_r v$. Hence

$$Au = AP_r v = AP_0 v + AP_r v = A(P_0 v P_r v) = Av = f.$$

This concludes the proof. □

Exercises

Exercise 10.1. Let $H := L^2(0,1)$ and $A : H \longrightarrow H$ is the multiplication operator $(Af)(t) := t \cdot f(t)$, $t \in (0,1)$. Show that A has no eigenvalues.

Exercise 10.2. Show that 0 is not an eigenvalue but an approximate eigenvalue of the integration operator J on $C[a,b]$.

Exercise 10.3. Consider the left shift L on ℓ^2. Show that $\lambda \in \mathbb{K}$ is an eigenvalue of L if and only if $|\lambda| < 1$.

Exercise 10.4. Let $(e_n)_{n \in \mathbb{N}}$ be the standard unit vectors on ℓ^2 and let $\lambda \in \mathbb{K}$ with $|\lambda| = 1$. Define $x_n := (1/\sqrt{n})(\lambda e_1 + \lambda^2 e_2 + \cdots + \lambda^n e_n)$. Use the sequence $(x_n)_{n \in \mathbb{N}}$ to show that λ is an approximate eigenvalue of the left shift L on ℓ^2.

Exercise 10.5. Consider the integration operator J on $L^2(a,b)$. Show that for every $\lambda \neq 0$, $\lambda I - J$ is invertible and compute a (Hilbert–Schmidt) integral kernel for $(\lambda I - J)^{-1}$.

Exercise 10.6. Let A be the multiplication operator $(Af)(t) = t \cdot f(t)$ on $L^2(0,1)$. Show that for every $\lambda \in \mathbb{C} \setminus [0,1]$ the operator $\lambda I - A$ is invertible and the inverse is given by

$$\left[(\lambda I - A)^{-1} f\right](t) = \frac{1}{\lambda - t} f(t) \qquad (t \in (0,1)).$$

Exercise 10.7 (Positive operators)**.** Let H be a Hilbert space. A bounded self-adjoint operator A on H is called **positive** if $\langle Af, f \rangle \geq 0$ for all $f \in H$. We write $A \geq 0$ if A is positive.

a) Show that if $A, B \in \mathcal{L}(H)$ are self-adjoint and $\alpha, \beta \in \mathbb{R}$ then $\alpha A + \beta B$ is self-sadjoint. Then show that if $A, B \geq 0$ and $\alpha, \beta \geq 0$ then also $\alpha A + \beta B \geq 0$.

b) Let A be self-adjoint such that $A \geq 0$, and let $C \in \mathcal{L}(H)$ be arbitrary. Show that C^*AC is selfadjoint and $C^*AC \geq 0$.

c) Show that A is self-adjoint and $\alpha \geq \|A\|$ then $\alpha \mathrm{I} \pm A \geq 0$.

d) Let A be a positive self-adjoint bounded operator on H. Show that

$$\|Af\|^2 \leq \|A\| \langle Af, f \rangle \tag{10.4}$$

for all $f \in H$. (Hint: Use (10.1) with $B = A$ and $\beta = \|A\|$.)

Exercise 10.8. Let A be a bounded self-adjoint operator and let λ be an approximate eigenvalue of A. Show that $\lambda \in \mathbb{R}$.

Exercise 10.9. Let H be a Hilbert space, let $A : H \longrightarrow H$ be a bounded linear operator, and let $F \subseteq H$ be an A-invariant closed subspace of H. Let $B := A\big|_F$ be the restriction of A to F. Show that the following assertions hold.

a) $\|B\|_{\mathcal{L}(F)} \leq \|A\|_{\mathcal{L}(H)}$.

b) If A is self-adjoint, then B is self-adjoint.

c) If A is compact, then B is compact.

Exercise 10.10. In the situation of the Spectral Theorem, show that

$$\ker(\lambda \mathrm{I} - A) = \operatorname{span}\{e_j \mid \lambda_j = \lambda\}$$

for each $\lambda \neq 0$.

Further Exercises

Exercise 10.11. Let E be a Banach space, $A \in \mathcal{L}(E)$, $\lambda, \mu \in \mathbb{K}$. such that $\lambda \mathrm{I} - A$ and $\mu \mathrm{I} - A$ are invertible. Show that

$$(\lambda \mathrm{I} - A)^{-1} - (\mu \mathrm{I} - A)^{-1} = (\mu - \lambda)(\lambda \mathrm{I} - A)^{-1} (\mu \mathrm{I} - A)^{-1}.$$

(This identity is called the **resolvent identity**.)

Exercise 10.12. Let H be a Hilbert space, $A \in \mathcal{L}(H)$ and $\lambda \in K$. Show that if $\lambda \mathrm{I} - A$ is invertible then $\overline{\lambda} - A^*$ is invertible and and

$$(\overline{\lambda} \mathrm{I} - A^*)^{-1} = (\lambda \mathrm{I} - A)^*.$$

Exercise 10.13. Let E be a Banach space, and let $A \in \mathcal{L}(E)$ be an invertible operator. Let $\lambda \in \mathbb{K} \setminus \{0\}$ such that $\lambda \mathrm{I} - A$ is invertible. Show that $\lambda^{-1}\mathrm{I} - A^{-1}$ is invertible with

$$\left(\lambda^{-1}\mathrm{I} - A^{-1}\right)^{-1} = \lambda \mathrm{I} - \lambda^2(\lambda \mathrm{I} - A)^{-1}.$$

Exercise 10.14. Let E be a Banach space and let $A \in \mathcal{L}(E)$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of scalars such that each $\lambda_n \mathrm{I} - A$ is invertible and

$$\lambda_n \to \lambda \quad \text{and} \quad \|(\lambda_n \mathrm{I} - A)^{-1}\| \to \infty.$$

Show that λ is an approximate eigenvalue of A. (Hint: By definition of the operator norm, find $z_n \in E$ such that $\|z_n\| \le 1$ and $\|R(\lambda_n, A)z_n\| \to \infty$. Define $x_n := R(\lambda_n, A)z_n / \|R(\lambda_n, A)z_n\|$ and show that $\|x_n\| = 1$ and $(\lambda \mathrm{I} - A)x_n \to 0$.)

Exercise 10.15. Let E be a Banach space and let $A \in \mathcal{L}(E)$. Show that 0 is not an approximate eigenvalue if and only if there is $\delta > 0$ such that

$$\|Ax\| \ge \delta \|x\| \qquad (x \in H).$$

Show that in this case $\ker(A) = \{0\}$ and $\operatorname{ran}(A)$ is a closed subspace of E.

Chapter 11

Some Applications

11.1. The Dirichlet-Laplace Operator

The operator

$$\Delta_0 : \mathrm{H}^2(0,1) \cap \mathrm{H}^1_0(0,1) \longrightarrow \mathrm{L}^2(0,1), \qquad \Delta u = u''.$$

is called the one-dimensional **Dirichlet-Laplacian**. It is a version of the Laplace operator (= second derivative) with Dirichlet boundary conditions $u(0) = u(1) = 0$. We abbreviate

$$\mathrm{D}(\Delta_0) := \mathrm{H}^2(0,1) \cap \mathrm{H}^1_0(0,1)$$

and call it the **domain** of the Dirichlet-Laplacian.

In Example 9.4 we have shown that Δ_0 is bijective, and its inverse is given by the Hilbert–Schmidt operator

$$\Delta_0^{-1} f = -G_0 f = -\int_0^1 g_0(\cdot, s) f(s) \,\mathrm{d}s$$

where

$$g_0(t,s) := \begin{cases} s(1-t) & 0 \le s \le t \le 1 \\ t(1-s) & 0 \le t \le s \le 1. \end{cases}$$

Since the operator G_0 is Hilbert–Schmidt, it is compact. Moreover, since g_0 is symmetric and real-valued, G_0 is self-adjoint. Moreover, $\ker(G_0) = \{0\}$, by construction.

To apply the Spectral Theorem, we need to find the non-zero eigenvalues and eigenvectors of G_0.

Lemma 11.1. *The real number $\lambda \neq 0$ is an eigenvalue of G_0 if and only if $-1/\lambda$ is an eigenvalue of Δ_0, and the eigenspaces coincide. Moreover,* *every non-zero eigenvalue of* G_0 *is strictly positive and every corresponding eigenfunction is in* $\mathrm{C}^\infty[0,1]$

Proof. If $G_0 u = \lambda u$ then $u = \lambda^{-1} G_0 u \in \mathrm{D}(\Delta_0)$ and $u'' = -\lambda^{-1} u$. Conversely, if this holds for some $u \in \mathrm{D}(\Delta_0)$, then also $u = -G_0 u'' = \lambda^{-1} G_0 u$ and thus $G_0 u = \lambda u$. This proves the first assertion.

For the second, suppose that $\lambda \neq 0$ and $u \in \mathrm{H}^2(0,1)$ such that $u'' = (-1/\lambda)u$. Then in particular $u'' \in \mathrm{H}^2(0,1) \subseteq \mathrm{C}[0,1]$, and hence $u \in \mathrm{C}^2[0,1]$. By $u'' = (-1/\lambda)u$ again, we now get $u \in \mathrm{C}^4[0,1]$ and iterating this we obtain $u \in \mathrm{C}^\infty[0,1]$. Finally, we write

$$\|u\|_2 = \langle u, u\rangle = -\lambda \langle u'', u\rangle = \lambda \langle u', u'\rangle$$

since $u \in \mathrm{C}^1_0[0,1]$. But $\langle u', u'\rangle \geq 0$ always, and thus either $\lambda > 0$ or $u = 0$. □

Employing the classical theory of differential equations we can now conclude that $G_0 u = \lambda u$ with $\lambda > 0$ if and only if

$$u(t) = \alpha \cos\left(\frac{t}{\sqrt{\lambda}}\right) + \beta \sin\left(\frac{t}{\sqrt{\lambda}}\right) \qquad (0 \leq t \leq 1)$$

for some constants α, β. The boundary condition $u(0) = 0$ forces $\alpha = 0$ and so $\ker(\lambda \mathrm{I} - G_0)$ is at most one-dimensional. On the other hand $u(1) = 0$ and $\beta \neq 0$ forces

$$\sin(1/\sqrt{\lambda}) = 0.$$

This yields the sequence of eigenvalues

$$\lambda_n = \frac{1}{n^2\pi^2} \qquad (n \geq 1)$$

with associated (normalized) eigenfunctions

$$e_n(t) = \frac{\sin(n\pi t)}{\sqrt{2}} \qquad (n \geq 1,\ t \in [0,1])$$

Since G_0 is injective, $\overline{\mathrm{ran}}(G_0) = H$ and according to the spectral theorem the system $(e_n)_{n=1}^\infty$ must be a maximal ONS for $\mathrm{L}^2(0,1)$. Moreover, the operator G_0 can be written as

$$(G_0 f)(t) = \int_0^1 g_0(t,s) f(s)\,\mathrm{d}s = \sum_{n=1}^\infty \left(\frac{1}{2n^2\pi^2}\int_0^1 f(s)\sin(n\pi s)\,\mathrm{d}s\right)\,\sin(n\pi t).$$

with $t \in (0,1)$ and $f \in \mathrm{L}^2(0,1)$. The general theory yields only convergence of the series in the L^2-norm, and *not pointwise*. However, in this particular
Ex.11.1 case, we see that the series converges even **uniformly** in $t \in [0,1]$.

It is even true that

$$g_0(t,s) = \sum_{n=1}^\infty \frac{\sin(n\pi \cdot t)\,\sin(n\pi s)}{2n^2\pi^2}$$

as a convergent series in $L^2((0,1)\times(0,1))$. Ex.11.2

11.2. One-dimensional Schrödinger Operators

We now perturb the Dirichlet-Laplacian by a multiplication operator. More precisely, we let $p \in C[0,1]$ be a fixed *positive* continuous function, and consider

$$S : H^2(0,1) \cap H^1_0(a,b) \longrightarrow L^2(a,b) \qquad Su = -u'' + pu.$$

It is called a one-dimensional **Schrödinger operator** with **potential function** p. We write $D(S) := H^2(0,1) \cap H^1_0(0,1)$ and call it the **domain** of S.

> **Advice/Comment:**
> This operator is a special case of a so-called *Sturm–Liouville* operator. We shall not treat general Sturm–Liouville operators here.

Let us first look at eigenvalues of S.

Lemma 11.2. *If $0 \neq u \in D(S)$ and $Su = \lambda u$, then $\lambda > 0$ and $u \in C^2[0,1]$. In particular, S is injective.*

Proof. If $Su = \lambda u$ then $u'' = pu - \lambda u \in C[0,1]$. Hence $u \in C^2[0,1]$. Then integration by parts yields

$$\begin{aligned} \lambda \|u\|_2^2 &= \langle \lambda u, u\rangle = \langle pu - u'', u\rangle = \langle pu, u\rangle - \langle u'', u\rangle \\ &= \int_0^1 p(s)\,|u(s)|^2 \,\mathrm{d}s + \|u'\|_2^2 \geq \|u'\|_2^2 \geq 0, \end{aligned}$$

since $p \geq 0$ by assumption. If $u \neq 0$ then also $u' \neq 0$ since $u(0) = 0$. Hence $\lambda > 0$. □

We shall now show that for every $f \in L^2(0,1)$ the **Sturm–Liouville problem**

$$u'' - pu = f \qquad u \in H^2(a,b) \cap H^1_0(a,b)$$

has a unique solution. In other words, the operator S is bijective. In fact, we shall construct a **Green's function** for its inverse operator S^{-1}. The method of doing this is classical. One chooses functions $u, v \in C^2[0,1]$ with the following properties:

$$\begin{aligned} u'' &= pu, \quad u(0) = 0, \quad u'(0) = 1 \\ v'' &= pv, \quad u(1) = 0, \quad u'(1) = 1 \end{aligned}$$

We have seen how to find u in Section 9.4 (take $g = 0$ there), and v can be found by similar methods.

Lemma 11.3. *With this choice of u and v, the* **Wronskian** $w := u'v - uv'$ *is a constant non-zero function.*

Proof. One easily computes $(u'v - uv')' = u''v - uv'' = puv - upv = 0$, so w is indeed a constant. If $w = 0$, then $u(1) = (uv')(1) = (u'v)(1) = 0$, and this means that $u \in \mathrm{D}(S)$. Since $u'' = pu$, we would have $Su = 0$, contradicting the injectivity of S (Lemma 11.2). □

Using these two functions we define $c := [J(uf)(1)]$ and

$$Bf := vJ(uf) - uJ(vf) + c\,u \qquad (f \in \mathrm{L}^2(0,1)).$$

Then $Bf \in \mathrm{H}^1_0(0,1)$. If we differentiate the function $vJ(uf)$ we obtain

$$[vJ(uf)]' = v'J(uf) + vuf$$

(we use the product rule for H^1-functions here, see Exercise 8.11). Doing the same for $vJ(uf)$ we obtain

$$(Bf)' = v'J(uf) - u'J(vf) + cu'$$

and this is in $\mathrm{H}^1(0,1)$ again. So $Bf \in \mathrm{H}^2(0,1)$ and we differentiate again, obtaining

$$(Bf)'' = v''J(uf) - u''(J(vf) + cu'' + (uv' - u'v)f = p(Bf) - wf.$$

Since w is a non-zero constant, we can divide by w and define $A := w^{-1}B$. Then $SAf = f$ and, whence S is also surjective and $S^{-1} = B$ is the inverse operator.

Lemma 11.4. *The operator $A = S^{-1}$ is a Hilbert–Schmidt operator with kernel function*

$$k(t,s) = \frac{1}{w}\begin{cases} v(s)u(t) & \text{if} \quad 0 \le s \le t \le 1 \\ v(t)u(s) & \text{if} \quad 0 \le t \le s \le 1. \end{cases}$$

Proof. We have

$$\begin{aligned} w(Af)(t) &= v(t)\int_0^t u(s)f(s)\mathrm{d}s - u(t)\int_0^t v(s)f(s)\,\mathrm{d}s + u(t)\int_0^1 v(s)f(s)\,\mathrm{d}s \\ &= \int_0^t v(t)u(s)f(s)\,\mathrm{d}s + \int_t^1 u(t)v(s)f(s)\,\mathrm{d}s \\ &= \int_0^1 \big(\mathbf{1}_{\{s\le t\}}(t,s)\,v(t)u(s) \;+\; \mathbf{1}_{\{t\le s\}}(t,s)\,u(t)v(s)\big)\,f(s)\,\mathrm{d}s. \end{aligned}$$

This proves the claim. □

The function is k is real-valued and symmetric, hence the operator $A = S^{-1}$ is self-adjoint and compact. Moreover, similar to the Dirichlet-Laplacian one shows that λ is an eigenvalue for A if and only $1/\lambda$ is an eigenvalue for S, with same eigenspaces. In particular, all eigenvalues are strictly positive. The Spectral Theorem yields the existence of an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of $\mathrm{L}^2(0,1)$ and a sequence of strictly positive scalars $(\lambda_j)_j$ such that Ex.11.3

$$Af = \sum_{j=1}^{\infty} \lambda_j \langle f, e_j\rangle e_j \tag{11.1}$$

with $\lambda_1 \ge \lambda_2 \ge \dots$ and $\lambda_j \searrow 0$. Since the integral kernel k is continuous on $[0,1]\times[0,1]$, one can say more: each $e_j \in \mathrm{C}[0,1]$, $\sum_j |\lambda_j|^2 < \infty$ and the series (11.1) converges in the sup-norm. Ex.11.4 Ex.11.5

11.3. The Heat Equation

In this section we look at the following partial differential equation (initial-boundary value problem) on $[0,\infty)\times[0,1]$:

$$\begin{cases} \partial_t u(t,x) = \partial_{xx}u(t,x) - p(x)u(t,x) & (t,x)\in(0,\infty)\times(0,1)\\ u(t,0) = u(t,1) = 0 & (t>0)\\ u(0,x) = f(x) & (x\in(0,1)) \end{cases} \tag{11.2}$$

where $0 \le p \in \mathrm{C}[0,1]$ is the potential and $f : (0,1) \longrightarrow \mathbb{K}$ is a given initial data. This is the one-dimensional **heat equation** for the Schrödinger operator with Dirichlet boundary conditions. If f is continuous it is reasonable to speak of a so-called "classical" solution, i.e., a function $u \in \mathrm{C}([0,\infty)\times[0,1]) \cap \mathrm{C}^{1,2}((0,\infty)\times(0,1))$ that solves the PDE in the ordinary sense. However, the most successful strategy is to allow for a very weak notion of solution (in order to make it easy to find one) and then in a second step investigate under which conditions on f this solution is a classical one.

To find a reasonable candidate for a solution, one shifts the problem from PDEs to functional analysis. We want our solution u to be a function $u : [0,\infty) \longrightarrow \mathrm{L}^2(0,1)$ satisfying

$$u(0) = f \in \mathrm{L}^2(0,1) \quad \text{and} \quad u(t)\in \mathrm{D}(S), \quad u_t(t) = -Su(t) \quad (t>0).$$

Here, the time derivative u_t is to be understood in a "weak sense", i.e., it is a function $u_t : [0,\infty) \longrightarrow \mathrm{L}^2(0,1)$ such that

$$\frac{d}{dt}\langle u(t), v\rangle = \langle u_t(t), v\rangle \qquad (t>0)$$

for all $v \in \mathrm{L}^2(0,1)$.

Equivalently, $-Au_t = u$ for all $t > 0$; writing the operator A in its associated Fourier expansion gives

$$u(t) = -A(u_t(t)) = \sum_{j=1}^{\infty} -\lambda_j \langle u_t(t), e_j \rangle e_j.$$

Since $(e_j)_{j\geq 1}$ is a maximal ONS in $\mathrm{L}^2(0,1)$, we can replace $u(t)$ on the left by its Fourier expansion to obtain

$$\sum_{j=1}^{\infty} \langle u(t), e_n \rangle e_n = \sum_{j=1}^{\infty} -\lambda_j \langle u_t(t), e_j \rangle e_j$$

and comparing Fourier coefficients we arrive at

$$\langle u(t), e_j \rangle = -\lambda_j \langle u_t(t), e_j \rangle \qquad (j \in \mathbb{N}, t > 0).$$

Employing our definition of the time-derivative above leads to the following infinite system of linear ODEs:

$$\frac{d}{dt} \langle u(t), e_j \rangle = \frac{-1}{\lambda_j} \langle u(t), e_j \rangle, \quad \langle u(0), e_j \rangle = \langle f, e_j \rangle \qquad (j \in \mathbb{N}).$$

This is clearly satisfiable by letting

$$u(t) := T(t)f := \sum_{j=1}^{\infty} e^{-t/\lambda_j} \langle f, e_j \rangle e_j \quad (t \geq 0).$$

It is now a quite tedious but manageable exercise in analysis to prove that the series actually defines a smooth function on $(0,\infty) \times [0,1]$ which satisfies the heat equation. Moreover, the initial condition is met in the sense that $\lim_{t \searrow 0} u(t) = f$ in $\mathrm{L}^2(0,1)$, but one can say more depending on whether f is continuous or has even higher regularity.

For each $t \geq 0$ the operator $T(t)$, which maps the initial datum f to the solution $u(t)$ at time t, is bounded. A little calculation shows that

$$T(0) = \mathrm{I} \quad \text{and} \quad T(t+s) = T(t)T(s), \quad (t, s \geq 0)$$

and that for fixed $f \in \mathrm{L}^2(0,1)$ the mapping

$$(0,\infty) \longrightarrow \mathrm{L}^2(0,1), \qquad t \longmapsto T(t)f$$

is continuous. It is an instance of a so-called (strongly continuous) **operator semigroup**. Because of the strong similarity with the scalar exponential function one sometimes writes

$$T(t) = \mathrm{e}^{-tS} \qquad (t \geq 0).$$

Advice/Comment:
The method sketched here is a step into the field of *Evolution Equations*. There one transforms finite-dimensional PDEs into ODE's in diverse Banach spaces and applies functional analytic methods in order to solve them or to study the asymptotic behaviour or other properties of their solutions.

11.4. The Norm of the Integration Operator

Several times we have encountered the operator J given by integration:

$$(Jf)(t) := \int_a^t f(s)\,\mathrm{d}s \qquad (t \in [a,b]).$$

This operator is often called also *the* **Volterra operator**. It is quite easy to compute its norm when considered as acting on $\mathrm{C}[a,b]$ with the supremum norm, see Exercise 9.9. But what is the norm of J when considered as an operator on $\mathrm{L}^2(a,b)$? Of course, J is an integral operator with kernel

$$k(t,s) = \mathbf{1}_{\{s\le t\}}(t,s)$$

and so one can estimate

$$\begin{aligned}\|J\|^2 \le \|k\|_{HS}^2 &= \int_a^b\int_a^b |k(t,s)|^2\,\mathrm{d}s\,\mathrm{d}t\\ &= \int_a^b\int_a^t \mathrm{d}s\,\mathrm{d}t = \int_a^b (t-a)\,\mathrm{d}t = \frac{(b-a)^2}{2}\end{aligned}$$

which gives $\|J\|_{\mathrm{L}^2\to\mathrm{L}^2} \le (b-a)/\sqrt{2}$. But we shall see that we do not have equality here.

The idea is to use the spectral theorem. However, J is not a self-adjoint operator and so one has to use a little trick, based on the following lemma.

Lemma 11.5. *Let A be an arbitrary bounded operator on a Hilbert space. Then A^*A and AA^* are (positive) self-adjoint operators with norm*

$$\|A^*A\| = \|AA^*\| = \|A\|^2.$$

Proof. One has $(A^*A)^* = A^*A^{**} = A^*A$, so A^*A is self-adjoint. It is also positive since $\langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 \ge 0$ for each $x \in H$. Clearly

$$\|A^*A\| \le \|A^*\|\,\|A\| = \|A\|^2$$

by Lemma 9.17. But on the other hand

$$\|Ax\|^2 = \langle Ax, Ax\rangle = \langle A^*Ax, x\rangle \le \|A^*Ax\|\,\|x\| \le \|A^*A\|\,\|x\|^2$$

for all $x \in H$, by Cauchy–Schwarz. Hence $\|A\|^2 \le \|A^*A\|$, by definition of the norm. For the statements about AA^* just replace A by A^* in these results. □

To apply the lemma we recall from Exercise 9.15 that

$$J^*f = \langle f, \mathbf{1}\rangle\,\mathbf{1} - Jf \qquad (f \in \mathrm{L}^2(a,b)).$$

By the above lemma, the operator $A := JJ^*$ is given by

$$\begin{aligned} Af(t) = JJ^* f(t) &= \langle f, \mathbf{1}\rangle\,(t-a) - J^2 f(t) \\ &= \int_a^b (t-a) f(s)\,\mathrm{d}s - \int_a^t (t-s)\,f(s)\,\mathrm{d}s \\ &= \int_a^b \min(t-a, s-a) f(s)\,\mathrm{d}s \end{aligned}$$

for $f \in \mathrm{L}^2(a,b)$, hence is induced by the kernel $k(t,s) := \min(t-a, s-a)$. Since A is a compact self-adjoint operator, by Lemma 10.10 its norm is equal to the largest modulus of an eigenvalue of A. The following lemma shows that our operator A is associated with the **Laplacian with mixed boundary conditions**.

Lemma 11.6. *Fix $\lambda \neq 0$ and $0 \neq u \in \mathrm{L}^2(a,b)$. Then $Au = \lambda u$ if and only if $u \in \mathrm{H}^2(a,b)$ satisfying*

$$u'' = -\lambda^{-1} u, \quad u(a) = 0 = u'(b).$$

Moreover, this is the case if and only if

$$\lambda = \left(\frac{2(b-a)}{(2n-1)\pi}\right)^2 \quad \text{and} \quad u(t) = \cos\left(\frac{(2n-1)\pi(t-a)}{2(b-a)}\right)$$

for some $n \in \mathbb{N}$.

Proof. Suppose that $Au = \lambda u$ with $\lambda \neq 0$. Since A integrates twice and adds a linear polynomial, as in Lemma 11.1 one concludes that $u \in \mathrm{C}^\infty[a,b]$ and $u'' = -\lambda^{-1}u$. Furthermore, $\lambda u(a) = (Au)(a) = 0$ and

$$\lambda u'(b) = (Au)'(b) = \langle u, \mathbf{1}\rangle - Ju(b) = 0.$$

To prove the converse, suppose that $u \in \mathrm{H}^2(a,b)$ satisfies $\lambda u'' = -u$ and $u(a) = 0 = u'(b)$. Then integrating twice yields

$$\lambda u = -J^2 u + c(t-a) + d$$

for some constants c, d. The boundary condition $u(a) = 0$ implies $d = 0$, and taking one derivative yields

$$0 = \lambda u'(b) = -Ju(b) + c$$

which gives $c = Ju(b) = \langle u, \mathbf{1}\rangle$. Together we have indeed $\lambda u = Au$. Next note that integration by parts together with the boundary conditions impy that

$$\|u\|_2 = \langle u, u\rangle = -\lambda \left\langle u'', u\right\rangle = \lambda \left\langle u', u'\right\rangle.$$

Then $u \neq 0$ forces $\lambda > 0$. By classical theory we find that

$$u(t) = \alpha \cos\left(\frac{t-a}{\sqrt{\lambda}}\right) + \beta \sin\left(\frac{t-a}{\sqrt{\lambda}}\right)$$

for some constants α, β. The boundary condition $u(a) = 0$ forces $\alpha = 0$ and

$$0 = u'(b) = \frac{1}{\sqrt{\lambda}} \cos\left(\frac{b-a}{\sqrt{\lambda}}\right).$$

This is the case if and only if

$$\frac{b-a}{\sqrt{\lambda}} = \frac{(2n-1)\pi}{2}$$

for some $n \in \mathbb{N}$. □

Now, back to our original question: we look for the biggest eigenvalue of A and find

$$\|J\|^2 = \|JJ^*\| = \|A\| = \left(\frac{2(b-a)}{\pi}\right)^2$$

and that gives

$$\|J\| = \frac{2(b-a)}{\pi}.$$

This is slightly smaller than $(b-a)/\sqrt{2} = \|J\|_{HS}$. Ex.11.6

11.5. The Best Constant in Poincaré's Inequality

In Chapter 8, Lemma 8.10 we encountered Poincaré's inequality

$$\|u\|_2 \le c \left\|u'\right\|_2 \qquad (u \in \mathrm{H}^1_0(a,b)), \tag{11.3}$$

and have seen there that one can choose $c = \|J\|$. We know now that $\|J\| = 2(b-a)/\pi$, but it is still not clear what the *best constant*

$$c_0 := \inf\left\{c \ge 0 \mid \|u\|_2 \le c \left\|u'\right\|_2 \quad \text{for all} \quad u \in \mathrm{H}^1_0(a,b)\right\}$$

actually is. The aim of this section is to determine c_0 as

$$c_0 = \frac{b-a}{\pi}.$$

As a first step we note the following.

Lemma 11.7. *The space* $\mathrm{H}^2(a,b) \cap \mathrm{H}^1_0(a,b)$ *is dense in* $\mathrm{H}^1_0(a,b)$.

Proof. We sketch the proof and leave details as an exercise. Note that $\mathrm{H}^1_0(a,b) = \{Jf \mid f \in \mathrm{L}^2(a,b),\ f \perp \mathbf{1}\}$. Take $f \in \mathrm{L}^2(a,b)$ with $f \perp \mathbf{1}$, find $f_n \in \mathrm{C}^1[a,b]$ such that $f_n \to f$ in $\|\cdot\|_2$. Then $Jf_n \to Jf$ in $\|\cdot\|_{\mathrm{H}^1}$. Let $g_n(t) := (Jf_n)(t) - \langle f_n, \mathbf{1}\rangle \cdot t$. Then $g \in \mathrm{C}^2[a,b] \cap \mathrm{H}^1_0(a,b)$ and $g_n \to Jf$ in $\|\cdot\|_{\mathrm{H}^1}$. □ Ex.11.7

The lemma shows that Poincaré's inequality (11.3) is equivalent to

$$\|u\|_2 \le c \left\|u'\right\|_2 \qquad (u \in \mathrm{H}^2(a,b) \cap \mathrm{H}^1_0(a,b)$$

By the product rule for H^1 (Exercise 8.11), it is also equivalent to

$$\|u\|_2^2 \le c^2 \left\langle u, -u'' \right\rangle \qquad (u \in \mathrm{H}^2(a,b) \cap \mathrm{H}^1_0(a,b).$$

(Note that the continuous(!) function uu' vanishes at the boundary.) But now recall that we have

$$\mathrm{H}^2(a,b) \cap \mathrm{H}^1_0(a,b) = \mathrm{D}(\mathcal{L}_0) = \{G_0 f \mid f \in \mathrm{L}^2(a,b)\}$$

is the domain of the Dirichlet–Laplacian. (Actually, we did it on $(0,1)$, but the arguments on the general interval (a,b) are analogous, with obvious changes.) So if we write $-f = u''$ and $u = G_0 f$ in the inequality above, it is equivalent to

$$\|G_0 f\|^2 \le c^2 \langle G_0 f, f\rangle \qquad (f \in \mathrm{L}^2(a,b)). \tag{11.4}$$

By Cauchy–Schwarz,

$$\langle G_0 f, f\rangle \le \|G_0 f\| \, \|f\| \le \|G_0\| \, \|f\|^2$$

and so (11.4) implies that $\|G_0\|^2 \le c^2 \|G_0\|$, whence $c^2 \ge \|G_0\|$. So the optimal c satisfies

$$c_0 \ge \sqrt{\|G_0\|}.$$

On the other hand, G_0 is a *positive* self-adjoint operator in the sense of Exercise 10.7. By part d) of that exercise we have

$$\|G_0 f\|^2 \le \|G_0\| \langle G_0 f, f\rangle \qquad (f \in \mathrm{L}^2(a,b)).$$

Hence Poincaré's inequality (11.3) is true with $c^2 = \|G_0\|$. To sum up, we have shown that

$$c_0^2 = \sqrt{\|G_0\|}.$$

Now, the Spectral Theorem tells us that $\|G_0\|$ equals the largest absolute value of an eigenvalue of G_0, which is $(b-a)^2/\pi^2$, with corresponding eigenfunction

$$e_1(t) = \sqrt{\frac{b-a}{2}} \sin\left(\frac{\pi(t-a)}{b-a}\right) \qquad (t \in [a,b]).$$

(Adapt the considerations about the operator on $(0,1)$ from above.) Hence indeed $c_0 = (b-a)/\pi$, with the function e_1 as extremal case.

Exercises

Exercise 11.1. Show that for every $f \in \mathrm{L}^2(0,1)$ the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2n^2\pi^2} \int_0^1 f(s) \sin(n\pi s) \, \mathrm{d}s \right) \sin(n\pi t).$$

converges uniformly in $t \in [0,1]$.

Exercise 11.2. Suppose that $(k_n)_n$ is a Cauchy sequence in $\mathrm{L}^2(X \times Y)$, suppose that $k \in \mathrm{L}^2(X \times Y)$ is such that $A_{k_n} f \to A_k f$ in $\mathrm{L}^2(X)$ for every $f \in \mathrm{L}^2(Y)$. Show that $k_n \to k$ in $\mathrm{L}^2(X \times Y)$.

Apply this to prove that

$$g_0(t,s) = \sum_{n=1}^{\infty} \frac{\sin(n\pi \cdot t) \, \sin(n\pi s)}{2n^2\pi^2}$$

as a convergent series in $\mathrm{L}^2((0,1) \times (0,1))$. (Hint: show that the series converges even uniformly on the square $0 \le, s, t \le 1$.)

Exercise 11.3. Let S be the Schrödinger operator considered in Section 11.2, and let $\lambda \neq 0$. Show that λ is an eigenvalue for $A = S^{-1}$ with eigenfunction $u \neq 0$, then $1/\lambda$ is an eigenvalue S with same eigenfunction u.

Exercise 11.4. Let $(e_j)_{j \in \mathbb{N}}$ be an ONS in $\mathrm{L}^2(a,b)$, let $\lambda \in \ell^\infty$ such that $\lambda_j \neq 0$ for all $j \in \mathbb{N}$. and let $A : \mathrm{L}^2(a,b) \longrightarrow \mathrm{L}^2(a,b)$ be given by

$$Af := \sum\nolimits_{j=1}^{\infty} \lambda_j \, \langle f, e_j \rangle \, e_j$$

for $f \in \mathrm{L}^2(a,b)$. Suppose further that $\operatorname{ran}(A) \subseteq \mathrm{C}[a,b]$ and $A : \mathrm{L}^2(a,b) \longrightarrow \mathrm{C}[a,b]$ is bounded. Show that

a) Each $e_j \in \mathrm{C}[a,b]$.

b) The series defining A converges uniformly, i.e., in the sup-norm.

c) Show that

$$\sum\nolimits_{j=1}^{\infty} |\lambda_j|^2 < \infty$$

(Hint: Use Exercise 9.41.)

Exercise 11.5. In the construction of the Green's function k for the Schrödinger operator S on $[0,1]$ with positive potential $p \ge 0$, show that that $u, u', v' \ge 0$ and $v \le 0$ on $[0,1]$. Conclude that the Green's function k is also positive.

Exercise 11.6. Discuss the spectral composition of the operator $A = JJ^*$ on $\mathrm{L}^2(a,b)$. In which sense do the appearing series converge?

Exercise 11.7. Fill in the details in the proof of Lemma 11.7.

Exercise 11.8. Let $Bu = u''$ defined on $\mathrm{D}(B) := \{u \in \mathrm{H}^2(a,b) \mid u'(a) = 0,\, u(b) = 0\}$. Show that

$$B : \mathrm{D}(B) \longrightarrow \mathrm{L}^2(a,b)$$

is bijective. Compute its inverse operator $A := B^{-1}$, show that it is a self-adjoint Hilbert–Schmidt operator mapping $\mathrm{L}^2(a,b)$ boundedly into $\mathrm{C}[a,b]$. Compute the eigenvalues and the corresponding eigenfunctions and discuss the spectral representation.

Exercise 11.9. Let $P : \mathrm{L}^2(0,1) \longrightarrow \{\mathbf{1}\}^\perp$ be the orthogonal projection. Show that the operator $A := PJ^2P$ is a self-adjoint Hilbert–Schmidt operator and compute its integral kernel. (See also Exercise 9.26.) Determine the eigenvalues of A and find an orthonormal basis of $\mathrm{L}^2(0,1)$ consisting of eigenfunctions of A.

Exercise 11.10. (more involved) Consider the operator $A = (1/2)(J^2 + J^{*2})$ on $\mathrm{L}^2(a,b)$. Determine its integral kernel. Show that $\operatorname{ran}(A) \subseteq \mathrm{H}^2(a,b)$ and $(Af)'' = f$. Find the right boundary conditions to characterize $\operatorname{ran}(A)$. Then determine the spectral decomposition of A.

Appendix A

Background

A.1. Sequences and Subsequences

Let X be an arbitrary non-empty set. Then each mapping

$$x : \mathbb{N} \longrightarrow X$$

is called a **sequence** in X. If x is such a sequence in X then one often writes

$$x_n \quad \text{instead of} \quad x(n)$$

for the n-th member and $x = (x_n)_{n\in\mathbb{N}} \subseteq X$ to denote the whole sequence.

Advice/Comment:
Note the difference between $x_n \in X$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$. The first denotes the n-th member of the sequence and is an element of X, the second denotes the sequence as a whole, and is an element of the set of functions from $\mathbb{N}$ to X.

Note also the difference between

$$(x_n)_{n\in\mathbb{N}} \quad \text{and} \quad \{x_n \mid n \in \mathbb{N}\}.$$

The first denotes the sequence, the second denotes the *range* of the sequence. For example, the range of the sequence $x_n := (-1)^n$, $n \in \mathbb{N}$, is

$$\{(-1)^n \mid n \in \mathbb{N}\} = \{1, -1\}.$$

If $x = (x_n)_{n\in\mathbb{N}} \subseteq X$ is a sequence in X, then a **subsequence** of x is a sequence of the form

$$y_n := x_{\pi(n)} \qquad (n \in \mathbb{N})$$

where $\pi : \mathbb{N} \to \mathbb{N}$ is a *strictly* increasing map. Intuitively, π selects certain members of the original sequence with increasing indices. One sometimes writes k_n instead of $\pi(n)$.

A.2. Equivalence Relations

An equivalence relation is the mathematical model of a fundamental operation of the human mind: forgetting differences between objects and identifying them when they share certain properties. For example, with respect to gender, all men are equal to each other, and all women are. This "being equal" is not at all hypothetical, since in certain circumstances people are actually treated as such. Indeed, a public toilet for instance treats all men the same (they may enter the left door, say) but differently from all women (who are only allowed to enter the right door). For a public toilet only gender counts, and nothing else.

Mathematically, an **equivalence relation** on a set X is a binary relation $\sim$ on X such that the following three axioms

1) $x \sim x$ **(reflexivity)**
2) $x \sim y \quad \to \quad y \sim x$ **(symmetry)**
3) $x \sim y,\ y \sim z \quad \to \quad x \sim z$ **(transitivity)**

are satisfied for all $x, y, z \in X$. If $\sim$ is an equivalence relation on a set X, then to each $x \in X$ one can define its **equivalence class**

$$[x] := [x]_\sim := \{y \in X \mid x \sim y\},$$

the set of all elements of X that are equivalent to x. Two such classes are either equal or disjoint: $[x] = [a]$ iff $x \sim a$. One collects all equivalence classes in a new set and defines

$$X/\!\sim \ := \ \{[x] \mid x \in X\}.$$

Suppose one has a binary operation $*$ on X such that if $x \sim a$ and $y \sim b$ then $x * y \sim a * b$. (One says that the operation $*$ is **compatible** with the equivalence relation $\sim$.) Then one can **induce** this operation on $X/\!\sim$ by defining

$$[x] * [y] := [x * y].$$

By hypothesis, this definition does not depend on the choice of representatives, hence is a good definition. One says that the operation $*$ on $X/\!\sim$ is **well-defined**.

This mechanism works with functions of more variables and with relations in general. In this way structural elements (relations, operations) are transported ("induced") on the new set $X/\!\sim$.

The standard examples are from algebra. For instance, if U is a subgroup of an abelian group G then one defines

$$x \sim_F y \quad :\Longleftrightarrow \quad x - y \in U$$

for $x, y \in G$. This is an equivalence relation on G (check it!). Let us denote the equivalence class containing $x \in G$ by $[x]$, as above. Then $[x] = [y]$ iff $x - y \in G$ and $[x] = x + U$ as sets. The set of equivalence classes

$$G/U := \{[x] \mid x \in G\}$$

is called the **factor group** or **quotient group**. It is itself an abelian group with respect to the operation

$$[x] + [y] := [x + y] \qquad (x, y \in G).$$

(Of course one has to check that these are well defined, i.e., that the sum is compatible with the equivalence relation.) The mapping

$$s : G \longrightarrow G/U, \quad sx := [x] \quad (x \in G)$$

is then a group homomorphism is called the **canonical surjection**.

A typical example occurs when $G = \mathbb{Z}$ and $U = n\mathbb{Z}$ for some natural number $n \in \mathbb{N}$. Other examples are quotient spaces of vector space (see below).

System of Representatives. Let $\sim$ be an equivalence relation on a non-empty set X. Each member of an equivalence class is called a **representative** for it. So each $y \in [x]$ is a representative for $[x]$.

A mapping $r : X/\sim \longrightarrow X$ assigning to each equivalence class a representative for it, i.e., such that $[r(t)] = t$ for all $t \in X/\sim$, is called a **system of representatives**. For example if one fixes a natural number $n \in \mathbb{N}$ then one can consider the quotient group $\mathbb{Z}/n\mathbb{Z}$ and $\{0, \dots, n-1\}$ would be a system of representatives for it.

An important theoretical tool in mathematics is the following set-theoretic axiom.

Axiom of Choice. *For each equivalence relation on a non-empty set X there exists a system of representatives.*

A.3. Ordered Sets

A **partial ordering** of a set X is a binary relation $\leq$ on X such that the following three axioms

1) $x \leq x$ **(reflexivity)**

2) $x \leq y$ and $y \leq x \quad \rightarrow \quad x = y$ **(antisymmetry)**

3) $x \le y,\ y \le z \quad \rightarrow \quad x \le z$ **(transitivity)**

are satisfied for all $x, y, z \in X$. A **(partially) ordered set** (poset) is a pair $(X, \le)$ where X is a set and $\le$ is a partial ordering of X. If one has in addition that

4) $x \le y \quad \text{or} \quad y \le x$

for all $x, y \in X$, then $(X, \le)$ is called a **totally ordered set**.

The applicability of the order concept is immense. Although $\le$ is the generic symbol for an ordering, in concrete situation other symbols may be used.

Examples A.1. a) If Ω is a set, then $X := \mathcal{P}(\Omega)$ (the power set of Ω) is partially ordered either by set inclusion $\subseteq$ or by set containment $\supseteq$.

b) $X := \mathbb{N}$ is partially ordered by $n|m$ (meaning that m is divisible by n).

c) $\mathbb{R}$ is totally ordered by the usual $\le$.

d) $\mathbb{R}^d$ is partially ordered by $x \le y$ being defined as $x_j \le y_j$ for all $j = 1, \dots, d$.

e) If $(X, \le)$ is a poset then $x \le_r y$ defined by $y \le x$ and usually written $x \ge y$, defines a new partial ordering on X, called the **reverse ordering**.

f) If $(X, \le)$ is a poset and $A \subseteq X$, then A is itself a poset with respect to the ordering induced by the ordering of X. For example, $\mathbb{N}$ is ordered by the ordering coming from $\mathbb{R}$.

Let $(X, \le)$ be an ordered set and let $A \subseteq X$. Then $x \in X$ is called an **upper bound** for A if $a \le x$ for all $a \in A$ (sometimes written as $A \le x$); and $x \in X$ is called a **lower bound** for A if $x \le a$ for all $a \in A$. A **greatest** element of A is an upper bound for A that moreover *belongs to* A. Analogously, a **least** element of A is a lower bound for A that belongs to A.

Greatest (or least) elements may not exist. However, if they exist, they are unique: if $a, b \in A$ are both greatest elements of A, then $a \le b$ and $b \le a$, so $a = b$ by antisymmetry.

The **supremum**, or the **least upper bound**, of A is the least element of the set of all upper bounds for A, provided such an element exists. In the case that it exists, it is denoted by $\sup A$ and it is characterised uniquely by the properties:

$$A \le \sup A \quad \text{and} \quad \forall\, x \in X : A \le x \rightarrow \sup A \le x.$$

Analogously, the greatest element of the set of all lower bounds of A, if it exists, is called an **infimum** of A.

An element $a \in A$ is called **maximal** in A if no other element of A is strictly greater, i.e., if it satisfies

$$\forall x \in A : (a \leq x \implies a = x).$$

Analogously, $a \in A$ is called **minimal** in A if no other element of A is strictly smaller. Maximal (minimal) elements may no exist, and when they exist they may not necessarily be unique. We give an important criterium for a poset to have a maximal element.

Theorem A.2 (Zorn's Lemma). *Let $(X, \leq)$ be a poset such that every totally ordered subset $A \subseteq X$ (a so-called "chain") has an upper bound. Then X has a maximal element.*

Proof. Zorn's Lemma is equivalent to the Axiom of Choice. See [**12**, Appendix] for a proof of Hausdorff's maximality theorem, which directly implies Zorn's lemma. □

A.4. Countable and Uncountable Sets

A set X is called **countable** if there is a surjective mapping $\varphi : \mathbb{N} \longrightarrow X$, i.e., X can be exhausted by a sequence: $X = \{x_n \mid n \in \mathbb{N}\}$. Note that in this case one can even find a *bijective* mapping between $\mathbb{N}$ and X, by discarding double occurences in the sequence $(x_n)_{n\in\mathbb{N}}$.

Clearly, finite sets are countable. Subsets of countable sets are countable. If X is a set and $A_n \subseteq X$ is countable for every $n \in \mathbb{N}$, then

$$A := \bigcup_{n\in\mathbb{N}} A_n \subseteq X$$

is countable, too. If X, Y are countable, then so is their Cartesian product

$$X \times Y = \{(x, y) \mid x \in X,\, y \in Y\}.$$

In particular, the sets $\mathbb{N}^2, \mathbb{N}^3, \dots$ are all countable.

A set that is not countable is called **uncountable**. Any non-void open interval of the real numbers is uncountable, by the famous diagonal proof of Cantor's.

A.5. The Completeness of the Real Numbers

As you may know, there is no science without presuppositions. One has to start somewhere, take something for granted. A proof is an argument that convinces us of a statement, but the very argument has to recur on our pre-established knowledge, things that we call "facts". However, these "facts" are only conditional to more basic "facts". In mathematics, the most basic "facts" are called *axioms*. Ideally, all other mathematical statements should rest logically on those axioms, and on nothing else.

You may know such axioms from elementary (synthetic) geometry. In the beginning of Euclid's *Elements* the basic concepts of *point, line, plane* are introduced together with some very basic statements about them, and all other statements are derived from these basic ones by means of logical arguments. This has been a role model for over 2000 years, until in the 19th century people became unsatisfied with the old approach. The discussion centered around the problem, what points and lines "really" are. (Euclid had given 'definitions', which were at best cloudy, and anyway never used as such in all his treatment.) Also, the invention (or discovery?) of non-euclidean geometries shattered the old dogma that mathematics is about the "real" (=physical) world. Finally, Hilbert in his 1899 treatise "Grundlagen der Geometrie" gave the first modern axiomatisation of geometry, cleared away the defects of Euclid, and moreover paved the way for a more flexible, non-(meta)physical view of mathematics. According to this new approach, axioms can be chosen freely depending on what you want to model with them. The actual *meaning* of the words is inessential, what matters are the relations described by the axioms.

Although being fundamental from a logical point of view, axioms have no conceptual priority. An axiom is not "more true" than any other statement, but axioms help us to *organise our knowlege*. In fact, within the modern approach there is no *intrinsic* criterium of truth other than the mere logical consistency. We cannot ask whether an axiom (or better: axiom system) is true or not, unless we mean by it whether it describes appropriately what it is supposed to describe.

In the (hi)story of mathematics, real numbers have been a pain in the neck ever since this concept has appeared. And indeed, what seems so natural to us was unknown for a long time. The ancient Greeks had a very different concept of a "number" than we do: they considered only the positive natural numbers as numbers (a concept associated with the process of counting) and objects as the rationals or the reals were not known. However, they had the related concepts of a length of a line segment and of a ratio of quantities, e.g., of lengths. Pythagoras discovered that the ratio of the length of the diagonal of a square and its side-length is not "rational", a fact that we express nowadays by saying that $\sqrt{2}$ is an irrational number.

The Greek fixation on geometry hindered very much the development of the number concept; for example, an algebraic equation like $x^2 + x = 3$ was inconceavable by the Greeks since x would represent a length of a line, and x^2 the area of a square. And how the hell could you add such things? So a big step towards the modern concept of number was by stripping off geometric interpretations and focusing on algebraic properties, a step that is already present in the work of Diophantus of Alexandria around 250 AD, but was properly established only after the important work of Arab mathematicians after 800 AD. (The very word "algebra" derives from Arab.) It was then Descartes in the 17th century who showed how to turn the things upside down and make numbers the fundamental entities; geometry was designed to become a subdiscipline of arithmetic.

In the 17th century the invention of the calculus by Newton and Leibniz revolutionised the world. They introduced something new, namely the concept of a *limit*, although quite intuitively in the beginning. Despite the obvious success of the new mathematics, many things remained unclear and were subject to harsh critique (e.g. by Berkeley). The many "infinities" around led to paradoxes, a fact that a modern student may understand by contemplating over the value of ∞/∞. In the beginning of the 19th century, the situation had become unbearable (see Bolzano's 1851 treatise "Paradoxien des Unendlichen"), also because since Fourier's theory of heat, the notion of a "function" was questioned. It was Georg Cantor, who finally led the way out by coining the notion of a "set", the most fundamental notion of mathematics to-day. Using this concept and some very intuitive properties, he showed how one may start from natural numbers and successively "construct" mathematical sets that have all the properties we expect from integers, rationals and real numbers. (The concept of "function" was given a precise set-theoretic definition by Dedekind shortly after, and Weierstrass gave the first proper definition of "limit" at around the same time.) The "arithmetisation" of analysis had been successful.

Axioms for the real numbers. Instead of constructing the real numbers from more elementary objects, one might as well give axioms for real numbers directly instead of for sets. These axioms usually consist of three groups which describe that the real numbers form a "complete, totally ordered, archimedean field". The different structural elements are

1) the algebraic structure: $\mathbb{R}$ is a *field*, i.e., one may add, subtract, multiply and divide according to the usual rules;
2) the order structure: $\mathbb{R}$ is *totally ordered*, and the order is compatible with the algebraic structure (e.g. if $x, y \geq 0$ then also $x + y \geq 0 \dots$);
3) the order is *archimedean*, i.e., if $0 < x, y$ then there is a natural number n such that $y < x + \cdots + x$ (n-times);
4) $\mathbb{R}$ is *complete*.

The completeness axiom (iv) comes in different forms, all equivalent in the presence of the other axioms. It is a matter of personal taste which one considers more fundamental, but the common spirit is the geometric intuition of the continuum: the real numbers have no holes.

Theorem A.3. *In the presence of the axioms* 1)–3) *the following statements are equivalent:*

(i) *Every bounded sequence of real numbers has a convergent subsequence.*

(ii) *Every Cauchy sequence of real numbers converges.*

(iii) *Every monotonic and bounded sequence of real numbers converges.*

(iv) *Every non-empty set of real numbers, bounded from above (below), has a supremum (infimum).*

Proof. (i)⇒(ii): This follows from the fact that a Cauchy sequence is bounded, and from the fact that a Cauchy sequence having a convergent subsequence must converge, see Lemma 5.2.c).

(ii)⇒(iii): If a monotonic sequence is not Cauchy, it has a subsequence with every two points having distance larger than a certain $\epsilon > 0$. The achimedean axiom prevents the sequence from being bounded. This shows that a monotonic and bounded sequence is Cauchy, and hence converges, by hypothesis.

(iii)⇒(iv): Suppose that A is a set of real numbers, bounded by b_0. Choose any $a_0 \in A$. (This works since A is assumend to be not empty.) Consider $x := (a_0 + b_0)/2$. Either x is also an upper bound for A, then set $a_1 := a_0$, $b_1 := x$; or x is not an upper bounded for A, and then there must exist $x < a_1 \in A$. In this case we set $b_1 := b_0$. In either situation, b_1 is an upper bound for A, $a_1 \in A$, and $b_1 - a_1 \leq (b_0 - a_0)/2$. Continuing inductively we construct an increasing sequence $a_n \leq a_{n+1}$ of elements in A and a decreasing sequence $b_{n+1} \leq b_n$ of upper bounds of A such that $b_n - a_n \to 0$. By hypothesis, both sequences converge to a and b, say. Clearly b is an upper bound for A, but since a_n comes arbitrarily close to b, there cannot be any strictly smaller upper bound of A. So $b = \sup A$.

(iv)⇒(i): Suppose that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Then one finds $a < b$ such that $x_n \in [a, b]$ for all n. Consider the set $A_n := \{x_k \mid k \geq n\}$. As $A_n \subseteq [a, b]$ there exists $a_n := \sup A_n \in [a, b]$, by hypothesis. The set $A := \{a_n \mid n \in \mathbb{N}\}$ is contained in $[a, b]$ and hence $x := \inf_n a_n$ exists. Using this, it is easy to find a subsequence of $(x_n)_{n\in\mathbb{N}}$ that converges to x. But since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, it itself must converge to x, see Lemma 5.2c). □

Property (i) is the usual metric completeness, (iii) is order completeness or the **Dedekind Axiom**, and property (ii) is usually stated under the name **Bolzano–Weierstrass**.

Corollary A.4. *The euclidean space* $\mathbb{R}^d$ *is complete.*

Proof. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the euclidean metric. Then, since $|x_{n,j} - x_{m,j}| \le \|x_n - x_m\|_2$, each coordinate sequence $(x_{nj})_{n\in\mathbb{N}}$, $j = 1, \dots, d$, is a Cauchy sequence in $\mathbb{R}$. By the completeness of $\mathbb{R}$, it must have a limit $x_{\infty,j}$. Is now easy to see that $x_\infty := (x_{\infty,1}, \dots, x_{\infty,d})$ is the limit of $(x_n)_{n\in\mathbb{N}}$. □

Corollary A.5 (Bolzano–Weierstrass)**.** *A subset* $A \subseteq \mathbb{R}^d$ *is (sequentially) compact iff it is closed and bounded.*

Proof. Suppose that A is sequentially compact; then A must be bounded, since a sequence which wanders off to "infinity" cannot have a convergent subsequence. It also must be closed, since if $(x_n)_{n\in\mathbb{N}} \subseteq A$ and $x_n \to x \in \mathbb{R}^d$ then by hypothesis there is a subsequence $(x_{n_k})_k$ which converges to some $a \in A$. But clearly $x_{n_k} \to x$ as well, and since limits are unique, $x = a \in A$. Hence A is closed, by Definition 4.1.

Conversely, suppose that A is closed and bounded, take any sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ and write $x_n = (x_{n1}, \dots, x_{nd})$. Then every coordinate sequence $(x_{nj})_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded. By the Bolzano-Weierstrass axiom/theorem, one finds $\pi_1 : \mathbb{N} \to \mathbb{N}$ strictly increasing, such that $x_{\infty,1} := \lim_n x_{\pi_1(n),1}$ exists. The same argument yields a subsequence $\pi_2 : \mathbb{N} \longrightarrow \mathbb{N}$ such that also $x_{\infty,2} := \lim_n x_{\pi_1(\pi_2(n)),2}$ converges. Continuing in this manner one finds $\pi_j : \mathbb{N} \to \mathbb{N}$ strictly increasing such that

$$x_{\infty,j} := \lim_{n\to\infty} x_{\pi_1 \dots \pi_j(n),j}$$

exists for every $j = 1, \dots j$. Setting $\pi := \pi_1 \circ \dots \circ \pi_d$, we have that $x_{\pi(n),j} \to x_{\infty,j}$ for all j, and hence

$$x_{\pi(n)} \to x_\infty := (x_{\infty,1}, \dots, x_{\infty,d})$$

in $\mathbb{R}^d$. Since A is closed, $x_\infty \in A$, and we are done. □

Note that one may exchange $\mathbb{R}$ for $\mathbb{C}$ in the previous two statements: also $\mathbb{C}^d$ is complete and a subset of $\mathbb{C}^d$ is compact iff it is closed and bounded. This is true since, metrically, $\mathbb{C}^d = \mathbb{R}^{2d}$.

A.6. Complex Numbers

Complex numbers can be constructed from the real numbers in many different ways. (Recall from the previous section that it is not important what complex numbers "really are", but which properties they satisfy; as different constructions lead to objects with the same properties, we may consider them equivalent and choose freely each one of them.) The easiest, probably, of these constructions goes back to Gauss and is based on the geometric piucture of complex numbers as points in the euclidean plane.

More precisely, we *define* the set of **complex numbers** $\mathbb{C}$ as

$$\mathbb{C} := \mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$$

the set of pairs of real numbers. If $z = (a, b)$ is a complex number, then $a := \operatorname{Re} z$ is the **real part** and $b := \operatorname{Im} z$ is the **imaginary part** of z. Clearly a complex number is uniquely determined by its real and imaginary part.

The algebraic operations on $\mathbb{C}$ are given by

$$\begin{aligned}(x, y) + (a, b) &:= (x + a, y + b)\\ (x, y) \cdot (a, b) &:= (xa - yb, xb + ya)\end{aligned}$$

for all $(x, y), (a, b) \in \mathbb{C}$. One then checks that all usual rules of computation (associativity and commutativity of addition and multiplication, distributivity) are valid. Then one realises that

$$(x, 0) + (y, 0) = (x + y, 0) \quad \text{and} \quad (x, 0) \cdot (y, 0) = (xy, 0)$$

for all real numbers $x \in \mathbb{R}$. So the complex numbers of the form $(x, 0)$ behave the "same" as the corresponding real numbers, whence there is no confusion writing x instead of $(x, 0)$. Defining the **imaginary unit** i by

$$\mathrm{i} := (0, 1)$$

and using the writing convention $x = (x, 0)$ for real numbers, we see that every complex number $z = (a, b)$ can be uniquely written as

$$z \;=\; a + \mathrm{i}b \;=\; \operatorname{Re} z + \operatorname{Im} z \cdot \mathrm{i}$$

Moreover, $\mathrm{i}^2 = -1$, and hence $1/\mathrm{i} = -\mathrm{i}$.

Two more operations play an imporant role for complex numbers. The first is **conjugation** defined by

$$\overline{z} := a - \mathrm{i}b \quad \text{if} \quad z = a + \mathrm{i}b,\ a, b \in \mathbb{R},$$

or, equivalently, by $\operatorname{Re} \overline{z} := \operatorname{Re} z, \operatorname{Im} \overline{z} := -\operatorname{Im} z$. Then

$$\operatorname{Re} z = \frac{1}{2}(z + \overline{z}) \quad \text{and} \quad \operatorname{Im} z = \frac{1}{2\mathrm{i}}(z - \overline{z}),$$

and $\overline{z} = z$ if and only if $z \in \mathbb{R}$. The following computation rules are easy to check:

$$\overline{z+w} = \overline{z} + \overline{w}, \quad \overline{zw} = \overline{z}\,\overline{w}, \quad z\overline{z} = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2.$$

The **modulus** or **absolute value** of a complex number $z = a + \mathrm{i}b$ is defined by

$$|z| = \sqrt{a^2+b^2} = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}.$$

If one pictures complex numbers geometrically as points in the plane, $|z|$ just gives the usual euclidean distance of z from the origin. It is then clear that we have

$$|\operatorname{Re} z| \le |z|, \quad |\operatorname{Im} z| \le |z|, \quad |\overline{z}| = |z| \quad \text{and} \quad z\overline{z} = |z|^2$$

for all complex numbers $z \in \mathbb{C}$. The last formula gives us a clue how to compute a *multiplicative inverse* for a non-zero complex number z. Indeed, $z \neq 0$ if and only if $|z| \neq 0$ and so the last formula becomes

$$z\left(|z|^{-2}\,\overline{z}\right) = 1$$

which amounts to

$$z^{-1} = |z|^{-2}\,\overline{z}.$$

A.7. Linear Algebra

A **vector space** over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is a set E together with an operation

$$E \times E \longrightarrow E, \qquad (x,y) \longmapsto x+y$$

called **addition**, an operation

$$\mathbb{K} \times E \longrightarrow E, \qquad (\lambda, x) \longmapsto \lambda x$$

called **scalar multiplication**, and a distinguished element $0 \in E$ called the **zero vector**, such that for $x, y, z \in E, \lambda, \mu \in \mathbb{K}$ the following statements hold:

1) $x + (y+z) = (x+y) + z$;
2) $x + y = y + x$;
3) $x + 0 = x$;
4) $x + (-1)x = 0$;
5) $(\lambda + \mu)x = \lambda x + \mu x$;
6) $\lambda(x+y) = \lambda x + \lambda y$;
7) $(\lambda\mu)x = \lambda(\mu x)$;
8) $1 \cdot x = x$.

(These are called the **vector space axioms**). Vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ are also called **linear spaces**. We use both expressions synomymously.

If E is a vector space, $\{v_1, \dots, v_n\} \subseteq E$ is a collection of vectors and $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ are scalars, the expression

$$\sum_{j=1}^{n} \lambda_j v_j = \lambda_1 v_1 + \cdots + \lambda_n v_n$$

is called a (finite) **linear combination** of the vectors v_j.

If E is a vector space and $F \subseteq E$ is a subset, then F is called a **(linear) subspace** of E if one has

$$0 \in F, \quad \text{and} \quad x, y \in F, \lambda \in \mathbb{K} \;\Rightarrow\; x + y \in F,\ \lambda x \in F.$$

It is then clear that F is a vector space in its own right (with respect to the induced operations and the same zero vector).

If $A \subseteq E$ is any subset there is a "smallest" linear subspace, called the **linear span** $\operatorname{span} A$ of A, consisting of all finite linear combinations of vectors in A, i.e.,

$$\operatorname{span} A := \Big\{ \sum_{a \in A} \lambda_a a \mid \lambda_a \in \mathbb{K},\ \#\{a \in A \mid \lambda_a \neq 0\} < \infty \Big\}.$$

(Note that we do not assume A to be a finite set here, so we have to set all coefficient $\lambda_a = 0$ except for finitely many.) If U is a subspace and $A \subseteq U$ is such that $\operatorname{span} A = U$, then A is said to **generate** U.

Let E be any $\mathbb{K}$-vector space. A family $(x_j)_{j \in J}$ of vectors in E (J an arbitrary non-empty index set) is called **linearly independent** if there is only the trivial way to write the zero vector 0 as a finite linear combination of the x_j. Formally:

$$\text{If} \quad \sum\nolimits_{j \in J} \lambda_j x_j = 0 \quad \text{where} \quad \lambda_j \in \mathbb{K},\ \#\{j \in J \mid \lambda_j \neq 0\} < \infty$$
$$\text{then} \quad \lambda_j = 0 \quad \forall\, j \in J.$$

An **(algebraic) basis** of E is a linearly independent family $(x_j)_{j \in J}$ such that $E = \operatorname{span}\{x_j \mid j \in J\}$. Equivalently, every element from E can be written in a *unique way* as a finite linear combination of the x_j.

A vector space E is called **finite-dimensional** if it has a finite basis, else **infinite-dimensional**. A famous theorem of Steinitz says that each two bases of a finite-dimensional vector space E have the same number of elements. This number is then called the **dimension** of E, $\dim E$.

Theorem A.6. *Let E be a vector space over $\mathbb{K}$ and let $A \subseteq B \subseteq E$ be subsets. If A is linearly independent and B generates E then there is a basis C of E such that $A \subseteq C \subseteq B$.*

Proof. Apply the following procedure: discard from B all vectors v such that $A \cup \{v\}$ is linearly dependent, and call this the new B. If then $A = B$, we are done. If not, then there is a vector $v \in B \setminus A$ such that $A \cup \{v\}$ is linearly independent. Call this set the new A and repeat the procedure.

If B is finite, this procedure must terminate and one ends up with a basis. In the case of infinite B one may have to make an infinite (and even uncountable) number of steps. Such a "transfinite" procedure is unwieldy and may be replaced by an application on Zorn's Lemma. To this end, let

$$X := \{C \mid A \subseteq C \subseteq B,\ B \text{ linearly independent}\}$$

and order X by ordinary set inclusion. If $C \in X$ is maximal, then each vector from B is linearly dependent on C (otherwise we could enlarge C). But since B is generating, C is then generating as well, and so C is the desired basis.

Hence a maximal element in X would solve the problem. To find it, we apply Zorn's Lemma A.2. Let $K \subseteq X$ be any totally ordered subset. We have to show that K has an upper bound in the poset X. Define

$$C_K := \bigcup_{C \in K} C.$$

Clearly $A \subseteq C_K \subseteq B$. To show that C_K is linearly independent, take $x_1, \dots, x_n \in C_K$. For any $1 \leq j \leq n$ we find a $C_j \in K$ such that $x_j \in C_j$. But since K is totally ordered, one of the C_j, say C_{j_0}, contains all the others, hence $x_1, \dots, x_n \in C_{j_0}$. But C_{j_0} is linearly independent, and so must be $\{x_1, \dots, x_n\}$. This shows that $C_K \in X$ and thus is an upper bound for $K \subseteq X$ in X. □

Linear Mappings. Let E, F be $\mathbb{K}$-vector spaces. A mapping $T : E \longrightarrow F$ is called **linear** if

$$T(x + y) = Tx + Ty \quad \text{and} \quad T(\lambda x) = \lambda T x$$

for all $x, y \in E, \lambda \in \mathbb{K}$. (Note that we often write Tx instead of $T(x)$.) Linear mappings are also called **linear operators**, especially when $E = F$. When $F = \mathbb{K}$, i.e., when $T : E \longrightarrow \mathbb{K}$ then T is called a **(linear) functional**.

If E, F, G are K vector spaces and $T : E \longrightarrow F$, $S : F \longrightarrow G$ are linear, then

$$ST := S \circ T : E \longrightarrow G$$

is linear as well. If $T : E \longrightarrow T$ is linear we write $T^0 := \mathrm{I}$ and $T^n = T \circ \ldots \circ T$ n-times, for $n \in \mathbb{N}$.

For a linear mapping $T : E \longrightarrow F$, its **kernel**

$$\ker T := \{x \in E \mid Tx = 0\}$$

is a linear subspace of E and its **range**

$$\operatorname{ran} T := \{Tx \mid x \in E\}$$

is a linear subspace of F. The kernel is sometimes also called the **null space** and is denoted by $N(T)$, the range is sometimes denoted by $R(T)$. The linear mapping $T : E \longrightarrow F$ is injective if and only if $\ker T = 0$ and surjective if and only if $\operatorname{ran} T = F$. If T is bijective then also T^{-1} is a linear mapping, and T is called an **isomorphism**.

Coordinatisation. Let E be a *finite-dimension* $\mathbb{K}$-vector space. Each ordered basis $B = \{b_1, \dots, b_d\}$ defines an isomorphism $\Phi : \mathbb{K}^d \longrightarrow E$ by

$$\Phi(x) := \sum_{j=1}^{d} x_j b_j \qquad (x = (x_1, \dots, x_d) \in \mathbb{K}^d).$$

The injectivity of T is due to the linear independence of B, the surjectivity due to the fact that B is generating. If $v = \Phi(x)$ then the tuple

$$\Phi^{-1}(v) = x = (x_1, \dots, x_d)$$

is called the **coordinate vector** of v with respect to the (ordered) basis B.

Direct Sums and Projections. Let E be a vector space and U, V linear subspaces. If

$$E = U + V := \{u + v \mid u \in U,\, v \in V\} \quad \text{and} \quad U \cap V = \{0\}$$

then we call E the **algebraic direct sum** of U and V and write

$$E = U \oplus V.$$

If $x \in E$ we can write $x = u + v$ with $u \in U$ and $v \in V$. Then condition $U \cap V = \{0\}$ implies that this representation of x is *unique*. Hence we may write

$$P_U x := u, \quad P_V x := v.$$

Then $P_U : E \longrightarrow U$, $P_V : E \longrightarrow V$ are linear, $P_U + P_V = \mathrm{I}$, $P_U^2 = P_U$, $P_V^2 = P_V$. The operators P_U, P_V are called the **canonical projections** associated with the *direct sum decomposition* $E = U \oplus V$.

Conversely, let $P : E \longrightarrow E$ satisfy $P^2 = P$. Then also $(\mathrm{I} - P)^2 = \mathrm{I} - P$ and $\operatorname{ran} P = \ker(\mathrm{I} - P)$. Defining $U := \operatorname{ran} P$ and $V := \ker P$ then

$$U + V = E, \quad \text{and} \quad U \cap V = \{0\}$$

hence $E = U \oplus V$ and $P = P_U, \mathrm{I} - P = P_V$.

Vector Spaces of Functions. Clearly $E = \mathbb{K}$ itself is a vector space over $\mathbb{K}$. If E is any $\mathbb{K}$-vector space and X is any non-empty set then the set of E-valued functions on X

$$\mathcal{F}(X;E) := \{f \mid f : X \longrightarrow E\}$$

is also a $\mathbb{K}$-vector space, with respect to the **pointwise operations**:

$$(f+g)(x) := f(x) + g(x)$$
$$(\lambda f)(x) := \lambda f(x)$$

whenever $f, g \in \mathcal{F}(X;E)$, $\lambda \in \mathbb{K}$. Note that for $X = \mathbb{N}$, an element $f \in \mathcal{F}(\mathbb{N};E)$ is nothing else than a **sequence** $(f(n))_{n\in\mathbb{N}}$ in E. If $X = \{1,\dots,d\}$ then $\mathcal{F}(X;\mathbb{K}) = \mathbb{K}^d$.

Suppose that E is a $\mathbb{K}$-vector space and X is non-empty set then each $a \in X$ defines a linear mapping

$$\delta_a : \mathcal{F}(X;E) \longrightarrow E$$

by **point evaluation** $\delta_a(f) := f(a)$, $f \in \mathcal{F}(X;E)$.

The Space of all Linear Mappings. Let E, F be vector spaces over $\mathbb{K}$. We write

$$\mathrm{Lin}(E;F) := \{T : E \longrightarrow F \mid T \text{ is linear}\}$$

the set of all linear mappings from E to F. This is clearly a subset of $\mathcal{F}(E;F)$, the vector space of *all* mappings from E to F.

Lemma A.7. *The set* $\mathrm{Lin}(E;F)$ *is a linear subspace of* $\mathcal{F}(E;F)$ *and hence a vector space. If* $E = F$ *then* $\mathrm{Lin}(E) := \mathrm{Lin}(E;E)$ *is an algebra, i.e., the multiplication (= concatenation) satisfies:*

$$R(ST) = (RS)T,$$
$$R(S+T) = RS + RT,$$
$$(R+S)T = RT + ST,$$
$$\lambda(ST) = (\lambda S)T = S(\lambda T),$$

with $R, S, T \in \mathrm{Lin}(E;F), \lambda \in \mathbb{K}$.

Quotient Spaces. If F is a linear subspace of a vector space E then one defines

$$x \sim_F y \quad :\Longleftrightarrow \quad x - y \in F$$

for $x, y \in E$. This is an equivalence relation on E (check it!) Let us denote the equivalence class containing $x \in E$ by $[x]$. Then $[x] = [y]$ iff $x - y \in F$ and $[x] = x + F$ as sets. The set of equivalence classes

$$E/F := \{[x] \mid x \in E\}$$

is called the **factor space** or **quotient space**. It is itself a vector space with respect to the operations

$$[x] + [y] := [x + y] \quad \text{and} \quad \lambda[x] := [\lambda x]$$

$(x, y \in E, \lambda \in \mathbb{K})$. (Of course one has to check that these are well defined.) The mapping

$$s : E \longrightarrow E/F, \quad sx := [x] \quad (x \in E)$$

is linear (by definition of addition and scalar multiplication on E/F) and is called the **canonical surjection**.

Sesquilinear Forms. Let E be a vector space over $\mathbb{K}$. A mapping $a : E \times E \longrightarrow \mathbb{K}$ is called a **sesquilinear form** on E, if

1) $a(\alpha f + \beta g, h) = \alpha a(f, h) + \beta a(g, h)$
2) $a(h, \alpha f + \beta g) = \overline{\alpha} a(h, f) + \overline{\beta} a(h, g)$

for all $f, g, h \in E$ and $\alpha, \beta \in \mathbb{K}$. By

$$q_a(f) := a(f, f) \qquad (f \in E)$$

we denote the associated **quadratic form**.

Advice/Comment:
1) says that $a(\cdot, h)$ is linear and 2) says that $a(h, \cdot)$ is "anti-linear" for each $h \in E$. Since for real numbers $\alpha \in \mathbb{R}$ one has $\overline{\alpha} = \alpha$, in the case that $\mathbb{K} = \mathbb{R}$ a sesquilinear form is simply bilinear, i.e., linear in each component.

A sesquilinear form a on E is called **hermitian** or **symmetric** if

$$a(g, f) = \overline{a(f, g)} \qquad (f, g \in E).$$

Symmetric sesquilinear forms have special properties:

Lemma A.8. *Let $a : E \times E \longrightarrow \mathbb{K}$ be a symmetric form on the $\mathbb{K}$-vector space E. Then the following assertions hold:*

a) $q_a(f) \in \mathbb{R}$
b) $q_a(f + g) = q_a(f) + 2 \operatorname{Re} a(f, g) + q_a(g)$
c) $q_a(f + g) - q_a(f - g) = 4 \operatorname{Re} a(f, g)$ **(polarisation identity)**
d) $q_a(f + g) + q_a(f - g) = 2q_a(f) + 2q_a(g)$ **(parallelogram law)**

for all $f, g \in E$.

Proof. By symmetry, $q_a(f) = \overline{q_a(f)}$; but only a real number is equal to its own conjugate. The sesquilinearity and symmetry of a yields

$$\begin{aligned} q_a(f+g) &= a(f+g, f+g) = a(f,f) + a(f,g) + a(g,f) + a(g,g) \\ &= q_a(f) + a(f,g) + \overline{a(f,g)} + q_a(g) = q_a(f) + 2\operatorname{Re} a(f,g) + q_a(g) \end{aligned}$$

since $z + \overline{z} = 2\operatorname{Re} z$ for every complex number $z \in \mathbb{C}$. This is b). Replacing g by $-g$ yields

$$q_a(f-g) = q_a(f) - 2\operatorname{Re} a(f,g) + q_a(g)$$

and addding this to b) yields d). Subtracting it leads to c). □

Suppose that $\mathbb{K} = \mathbb{R}$. Then the polarisation identity reads

$$a(f,g) = \frac{1}{4}(q_a(f+g) - q_a(f-g)) \qquad (f, g \in E).$$

This means that the values of the form a are uniquely determined by the values of its associated quadratic form. The same is true in the case $\mathbb{K} = \mathbb{C}$, since in this case

$$a(f,g) = \operatorname{Re} a(f,g) + \mathrm{i} \operatorname{Re} a(f, \mathrm{i}g)$$

for all $f, g \in E$.

A.8. Set-theoretic Notions

According to G.Cantor, a set is a "collection of well-distinguished objects of our mind". This is not a mathematical definition, but helps to set up the decisive properties of sets. The basic relation in the universe of sets is the $\in$-relation: $a \in A$ indicates that some object a (which may be a set itself) is an **element** of (= is contained in) the set A.

One writes $A \subseteq B$ and calls A a **subset** of B if every element of A is also an element of B:

$$A \subseteq B \quad :\Longleftrightarrow \quad \forall x : x \in A \implies x \in B.$$

Two sets are **equal** if the have the same elements:

$$A = B \quad :\Longleftrightarrow \quad A \subseteq B \quad \text{and} \quad B \subseteq A.$$

The **emtpy set** is the set $\emptyset$ that has *no* elements. If $a, b, c \ldots$ are objects, then we denote by

$$\{a, b, c \ldots\}$$

the set that contains them. In particular, $\{a\}$ is the **singleton** whose only element is a. If X is a set and P is a *property* that an element of X may or may not have, then

$$\{x \mid x \in X,\ x \text{ has P}\}$$

denotes the set whose elements are precisely the elements of X that have P. The **power set** of a set X is the unique set $\mathcal{P}(X)$ whose elements are precisely the subsets of X:

$$\mathcal{P}(X) := \{A \mid A \subseteq X\}$$

Note that always $\emptyset, X \in \mathcal{P}(X)$. For subsets $A, B \subseteq X$ we define their **union, intersection, difference** as

$$\begin{aligned} A \cup B &:= \{x \in X \mid x \in A \text{ or } x \in B\} \\ A \cap B &:= \{x \in X \mid x \in A \text{ and } x \in B\} \\ A \setminus B &:= \{x \in X \mid x \in A \text{ but } \ x \notin B\}. \end{aligned}$$

The **complement** in X of a subset $A \subseteq X$ is

$$A^c := X \setminus A = \{x \in X \mid x \notin A\}.$$

If X, Y are sets we let

$$X \times Y := \{(x, y) \mid x \in X, y \in Y\}$$

the set of all **ordered** pairs of elements in X and Y. The set $X \times Y$ is also called the **Cartesian product** of X and Y. Each subset

$$R \subseteq X \times Y$$

is called a (binary) **relation**. Instead of writing $(x, y) \in R$ one often has other notations, e.g., $x \leq y$, $x \sim y$, $\dots$, depending on the context.

A **mapping** $f : X \longrightarrow Y$ is an assignment that associates with each element $x \in X$ a *value* or *image* $f(x) \in Y$. Set-theoretically, we can *identify* a mapping f with its **graph**

$$\operatorname{graph} f = \{(x, y) \mid x \in X, y = f(x)\} \ \subseteq \ X \times Y.$$

This is a special kind of a binary relation R , satisfying

$$\begin{aligned} &(1) \quad \forall\, x \in X \ \exists\, y \in Y : (x, y) \in R; \\ &(2) \quad \forall\, x \in X \ \forall y, y' \in Y : (x, y), (x, y') \in R \implies y = y'. \end{aligned}$$

Properties (1) and (2) say that this relation is **functional**. Every functional relation is the graph of a mapping.

If $f : X \longrightarrow Y$ is a mapping then we call $X = \operatorname{dom}(f)$ the **domain** of f and Y the **codomain** or **target set**. We write

$$\begin{aligned} f(A) &:= \{f(x) \mid x \in A\} = \{y \in Y \mid \exists\, x \in A : f(x) = y\} \\ f^{-1}(B) &:= \{x \in X \mid f(x) \in B\} \end{aligned}$$

for subsets $A \subseteq X, B \subseteq Y$ and call it the **image** of A and the **inverse image** of B under f. The mapping $f : X \longrightarrow Y$ is called **surjective** if $f(X) = Y$, and **injective** if $f(x) = f(y)$ implies that $f(x) = f(y)$, for all

$x, y \in X$. A mapping f which is injective *and* surjective, is called **bijective**. In this case one can form its **inverse**, denoted by $f^{-1} : Y \longrightarrow X$.

An **indexed family** of elements of a set X is simply a mapping $J \longrightarrow X$, where we call $J \neq \emptyset$ the **index set**. One often writes $(x_j)_{j\in J} \subseteq X$ to denote this mapping, in particular when one does not want to use an own name for it (like "f" or so).

Given an indexed family $(A_j)_{j\in J} \subseteq \mathcal{P}(X)$ of subsets of a set X one considers their **union** and their **intersection**

$$\bigcup_{j\in J} A_j = \{x \in X \mid \exists\, j \in J : x \in A_j\}$$

$$\bigcap_{j\in J} A_j = \{x \in X \mid \forall\, j \in J : x \in A_j\}.$$

De Morgan's Laws say that

$$\Big(\bigcup_{j\in J} A_j\Big)^c = \bigcap_{j\in J} A_j^c \quad \text{and} \quad \Big(\bigcap_{j\in J} A_j\Big)^c = \bigcup_{j\in J} A_j^c.$$

If $f : X \longrightarrow Y$ is a mapping and then

$$f\Big(\bigcup_{j\in J} A_j\Big) = \bigcup_{j\in J} f(A_j) \quad \text{and} \quad f\Big(\bigcap_{j\in J} A_j\Big) \subseteq \bigcap_{j\in J} f(A_j)$$

Attention: the inclusion on the right-hand side is proper in general!

If $f : X \longrightarrow Y$ is a mapping and $(B_j)_{j\in J} \subseteq \mathcal{P}(Y)$ is an indexed family of subsets of Y then

$$f^{-1}\Big(\bigcup_{j\in J} B_j\Big) = \bigcup_{j\in J} f^{-1}(B_j) \quad \text{and} \quad f^{-1}\Big(\bigcap_{j\in J} B_j\Big) = \bigcap_{j\in J} f^{-1}(B_j)$$

Appendix B

Some Proofs

B.1. The Weierstrass Theorem

Let us restate the result from Chapter 3.

Theorem B.1 (Weierstrass). *Let $[a, b]$ be a compact interval in $\mathbb{R}$. Then the space of polynomials $\mathrm{P}[a, b]$ is dense in $\mathrm{C}[a, b]$ for the sup-norm.*

Proof. Since every $f \in \mathrm{C}([a, b]; \mathbb{C})$ can be written as $f = u + iv$, with $u, v \in \mathrm{C}([a, b]; \mathbb{R})$, and such a representation is equally true for polynomials, we see that the theorem for $\mathbb{C}$-valued functions follows easily from the real version. So we may suppose $\mathbb{K} = \mathbb{R}$ in the following.

We introduce the special polynomials $\mathbf{1}, \mathbf{t}, \mathbf{t}^2$ by

$$\mathbf{1}(t) := 1, \quad \mathbf{t}(t) = t, \quad \mathbf{t}^2 := t^2 \qquad (t \in \mathbb{R}).$$

Moreover, we write $f \leq g$ for functions f, g on an interval $[a, b]$ as an abbreviation of $f(t) \leq g(t)$ for all $t \in [a, b]$.

Without loss of generality we may suppose that $[a, b] = [0, 1]$; else we use the change of variables

$$s = \frac{t-a}{b-a},\ t \in [a, b], \quad t = a + s(b-a),\ s \in [0, 1]$$

which yields an isometric isomorphism of $\mathrm{C}[a, b]$ onto $\mathrm{C}[0, 1]$, mapping $\mathrm{P}[a, b]$ onto $\mathrm{P}[0, 1]$.

For an arbitrary function $f : [0, 1] \longrightarrow \mathbb{R}$ consider its n-th **Bernstein polynomial** $B_n(f, \cdot)$, defined by

$$B_n(f, t) := \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} f(k/n) \qquad (t \in [0, 1]).$$

The operation

$$B_n : \mathrm{C}[0,1] \longrightarrow \mathrm{P}[0,1], \quad f \longmapsto B_n(f,\cdot)$$

is obviously linear; it is also *monotonic*, by which we mean that if $f \le g$, then also $B_n f \le B_n g$. We shall show that if f is continuous, $B_n f \to f$ uniformly on $[0,1]$.

To this aim, we first compute the Bernstein polynomials of three special functions.

$$B_n(\mathbf{1},t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} = (t+(1-t))^n = 1^n = 1$$

and hence $B_n\mathbf{1} = \mathbf{1}$ for all $n \in \mathbb{N}$.

$$\begin{aligned} B_n(\mathbf{t},t) &= \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k}(k/n) = \sum_{k=1}^{n} \binom{n-1}{k-1} t^k (1-t)^{n-k} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} t^{k+1}(1-t)^{n-1-k} = t(t+(1-t))^{n-1} = t \end{aligned}$$

and hence $B_n\mathbf{t} = \mathbf{t}$ for all $n \in \mathbb{N}$. Finally,

$$B_n(\mathbf{t}^2,t) = \frac{t(1-t)}{n} + t^2 \qquad (t \in [0,1], n \in \mathbb{N})$$

(We leave the computation as Exercise B.1 below). Now take $f \in \mathrm{C}[0,1]$ and fix $\epsilon > 0$. Since $[0,1]$ is compact, f is uniformly continuous. This means that we find $\delta > 0$ such that

$$|s-t| \le \delta \quad \Longrightarrow \quad |f(s)-f(t)| \le \epsilon \qquad (s,t \in [0,1]).$$

Define $\alpha := 2\,\|f\|_\infty/\delta^2$. Then if $s,t \in [0,1]$ are such that $|s-t| \ge \delta$, then

$$|f(s)-f(t)| \le |f(s)| + |f(t)| \le 2\,\|f\|_\infty = \alpha\delta^2 \le \alpha(s-t)^2.$$

Combining both yields

$$|f(s)-f(t)| \le \epsilon + \alpha(s-t)^2 \qquad (s,t \in [0,1]).$$

Fix $s \in [0,1]$ and define $h_s(t) := (s-t)^2$. Then we may write

$$-\epsilon - \alpha h_s \le f(s)\mathbf{1} - f \le \epsilon + \alpha h_s.$$

We now take Bernstein polynomials of all these functions. This yields

$$B_n(-\epsilon\mathbf{1} - \alpha h_s) \le B_n(f(s)\mathbf{1} - f) \le B_n(\epsilon\mathbf{1} + \alpha h_s).$$

Since taking Bernstein polynomials is linear and $B_n(\mathbf{1},t) = 1$, we get

$$-\epsilon - \alpha B_n(h_s,t) \le f(s) - B_n(f,t) \le \epsilon + \alpha B_n(h_s,t) \qquad (t \in [0,1], n \in \mathbb{N}).$$

Recall that $h_s = s^2\mathbf{1} - 2s\mathbf{t} + \mathbf{t}^2$ and that we computed $B_n(g, \cdot)$ for $g = \mathbf{1}, \mathbf{t}, \mathbf{t}^2$ already above. Using these results we obtain

$$|f(s) - B_n(f,t)| \le \epsilon + \alpha B_n(h_s, t) = \epsilon + \alpha\left(s^2 - 2st + \frac{t(1-t)}{n} + t^2\right)$$

for all $t \in [0,1], n \in \mathbb{N}$. Equating $s = t$ here finally yields

$$|f(s) - B_n(f,s)| \le \epsilon + \alpha\frac{s(1-s)}{n} \le \epsilon + \frac{\alpha}{n}$$

for all $s \in [0,1]$. Taking suprema yields $\|f - B_n f\|_\infty \le \epsilon + \alpha/n \le 2\epsilon$ as $n \ge \alpha/\epsilon$. □

Exercise B.1. Verify that

$$B_n(\mathbf{t}^2, t) = \frac{t(1-t)}{n} + t^2 \qquad (t \in [0,1], n \in \mathbb{N}).$$

Show directly that $B_n(\mathbf{t}^2, \cdot) \to \mathbf{t}^2$ uniformly on $[0,1]$.

B.2. A Series Criterion for Completeness

In Theorem 5.16 it is shown that in a Banach space each absolutely convergent series converges. Here we shall prove the following converse statement.

Theorem B.2. *Let $(E, \|\cdot\|)$ be a normed vector space such that every absolutely convergent series converges in E. Then E is complete, i.e., a Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subseteq E$ be a Cauchy sequence in E. By Lemma 5.2.c) it suffices to find a subsequence that converges. Pick successively $(f_{n_k})_{k\in\mathbb{N}}$ such that

$$\left\| f_{n_k} - f_{n_{k+1}} \right\| \le 2^{-k} \qquad (k \in \mathbb{N}).$$

(This is possible since $(f_n)_{n\in\mathbb{N}}$ is Cauchy!) For

$$g_k := f_{n_k} - f_{n_{k+1}}$$

one has therefore $\sum_{k=1}^\infty \|g_k\| < \infty$. By assumption,

$$g := \lim_{N\to\infty} \sum\nolimits_{k=1}^N g_k = \lim_{N\to\infty} \sum\nolimits_{k=1}^N f_{n_k} - f_{n_{k+1}} = \lim_{N\to\infty} f_{n_1} - f_{n_{N+1}}$$

exists in E. But this implies that $f_{n_N} \to f_{n_1} - g$ as $N \to \infty$, and we are done. □

B.3. Density Principles

We prove the statements from the Intermezzo.

Theorem B.3 ("dense in dense is dense"). *Let E be a normed space, let F, G be linear subspaces. If F is dense in E and $F \subseteq \overline{G}$, then G is dense in E.*

Proof. Let $f \in E$ and $n \in \mathbb{N}$. Then since F dense in E one finds $g \in F$ such that $\|g_n - f\| < 1/n$. Since $F \subseteq \overline{G}$ by hypothesis, $g_n \in \overline{G}$ and so there is $h_n \in G$ such that $\|g_n - h_n\| < 1/n$, Hence by the triangle inequality

$$\|f - h_n\| \le \|f - g_n\| + \|g_n - h_n\| < 2/n \to 0$$

and so $f \in \overline{G}$, as was to show.

(A different proof: Since $F \subseteq \overline{G}$ one has $E = \overline{F} \subseteq \overline{\overline{G}} = \overline{G}$, by Lemma 3.17.) □

Theorem B.4 ("dense is determining"). *Let E, F be normed spaces and let $T, S : E \longrightarrow F$ be bounded linear mappings. If $G \subseteq E$ is dense in E and $Tf = Sf$ for all $f \in G$, then $S = T$.*

Proof. This was also the subject of Exercise 4.15. Since the operator $T - S$ is bounded, $\ker(T - S)$ is closed. By hypothesis, $\ker(T - S)$ contains G, hence it contains also $\overline{G} = E$. So $T = S$. □

Theorem B.5 ("the image of dense is dense in the image"). *Let E, F be normed spaces and let $T : E \longrightarrow F$ be a bounded linear mapping. If $G \subseteq E$ is dense in E, then $T(G)$ is dense in $T(E)$.*

Proof. Let $f \in T(E)$. Then there is $g \in E$ such that $Tg = f$. Since G is dense in E, there is a sequence $(h_n)_{n\in\mathbb{N}} \subseteq G$ such that $h_n \to g$ in E. Since T is bounded (=continuous), $Th_n \to Tg = f$. But $(Th_n)_{n\in\mathbb{N}} \subseteq T(G)$, and so $f \in \overline{T(G)}$. □

Theorem B.6 ("dense implies dense in a weaker norm"). *Let G be a linear subspace of a linear space E, and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on E such that there is a constant with $\|f\|_1 \le c\,\|f\|_2$ for all $f \in G$. If G is $\|\cdot\|_2$-norm dense in E, then it is $\|\cdot\|_1$-norm dense in E, too.*

Proof. This is a consequence of the preceding theorem with $T = \mathrm{I}$ is the identity operator, $F = E$ as linear spaces, but with E carrying the norm $\|\cdot\|_2$ and $F = E$ carrying the norm $\|\cdot\|_1$. □

Theorem B.7 ("dense convergence implies everywhere convergence"). *Let E, F be normed spaces and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear mappings $T_n : E \longrightarrow F$ such that there is $C \geq 0$ such that*

$$\|T_n f\| \leq C\,\|f\| \qquad (f \in E, n \in \mathbb{N}) \tag{B.1}$$

If $T : E \longrightarrow F$ is a bounded linear mapping such that

$$Tf = \lim_{n\to\infty} T_n f \tag{B.2}$$

for all $f \in G$, and G is dense in E, then (B.2) *holds for all $f \in E$.*

Proof. Fix C' such that $\|Tf\| \leq C'\,\|f\|$ for all $f \in E$. (One can take $C' = C$ here; do you see, why?) It suffices to show that the set

$$M := \{f \in E \mid T_n f \to Tf\}$$

is closed. Indeed, by hypothesis $G \subseteq M$, so that $E = \overline{G} \subseteq \overline{M} \subseteq M$. To show that M is closed, let $(f_m)_{m\in\mathbb{N}} \subseteq M$ and suppose that $f_m \to f \in E$. Then for all $n, m \in \mathbb{N}$

$$\begin{aligned}\|T_n f - Tf\| &\leq \|T_n f - T_n f_m\| + \|T_n f_m - Tf_m\| + \|Tf_m - Tf\| \\ &\leq \|T_n\|\,\|f - f_m\| + \|T_n f_m - Tf_m\| + C'\,\|f_m - f\| \\ &\leq (C + C')\,\|f_m - f\| + \|T_n f_m - Tf_m\|\,.\end{aligned}$$

Fixing $m \in \mathbb{N}$ and letting $n \to \infty$ yields

$$\limsup_{n\to\infty} \|T_n f - Tf\| \leq (C + C')\,\|f_m - f\|\,,$$

since $T_n f_m \to Tf_m$. No letting $m \to \infty$ we obtain $\limsup_{n\to\infty} \|T_n f - Tf\| = 0$, i.e, $f \in M$. □

Recall that a sequence of linear operators that satisfies the conditions (B.1) is called **uniformly bounded**.

Theorem B.8 ("densely defined and bounded extends"). *Let E be a normed spaces, let F be a Banach space and let $G \subseteq E$ be a dense linear subspace. Furthermore, let $T : G \longrightarrow F$ be a bounded linear operator. Then T extends uniquely to a bounded linear operator $T^\sim : E \longrightarrow F$, with the same bound.*

By "the same bound" we mean that if $\|Tf\| \leq c\,\|f\|$ for all $f \in G$, then also $\|T^\sim f\| \leq c\,\|f\|$ for all $f \in E$.

Proof. Let $c > 0$ be such that $\|Tf\| \leq c\,\|f\|$ for all $f \in G$. Uniqueness follows from Theorem B.4 above. To define the extension, take $f \in E$ arbitrary. Since G is dense in E, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq G$ such that $f_n \to f$. In particular, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E. Since T is bounded as an operator from G to F, we have

$$\|Tf_n - Tf_m\| \leq c\,\|f_n - f_m\|$$

and this shows that $(Tf_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in F. Since F is a Banach space, the limit $\lim_{n\to\infty} Tf_n$ exists in F.

Now we would like to *define* $T^{\sim}f := \lim_{n\to\infty} Tf_n$. To be able to do this, we must make sure that this definition does only depend on f and not on the chosen approximation $(f_n)_{n\in\mathbb{N}}$. This is easy: if also $g_n \to f$ then $g_n - f_n \to 0$ and so $Tf_n - Tg_n = T(f_n - g_n) \to T0 = 0$, by continuity. So $T^{\sim}$ is well-defined. Since for $f \in G$ we can choose $f_n := f$ for all $n \in \mathbb{N}$, it is clear that $T^{\sim}f = Tf$ if $f \in G$; hence $T^{\sim}$ is indeed an extension of T.

We show that $T^{\sim}$ is linear. If $f, g \in E$ and $\alpha \in \mathbb{K}$, choose $f_n, g_n \in G$ with $f_n \to f$ and $g_n \to g$. Then $f_n + \alpha g_n \in G$ and $f_n + \alpha g_n \to f + \alpha g$, hence by definition

$$T^{\sim}(f + \alpha g) = \lim_{n\to\infty} T(f_n + \alpha g_n) = \lim_{n\to\infty} Tf_n + \alpha T g_n = T^{\sim}f + \alpha T^{\sim}g.$$

To show that $T^{\sim}$ is bounded, take again $f_n \in G$ $f_n \to f \in E$. Then $\|f_n\| \to \|f\|$ and $\|Tf_n\| \to \|T^{\sim}f\|$ by the continuity of the norm, and so from $\|Tf_n\| \leq c\,\|f_n\|$ for all n it follows that $\|T^{\sim}f\| \leq c\,\|f\|$. The proof is complete. □

By the theorem, there is no danger if we write again T in place of $T^{\sim}$. Note that it is absolutely essential to have the completeness of the space F.

Finally, here is a slight variation of Theorem B.7, but again: here it is essential that F is a Banach space.

Theorem B.9 ("dense convergence implies everywhere convergence (2)")**.** *Let E be a normed space, let F be a Banach space and let $(T_n)_{n\in\mathbb{N}}$ be a uniformly bounded sequence of bounded linear mappings $T_n : E \longrightarrow F$. Then if the limit*

$$\lim_{n\to\infty} T_n f \tag{B.3}$$

exists for each $f \in G$, and G is dense in E, then the limit (B.3) *exists for each $f \in E$, and a bounded linear operator $T : E \longrightarrow F$ is defined by $Tf := \lim_{n\to\infty} T_n f$, $f \in E$.*

Proof. Let $c > 0$ such that $\|T_n f\| \leq c\,\|f\|$ for all $n \in \mathbb{N}$ and $f \in E$. Define the operator $T : G \longrightarrow F$ by $Tf := \lim_{n\to\infty} Tf_n$ for $f \in G$. Then it is obvious that T is linear and bounded, similar as in the proof of the Theorem B.8. By this very theorem, T extends uniquely to a bounded linear operator $T : E \longrightarrow F$. Now Theorem B.7 can be applied to conclude that $T_n f \to T^f$ for *all* $f \in E$, since G is dense. □

B.4. The Completeness of L^1 and L^2

In this section we give a proof for the completeness of the spaces $\mathrm{L}^2(X)$, where $X \subseteq \mathbb{R}$ is an interval. We use only elementary properties of the Lebesgue integral and the following fundamental result, assumed without proof.

Again, let $X \subseteq \mathbb{R}$ is an arbitrary interval.

Theorem B.10 (Monotone Convergence). *Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}_+(X)$ be an increasing sequence, i.e., $f_n \le f_{n+1}$ for $n \in \mathbb{N}$. Let $f(x) := \lim_{n\to\infty} f_n(x)$ be the pointwise limit. If*

$$\sup_{n\in\mathbb{N}} \int_X f_n \,\mathrm{d}\lambda < \infty$$

Then $f < \infty$ almost everywhere, $f \in \mathrm{L}^1(X)$ and

$$\lim_{n\to\infty} \int_X f_n \,\mathrm{d}\lambda = \int_X f \,\mathrm{d}\lambda.$$

Using this we can now give a proof of the Dominated Convergence Theorem.

Theorem B.11 (Dominated Convergence). *Let $p \in \{1, 2\}$ and let the sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{L}^p(X)$ be such that $f(x) := \lim_{n\to\infty} f_n(x)$ exists for almost all $x \in X$. If there is $0 \le g \in \mathrm{L}^p(X)$ such that $|f_n| \le g$ almost everywhere, for each $n \in \mathbb{N}$, then $f \in \mathrm{L}^p(X)$ and $\|f_n - f\|_p \to 0$. If $p = 1$, we have also*

$$\lim_{n\to\infty} \int_X f_n \,\mathrm{d}\lambda = \int_X f \,\mathrm{d}\lambda.$$

Proof. Consider $h_k := |f_k - f|^p \le 2^p g^p$. For each n is therefore $h_n^* := \sup_{k\ge n} h_k \le 2^p g^p$, and these relations are to be read pointwise almost everywhere. Clearly $h_n^* \searrow 0$ pointwise a.e.. But

$$\int_X h_n^* \,\mathrm{d}\lambda \le 2^p \int_X g^p \,\mathrm{d}\lambda = 2^p \|g\|_p^p < \infty$$

by hypothesis. Since $h_1^* - h_n^* \searrow H_1$, by the Monotone Convergence Theorem we have $\int_X (h_1^* - h_n^*) \nearrow \int_X h_1^*$. This implies that $\int_X h_n^* \to 0$. But

$$\|f_n - f\|_p^p = \int_X |f_n - f|^p \,\mathrm{d}\lambda = \int_X h_n \mathrm{d}\lambda \le \int_X h_n^* \,\mathrm{d}\lambda \to 0.$$

Hence $\|f_n \to f\|_p \to 0$ as claimed. In the case $p = 1$ we have in addition

$$\left| \int_X f_n \,\mathrm{d}\lambda - \int_X f \,\mathrm{d}\lambda \right| = \left| \int_X f_n - f \,\mathrm{d}\lambda \right| \le \int_X |f_n - f| \,\mathrm{d}\lambda \to 0$$

as asserted. □

Based on the Dominated Convergence Theorem, we can prove that L^p is complete, and a little more.

Theorem B.12 (Completeness of L^p). *Let $p \in \{1, 2\}$ and let $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{L}^p(X)$ be a $\|\cdot\|_p$-Cauchy sequence. Then there are functions $f, F \in \mathrm{L}^p(X)$ and a subsequence $(f_{n_k})_k$ such that*

$$|f_{n_k}| \le F \quad \text{a.e.} \quad \text{and} \quad f_{n_k} \to f \quad \text{a.e.}.$$

Moreover $\|f_n - f\|_p \to 0$. In particular, $\mathrm{L}^p(X)$ is a Banach space.

Proof. Note first that if we have found f, F and the subsequence with the stated properties, then $\|f_{n_k} - f\|_p \to 0$ by Dominated Convergence, and hence $\|f_n - f\|_p \to 0$ since the sequence $(f_n)_{n\in\mathbb{N}}$ is $\|\cdot\|_p$-Cauchy.

We find the subsequence in the following way. By using the Cauchy property we may pick $n_k < n_{k+1}$, $k \in \mathbb{N}$, such that $\left\|f_{n_k} - f_{n_{k+1}}\right\|_p < \frac{1}{2^k}$. To facilitate notation , let $g_k := f_{n_k}$. Then for every $N \in \mathbb{N}$

$$\int_X \left(\sum_{k=0}^N |g_k - g_{k+1}|\right)^p \mathrm{d}\lambda = \left\|\sum_{k=0}^N |g_k - g_{k+1}|\right\|_p^p \le \left(\sum_{k=0}^N \|g_k - g_{k+1}\|_p\right)^p$$
$$\le \left(\sum_{k=0}^\infty \frac{1}{2^k}\right)^p = 2^p$$

Define $g := \sum_{k=0}^\infty |g_k - g_{k+1}|$. Then by the Monotone Convergence Theorem

$$\int_X g^p = \lim_{N\to\infty} \int_X \left(\sum_{k=0}^N |g_k - g_{k+1}|\right)^p \mathrm{d}\lambda \;\le\; 2^p$$

and hence $g \in \mathrm{L}^p(X)$. Also by the Monotone Convergence Theorem we have $g^p < \infty$ almost everywhere, implying that

$$\sum_{k=0}^\infty |g_k - g_{k+1}| < \infty \quad \text{a.e.}.$$

Hence for almost all $x \in X$ the series

$$h(x) := \sum_{k=0}^\infty g_k(x) - g_{k+1}(x) = g_0(x) - \lim_k g_k(x)$$

exists. Hence $g_k \to f := g_0 - h$ almost everywhere. And

$$|g_k| \le |g_0| + \sum_{j=0}^{k-1} |g_j - g_{j+1}| \le |g_0| + g \quad \text{a.e.}.$$

Thus if we set $F := |g_0| + g$, the theorem is completely proved. □

Appendix C

General Orthonormal Systems

C.1. Unconditional Convergence

In general, even countable orthonormal systems do not come with a canonical ordering. For example the trigonometric system

$$e_n(t) := e^{2\pi i n \cdot t} \qquad (t \in [0,1],\ n \in \mathbb{Z})$$

in $\mathrm{L}^2(0,1)$ is indexed by the integers, and there are of course many ways to count all integers. Fortunately, the next result shows that the convergence in Theorem 7.13 is independent of the arrangement of the summands.

Lemma C.1. *Let $(e_j)_{j\in\mathbb{N}}$ be an ONS in a Hilbert space H and let $f \in H$. Then*

$$\sum_{j=1}^{\infty} \langle f, e_j \rangle\, e_j = \sum_{j=1}^{\infty} \langle f, e_{\pi(j)} \rangle\, e_{\pi(j)}$$

for every permutation (= bijective map) $\pi : \mathbb{N} \longrightarrow \mathbb{N}$.

Proof. Note that of course $(e_{\pi(j)})_{j\in\mathbb{N}}$ is also an ONS, and so both series converge, by Theorem 7.13. By the very same theorem,

$$\sum_{j=1}^{\infty} \langle f, e_j \rangle\, e_j = P_F f \quad \text{and} \quad \sum_{j=1}^{\infty} \langle f, e_{\pi(j)} \rangle\, e_{\pi(j)} = P_G f,$$

where

$$F = \overline{\operatorname{span}\{e_j \mid j \in \mathbb{N}\}} \quad \text{and} \quad G = \overline{\operatorname{span}\{e_{\pi(j)} \mid j \in \mathbb{N}\}}$$

Since obviously $F = G$, the lemma is proved. □

From Lemma C.1 we can conclude that if $(e_i)_{i\in I}$ is an ONS in a Hilbert space H, and I is countable, one can use *any* bijection $\pi : \mathbb{N} \longrightarrow I$ to define

$$\sum\nolimits_{i\in I} \langle f, e_i\rangle\, e_i := \sum\nolimits_{j=1}^{\infty} \left\langle f, e_{\pi(j)}\right\rangle e_{\pi(j)} \qquad (f \in H).$$

Remark C.2. The property shown in Lemma C.1 is called **unconditional summability**. Not every convergent series is unconditionally convergent. For example, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges in $\mathbb{R}$, but not unconditionally: one can rearrange the summands to get a non-convergent series. This follows from a famous theorem of Riemann, saying that in finite-dimensional spaces unconditional summability is the same as *absolute* summability. And indeed, the sum

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

is not convergent. Riemann's theorem is only true in finite dimensions, and here is an example: Let $(e_n)_{n\in\mathbb{N}}$ be an ONS in a Hilbert space H and $\lambda : \mathbb{N} \longrightarrow \mathbb{R}$. The previous results show that the series $\sum_n \lambda_n e_n$ converges unconditionally if and only if $\lambda \in \ell^2$. However, by definition it converges absolutely if

$$\sum_n |\lambda_n| = \sum_n |\lambda_n| \, \|e_n\| = \sum_n \|\lambda_n e_n\| < \infty,$$

hence if and only if $\lambda \in \ell^1$. Since the sequence $(1/n)_{n\in\mathbb{N}}$ is square summable but not absolutely summable, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} e_n$$

converges unconditionally but not absolutely in H.

Remark C.3 (The case of the integers)**.** In the case of the integers (as in the trigonometric system) we can also go a different way. The infinite double series

$$\sum\nolimits_{n=-\infty}^{\infty} f_n$$

can be interpreted as a double limit $\lim\limits_{n,m\to\infty} \sum\nolimits_{k=-m}^{n} f_k$, where (in a general metric space (Ω, d)) $x = \lim_{n,m\to\infty} x_{nm}$ simply means

$$\forall\, \epsilon > 0 \;\exists N \in \mathbb{N} \;\forall\, n, m \geq N : d(x_{nm}, x) < \epsilon.$$

C.2. Uncountable Orthonormal Bases

We now turn to the question how to deal with general, that is, possibly uncountable ONS. So let H be a Hilbert space and let $(e_i)_{i\in I} \subseteq H$ be an arbitrary ONS therein. For $f \in H$ define

$$I_f := \{i \in I \mid \langle f, e_i\rangle \neq 0\}.$$

Lemma C.4. *In the situation above, the set I_f is at most countable.*

Proof. Consider the set $I_n := \{i \in I \mid |\langle f, e_i\rangle|^2 \geq \frac{1}{n}\}$. Then if $J \subseteq I_n$ is finite,

$$\frac{\operatorname{card} J}{n} \leq \sum_{j\in J} |\langle f, e_j\rangle|^2 \leq \|f\|^2,$$

by Bessel's inequality, and hence $\operatorname{card} I_n \leq n\|f\|^2$ is finite. Therefore, $I_f = \bigcup_{n\in\mathbb{N}} I_n$ is at most countable. □

For $f \in H$ we can now define the term

$$\sum\nolimits_{i\in I} \langle f, e_i\rangle\, e_i := \sum\nolimits_{i\in I_f} \langle f, e_i\rangle\, e_i$$

and for the sum on the right side we can use any enumeration of the (at most countably many) members of I_f. By what we have seen above, this defines an element of H unambiguously. Then the analogue of Theorem 7.13 holds, replacing everywhere '$\sum_{j=1}^{\infty}$' by '$\sum_{j\in I}$' and '$j \in \mathbb{N}$' by '$j \in I$'. In particular, the analogue of Corollary 7.14 is true:

Theorem C.5. *Let H be a Hilbert space, let $(e_j)_{j\in I}$ be an ONS in H. Then the following assertions are equivalent.*

(i) $\{e_j \mid j \in I\}^{\perp} = \{0\}$;

(ii) $\operatorname{span}\{e_j \mid j \in I\}$ *is dense in* H;

(iii) $f = \sum_{j\in I} \langle f, e_j\rangle\, e_j$ *for all* $f \in H$;

(iv) $\|f\|^2 = \sum_{j\in I} |\langle f, e_j\rangle|^2$ *for all* $f \in H$;

(v) $\langle f, g\rangle_H = \sum_{j\in I} \langle f, e_j\rangle\, \overline{\langle g, e_j\rangle}$ *for all* $f, g \in H$.

Analogously, we call the $(e_j)_{j\in I}$ a **maximal ONS** (or an **orthonormal basis**), if it satisfies the equivalent conditions from the theorem.

Of course, the question arises whether a maximal ONS does always exist. This is indeed the case, by an application of Zorn's Lemma. Moreover, some other set-theoretic reasoning also reveals that two maximal ONS's must be of the same "size" (cardinality).

Exercise C.1. Let H be a separable Hilbert space. Show that every orthonormal system in H is at most countable.

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