THE HOLOMORPHIC FUNCTIONAL CALCULUS APPROACH TO OPERATOR SEMIGROUPS

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Abstract. In this article we construct a holomorphic functional calculus for operators of half-plane type and show how key facts of semigroup theory (Hille-Yosida and Gomilko-Shi-Feng generation theorems, Trotter-Kato approximation theorem, Euler approximation formula, Gearhart-Prüss theorem) can be elegantly obtained in this framework. Then we discuss the notions of bounded H^∞-calculus and m-bounded calculus on half-planes and their relation to weak bounded variation conditions over vertical lines for powers of the resolvent. Finally we discuss the Hilbert space case, where semigroup generation is characterised by the operator having a strong m-bounded calculus on a half-plane.

1. Introduction

The theory of strongly continuous semigroups is more than 60 years old, with the fundamental result of Hille and Yosida dating back to 1948. Several monographs and textbooks cover material which is now canonical, each of them having its own particular point of view. One may focus entirely on the semigroup, and consider the generator as a derived concept (as in [11]) or one may start with the generator and view the semigroup as the inverse Laplace transform of the resolvent (as in [2]).

Right from the beginning, *functional calculus* methods played an important role in semigroup theory. Namely, given a C_0-semigroup $T = (T(t))_{t\geq 0}$ which is uniformly bounded, say, one can form the averages

$$T_\mu := \int_0^\infty T(s)\,\mu(\mathrm{d}s) \qquad \text{(strong integral)}$$

for $\mu \in \mathrm{M}(\mathbb{R}_+)$ a complex Radon measure on $\mathbb{R}_+ = [0, \infty)$. If $-A$ denotes the generator of the semigroup, then one wants to interpret $T(t) = e^{-tA}$, and hence it is reasonable to define

$$f(A) := T_\mu, \quad \text{where} \quad f(z) = \mathcal{L}\mu(z) = \int_0^\infty e^{-sz}\,\mu(\mathrm{d}s) \qquad (\operatorname{Re} z > 0)$$

is the *Laplace transform* of μ. This functional calculus is called the *Hille–Phillips* calculus, and it is essentially based on methods from real analysis.

On the other hand, during the last two decades the theory of *holomorphic functional calculus* has proved to be an indispensable tool to deal with abstract evolution equations, above all in the discussion of maximal regularity [1, 23, 19]. Despite their success, these methods have been mainly restricted to sectorial operators and hence to holomorphic semigroups. DeLaubenfels [8, 9] devised a fairly general approach, however without large influence at the time. The holomorphic functional calculus for strip-type operators was developed in [14, 15], basically to treat C_0-groups that arise as imaginary powers of a sectorial operator.

In contrast to this progress, a treatment of general C_0-semigroups by means of a holomorphic functional calculus approach is missing, and the aim of this paper is to close this gap. (Some parts of the presented material are contained in an unpublished note [17] by the second author, but appear here in published form for

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the first time. The mere construction of the calculus can be found already in the papers of deLaubenfels mentioned above.)

The fundamental difference between the Hille–Phillips calculus and the holomorphic functional calculi is that the former starts with the semigroup, whereas the latter depart purely from resolvents. As such, holomorphic functional calculus can be used to establish *generation theorems* and we shall incorporate the two most important ones (the Hille–Yosida and the Gomilko–Shi–Feng theorems). In functional calculus terms, the generator property is (more or less) equivalent to $(e^{-tz})(A)$ being a bounded operator for each $t > 0$ (see Proposition 2.5 for a precise statement). In our *holomorphic* set-up, the operator $(e^{-tz})(A)$ is essentially given as the *inverse Laplace transform* of the resolvent, so the present paper can be seen as a systematisation of this approach to generation theorems that has often been used in an ad hoc way in the past.

The notion of bounded H^∞-calculus on sectors (or strips) has been extensively studied and it is known to have many applications to evolution equations [19]. It can be characterised by weak quadratic, or weak bounded variation, estimates [7, 23, 3], or by quadratic, or square function, estimates which are particularly simple in the case of Hilbert spaces. There is a corresponding definition of bounded H^∞-calculus on half-planes, but similar characterisations are not valid (see Section 7). Instead we show in Theorem 6.4 that a stronger form of weak bounded variation is equivalent to a weaker form of holomorphic functional calculus on half-planes. The stronger form of weak bounded variation is the condition shown by Gomilko [12] and Shi and Feng [26] to be sufficient for generation of a C_0-semigroup. For the weaker form of functional calculus, observe that if $f(z)$ is bounded and holomorphic on the right half-plane $\mathrm{R}_a := \{z \in \mathbb{C} \mid \operatorname{Re} z > a\}$, then $\|f'(z)\| \le (b-a)^{-1}\|f\|_{\mathrm{H}^\infty(\mathrm{R}_a)}$ whenever $\operatorname{Re} z > b > a$. Thus if A is an operator with bounded H^∞-calculus on R_b, it satisfies the following property which we call 1-bounded calculus:

$$\|f'(A)\| \le \frac{C}{b-a}\|f\|_{\mathrm{H}^\infty(\mathrm{R}_a)}$$

whenever $a < b$ and $f \in \mathrm{H}^\infty(\mathrm{R}_a)$, where C is independent of a. In contrast to strips, this type of functional calculus is not equivalent to bounded H^∞-calculus on half-planes, but it is equivalent to the condition of the Gomilko–Shi–Feng theorem.

1.1. **Some Notations and Definitions.** For a closed linear operator A on a complex Banach space X we denote by $\operatorname{dom}(A)$, $\operatorname{ran}(A)$, $\ker(A)$, $\sigma(A)$ and $\varrho(A)$ the *domain*, the *range*, the *kernel*, the *spectrum* and the *resolvent set* of A, respectively. The norm-closure of the range is written as $\overline{\operatorname{ran}}(A)$. The space of bounded linear operators on X is denoted by $\mathcal{L}(X)$. For two possibly unbounded linear operators A, B on X their *product* AB is defined on its natural domain $\operatorname{dom}(AB) := \{x \in \operatorname{dom}(B) \mid Bx \in \operatorname{dom}(A)\}$. An inclusion $A \subseteq B$ denotes inclusion of graphs, i.e., it means that B extends A. A possibly unbounded operator A on X *commutes* with a bounded operator $T \in \mathcal{L}(X)$ if $\operatorname{graph}(A)$ is $T\times T$-invariant, or equivalently if $TA \subseteq AT$.

We let $\mathbb{R}_+ := [0, \infty)$ and write $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ for the open, and $\overline{\mathbb{C}}_+ := \{z \in \mathbb{C} \mid \operatorname{Re} z \ge 0\}$ for the closed, right half-planes. More generally, for $\omega \in [-\infty, \infty]$ we denote by

$$\mathrm{L}_\omega := \{z \in \mathbb{C} \mid \operatorname{Re} z < \omega\}, \qquad \mathrm{R}_\omega := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega\},$$

the open left and right half-planes defined by the abscissa $\operatorname{Re} z = \omega$, where in the extremal cases one half-plane is understood to be empty, and the other is the whole complex plane. For a domain Ω, let $\mathcal{O}(\Omega)$ be the space of all holomorphic functions $f : \Omega \to \mathbb{C}$, and $\mathrm{H}^\infty(\Omega)$ be the subspace of all bounded holomorphic functions. We denote the supremum norm on $\mathrm{H}^\infty(\Omega)$ by $\|\cdot\|_{\mathrm{H}^\infty(\Omega)}$ or $\|\cdot\|_{\mathrm{H}^\infty}$, or simply $\|\cdot\|_\infty$ if the context excludes ambiguities.

In the present paper we shall use standard results about vector-valued holomorphic functions as collected in [2, Appendix A].

2. Holomorphic Functional Calculus on a Half-Plane

It is a standard fact from semigroup theory that if $-A$ generates a strongly continuous semigroup, then the spectrum $\sigma(A)$ of A is located in a right half-plane of $\mathbb{C}$, and the resolvent $R(\lambda, A) = (\lambda - A)^{-1}$ is uniformly bounded in the complementary left half-plane. We shall take this "spectral picture" as a starting point.

Definition 2.1. An operator A on a Banach space X is said to be *of half-plane type* $\omega \in (-\infty, \infty]$ (in short: $A \in \mathrm{HP}(\omega)$) if $\sigma(A) \subseteq \overline{\mathrm{R}_\omega}$ and

$$M_\alpha := M_\alpha(A) := \sup\{\|R(z, A)\| \mid \operatorname{Re} z \le \alpha\} < \infty \qquad \text{for every } \alpha < \omega.$$

An operator A is of *strong* half-plane type ω (in short: $A \in \mathrm{SHP}(\omega)$) if

$$M'_\alpha := M'_\alpha(A) := \sup\{|\alpha - \operatorname{Re} z| \, \|R(z, A)\| \mid \operatorname{Re} z \le \alpha\} < \infty$$

for every $\alpha < \omega$.

An operator A is said to be *of (strong) half-plane type* if it is of (strong) half-plane type ω for some $\omega \in (-\infty, \infty]$. One writes $A \in \mathrm{HP}(X)$ or $A \in \mathrm{SHP}(X)$, respectively.

Note that $\mathrm{SHP}(\omega) \subseteq \mathrm{HP}(\omega)$. If A is of half-plane type, then it is of half-plane type $s_0(A)$, where

$$s_0(A) := \max\{\omega \mid A \in \mathrm{HP}(\omega)\} = \sup\left\{\alpha \mid \sup_{\operatorname{Re} z \le \alpha} \|R(z, A)\| < \infty\right\}$$

is the *abscissa of uniform boundedness* of the operator A.

For $\omega \in \mathbb{R}$ we define

$$\mathcal{E}(\mathrm{R}_\omega) := \left\{f \in \mathcal{O}(\mathrm{R}_\omega) \mid f(z) = O\left(|z|^{-(1+s)}\right) \text{ as } |z| \to \infty, \text{ for some } s > 0\right\}.$$

Then $\mathcal{E}(\mathrm{R}_\omega)$ contains the functions $(\lambda - z)^{-1}(\mu - z)^{-1}$ whenever $\operatorname{Re} \lambda, \operatorname{Re} \mu < \omega$. For $f \in \mathcal{E}(\mathrm{R}_\omega)$ we have the following version of Cauchy's integral theorem.

Lemma 2.2. *Let $f \in \mathcal{E}(\mathrm{R}_\omega)$ and let $\omega < \delta$. Then*

$$f(a) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \delta} \frac{f(z)}{z - a} \, \mathrm{d}z \qquad (\delta < \operatorname{Re} a)$$

and

$$0 = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \delta} f(z) \, \mathrm{d}z.$$

The direction of integration is top down, i.e., from $\delta + i\infty$ to $\delta - i\infty$.

Proof. Fix $\operatorname{Re} a > \delta$. To establish the formula we employ the usual Cauchy theorem with the contour being the boundary of the rectangle $\operatorname{Im} z \in [-R, R], \operatorname{Re} z \in [\delta, \delta']$, for $\delta' > \operatorname{Re} a$, and $R > 0$ large. If we let $R \to \infty$ we see that

$$f(a) = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \delta} \frac{f(z)}{z - a} \, \mathrm{d}z - \frac{1}{2\pi i} \int_{\operatorname{Re} z = \delta'} \frac{f(z)}{z - a} \, \mathrm{d}z.$$

(The decay of $f(z)$ as $|\operatorname{Im} z| \to \infty$ causes the integrals to converge; and the integrals over the upper and lower rectangle sides vanish as R becomes large.) As a consequence of this representation we see that the value of the second integral does not depend on δ'. So we may let $\delta' \to +\infty$ without changing its value. But then this value has to be zero, because of the decay of f. The arguments in the second case are similar. $\square$

Now, let A be an operator of half-plane type, and let $\omega < \delta < s_0(A)$. Since the resolvent $R(\cdot, A)$ is bounded on the vertical line $\{\operatorname{Re} z = \delta\}$, the integral

$$\Phi(f) := f(A) := \frac{1}{2\pi i} \int_{\operatorname{Re} z=\delta} f(z) R(z, A)\, \mathrm{d}z$$

converges absolutely. By virtue of Cauchy's theorem (for vector-valued functions) and arguments similar to the proof of Lemma 2.2, the definition of $f(A)$ is independent of $\delta \in (\omega, s_0(A))$.

Proposition 2.3. *The so-defined mapping $\Phi : \mathcal{E}(\mathrm{R}_\omega) \to \mathcal{L}(X)$ satisfies the following properties:*

a) *Φ is a homomorphism of algebras.*
b) *If $T \in \mathcal{L}(X)$ commutes with A, i.e., $TA \subseteq AT$, it commutes with every $\Phi(f)$, $f \in \mathcal{E}$.*
c) *$\Phi(f(z)(\lambda - z)^{-1}) = \Phi(f)R(\lambda, A)$ whenever $\operatorname{Re}\lambda < \omega$.*
d) *$\Phi((\lambda - z)^{-1}(\mu - z)^{-1}) = R(\lambda, A)R(\mu, A)$ whenever $\operatorname{Re}\lambda, \operatorname{Re}\mu < \omega$.*

Proof. a) Additivity is clear. Multiplicativity follows from a combination of Fubini's theorem, the resolvent identity and Lemma 2.2. The computation is the same as in the classical Dunford–Riesz setting, see [6, VII.4.7].
b) is obvious.
c) By the resolvent identity and Lemma 2.2

$$\begin{aligned}
\Phi(f)R(\lambda, A) &= \frac{1}{2\pi i} \int_{\operatorname{Re} z=\delta} f(z) R(z, A) R(\lambda, A)\, \mathrm{d}z \\
&= \frac{1}{2\pi i} \int_{\operatorname{Re} z=\delta} \frac{f(z)}{\lambda - z} [R(z, A) - R(\lambda, A)]\, \mathrm{d}z \\
&= \frac{1}{2\pi i} \int_{\operatorname{Re} z=\delta} \frac{f(z)}{\lambda - z} R(z, A)\, \mathrm{d}z = \Phi\left(\frac{f(z)}{\lambda - z}\right).
\end{aligned}$$

d) We only give an informal argument and leave the details to the reader. In the integral

$$\frac{1}{2\pi i} \int_{\operatorname{Re} z=\delta} \frac{1}{(\lambda - z)(\mu - z)} R(z, A)\, \mathrm{d}z$$

we shift the path to the left, i.e., let $\delta \to -\infty$. When one passes the abscissas $\delta = \operatorname{Re}\lambda$ and $\delta = \operatorname{Re}\mu$, the residue theorem yields some additive contributions which sum up to $R(\lambda, A)R(\mu, A)$ by the resolvent identity; if $\delta < \operatorname{Re}\lambda, \operatorname{Re}\mu$, the integral does not change any more as $\delta \to -\infty$ and hence it is equal to zero. ☐

Denote by $\mathcal{M}(\mathrm{R}_\omega)$ the field of meromorphic functions on the right half-plane R_ω. Then the triple $(\mathcal{E}(\mathrm{R}_\omega), \mathcal{M}(\mathrm{R}_\omega), \Phi)$ is a meromorphic functional calculus in the sense of [19, Section 1.3]. A meromorphic function f is called *regularisable* if there is a function $e \in \mathcal{E}$ such that $ef \in \mathcal{E}$ and $e(A)$ is injective. In this case one defines

$$f(A) := e(A)^{-1}(ef)(A),$$

which is a closed operator. This definition does not depend on the chosen *regulariser* $e \in \mathcal{E}$ (cf. [19, Section 1.2.1] or [18]).

The basic rules governing this functional calculus are the same as for any meromorphic functional calculus, see [19, Theorem 1.3.2]. The two most important of these are the laws for sums:

$$f(A) + g(A) \subseteq (f + g)(A)$$

and products

$$f(A)g(A) \subseteq (fg)(A), \qquad \operatorname{dom}((fg)(A)) \cap \operatorname{dom}(g(A)) = \operatorname{dom}(f(A)g(A)).$$

In particular, one has $f(A) + g(A) = (f + g)(A)$ and $f(A)g(A) = (fg)(A)$ whenever $g(A) \in \mathcal{L}(X)$.

Note that every bounded holomorphic function $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$ is regularisable, namely by the function $e(z) := (\mu - z)^{-2}$, where $\operatorname{Re}\mu < \omega$. This is because $f(z)(\mu - z)^{-2}$ decreases quadratically as $|z| \to \infty$ and $e(A) = R(\mu, A)^2$ is clearly injective. In particular, for each $t \ge 0$ the operator

$$e^{-tA} := (e^{-tz})(A)$$

is defined as a closed operator and $\operatorname{dom}(A^2) \subseteq \operatorname{dom}(e^{-tA})$, $t \ge 0$.

Lemma 2.4. (Complex inversion formula)
Let A be an operator of half-plane type, and let $\omega < s_0(A)$. Then for each $x \in \operatorname{dom}(A^2)$ the function

$$(t \longmapsto e^{-tA}x) : [0, \infty) \longrightarrow X$$

is continuous and satisfies $\sup_{t>0} \left\| e^{\omega t} e^{-tA} x \right\| < \infty$. Its Laplace transform is

$$\int_0^\infty e^{-\lambda t} e^{-tA} x \,\mathrm{d}t = R(\lambda, -A)x = (\lambda + A)^{-1}x \qquad (\operatorname{Re}\lambda > -\omega).$$

Moreover, $e^{-tA}x$ is also given by the improper *integral*

$$e^{-tA}x = \frac{-1}{2\pi} \int_{-\infty}^{\infty} e^{-(\omega + is)t} R(\omega + is, A)x \,\mathrm{d}s \qquad (t > 0). \tag{2.1}$$

Proof. Fix $\mu < \omega$ and write $e^{-tA}x = (e^{-tz}/(\mu - z)^2)(A)[(\mu - A)^2 x]$. Then the continuity in t is clear from Lebesgue's theorem, and the bound is a simple estimate. An application of Fubini's theorem establishes the claim about the Laplace transform. To establish (2.1) we fix $t > 0$ and $\mu < \omega$ and note that

$$\int_{\operatorname{Re} z = \omega} \frac{e^{-zt}}{\mu - z} \,\mathrm{d}z = 0$$

in the improper sense. Indeed,

$$\frac{e^{-zt}}{\mu - z} = \frac{e^{-zt}}{t(\mu - z)^2} - \left(\frac{e^{-zt}}{t(\mu - z)} \right)',$$

the (improper) integral over the right-hand side being zero by Lemma 2.2. With this information at hand, we use the formula $(\mu - A)R(z, A) = (\mu - z)R(z, A) + I$ twice and compute

$$\begin{aligned} e^{-tA}x &= \left(\frac{e^{-tz}}{(\mu - z)^2} \right)(A)(\mu - A)^2 x = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \omega} \frac{e^{-tz}}{(\mu - z)^2} R(z, A)(\mu - A)^2 x \,\mathrm{d}z \\ &= \frac{1}{2\pi i} \int_{\operatorname{Re} z = \omega} \frac{e^{-tz}}{\mu - z} R(z, A)(\mu - A)x \,\mathrm{d}z = \frac{1}{2\pi i} \int_{\operatorname{Re} z = \omega} e^{-tz} R(z, A)x \,\mathrm{d}z. \end{aligned}$$

This concludes the proof. □

Note that if A is a densely defined operator with non-empty resolvent set, then $\operatorname{dom}(A^n)$ is dense in X for each $n \ge 2$. Indeed, $\operatorname{dom}(A^n) = \operatorname{ran} T^n$ for $T := R(\mu, A)$ and any $\mu \in \varrho(A)$; then inductively one obtains

$$\operatorname{ran} T = T(\overline{\operatorname{ran}}\, T^n) \subseteq \overline{T(\operatorname{ran} T^n)} = \overline{\operatorname{ran}}\, T^{n+1}$$

by the continuity of T, hence $X = \overline{\operatorname{ran}}\, T = \overline{\operatorname{ran}}\, T^{n+1}$.

Proposition 2.5. *Let A be an operator of half-plane type. Then $-A$ is the generator of a C_0-semigroup T if and only if A is densely defined and e^{-tA} is a bounded operator for all $t \in [0, 1]$ satisfying $\sup_{t \in [0,1]} \left\| e^{-tA} \right\| < \infty$. In this case, $T(t) = e^{-tA}$ for all $t \ge 0$.*

Proof. Let $-A$ generate a C_0-semigroup $(T(t))_{t\geq 0}$. Then A is densely defined, so $\mathrm{dom}(A^2)$ is dense. Lemma 2.4 yields that $R(\cdot,-A)x$ is the Laplace transform of $(t \mapsto e^{-tA}x)$ for $x \in \mathrm{dom}(A^2)$. By the uniqueness of Laplace transforms, $T(t)x = e^{-tA}x$, $t \geq 0$. Since $\mathrm{dom}(A^2)$ is dense and e^{-tA} is a closed operator, $e^{-tA} = T(t)$ is a bounded operator. The uniform boundedness for $t \in [0,1]$ is a standard property of C_0-semigroups.

Conversely, suppose that A is densely defined and $T(t) := e^{-tA}$ is a bounded operator for all $t \geq 0$. Then T is a semigroup (by general functional calculus) and $\mathrm{dom}(A^2)$ is dense. From the uniform boundedness $\sup_{t\in[0,1]} \|T(t)\| < \infty$ and the semigroup property, one concludes easily that $(T(t))_{t\geq 0}$ is uniformly bounded on compact intervals. Lemma 2.4 and the density of $\mathrm{dom}(A^2)$ imply that $(T(t))_{t\geq 0}$ is strongly continuous. Its Laplace transform coincides with the resolvent of $-A$ on $\mathrm{dom}(A^2)$ (Lemma 2.4), and hence on X by density. So $-A$ is the generator of T. □

Remarks 2.6.
a) We do not know if one can omit the boundedness assumption from Proposition 2.5.
b) The complex inversion formula (2.1) is well known when $(e^{-tA})_{t\geq 0}$ is a C_0-semigroup (see [21, Theorem 11.6.1], for example). In the opposite direction, many instances in the literature (for example, [21, Theorem 12.6.1]) use it as a starting point for generation theorems. Lemma 2.4 shows that our approach works in the latter direction. Moreover, it has the advantage of replacing tedious arguments involving closures and improper integrals by a clean algebraic extension procedure applied to the convenient functional calculus for elementary functions.

2.1. **Compatibility with the Sectorial Calculus.**

For $0 < \omega \leq \pi$ denote by

$$\mathrm{S}_\omega := \{z \in \mathbb{C} \mid 0 \neq z, \ \ |\arg z| < \omega\}$$

the open sector symmetric about the positive real axis with vertex 0 and angle 2ω. The degenerate case is $\mathrm{S}_0 := (0,\infty)$.

An operator A on a Banach space X is called *sectorial* of angle $\varphi \in [0,\pi)$ if $\sigma(A) \subseteq \overline{\mathrm{S}_\varphi}$ and for each $\omega \in (\varphi,\pi)$ one has

$$M(\omega,A) := \sup\{\|zR(z,A)\| \ \mid z \in \mathbb{C} \setminus \overline{\mathrm{S}_\omega}\} < \infty.$$

The minimal φ such that A is sectorial of angle φ is called the *angle of sectoriality.* There is a natural holomorphic functional calculus for sectorial operators, discussed at length in [19].

Examples 2.7.
1) If A satisfies an estimate $\|R(z,A)\| \leq M\,|\mathrm{Re}\, z|^{-1}$ for $\mathrm{Re}\, z < 0$, then A is sectorial of angle $\pi/2$. In particular, if $-A$ generates a bounded semigroup or if A is an operator of strong half-plane type $\omega > 0$, then A is sectorial of angle $\pi/2$.
2) On the other hand, if A is invertible and sectorial of angle $\varphi < \pi/2$, then A is of half-plane type ω for some $\omega > 0$.

We note the following result about compatibility of functional calculi.

Proposition 2.8. *Suppose that A is of half-plane type $\omega > 0$ and that f is a holomorphic function on $\mathrm{R}_0 = \mathrm{S}_{\pi/2}$. If A is also sectorial and $f(A)$ is defined in either calculus, then both definitions lead to the same operator.*

Proof. We only sketch the basic ingredients of the proof and leave the details to the reader. Let $\omega_s(A)$ be the sectoriality angle of A. Suppose first that $\omega_s(A) \geq \pi/2$. Let $\omega_s(A) < \varphi < \pi$ and take f in the class $\mathrm{H}_0^\infty(\mathrm{S}_\varphi)$ of holomorphic functions on S_φ defined in [19, p.28]. Then $g(z) := f(z)(1+z)^{-1} \in \mathcal{E}(\mathrm{R}_0)$. But $g(A)$ is the same in either calculus by contour deformation. Hence $f(A)$ is the same in either calculus. This establishes a *morphism* of the calculus on the sector S_φ to the calculus on the half-plane R_0, and the compatibility for other functions follows from [19, Prop.1.2.7].

In the case that $\omega_s(A) < \pi/2$ take $f \in \mathcal{E}(\mathrm{R}_0)$. Then f is in the elementary calculus for *invertible* sectorial operators, see [19, Sec.2.5.1] and $f(A)$ is the same in either calculus by contour deformation. Again we have a morphism and that establishes the claim. □

3. The Hille–Yosida Theorem

The Hille–Yosida theorem is one of the most fundamental results in the "elementary" theory of C_0-semigroups. We show that it is a straightforward consequence of the following general fact of functional calculus theory.

Theorem 3.1. (Convergence Lemma)
Let A be a densely defined operator of half-plane type on a Banach space X, and let $\omega < s_0(A)$. Let I be a directed set, and let K be any index set. Suppose that $(f_{\iota,\kappa})_{(\iota,\kappa)\in I\times K} \subseteq \mathrm{H}^\infty(\mathrm{R}_\omega)$ has the following properties:

1) $\sup_{\iota,\kappa} \|f_{\iota,\kappa}\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)} < \infty$;
2) $f_{\iota,\kappa}(A) \in \mathcal{L}(X)$ *for all* ι, κ, *and* $\sup_{\iota,\kappa} \|f_{\iota,\kappa}(A)\| < \infty$;
3) $f_\kappa(z) := \lim_\iota f_{\iota,\kappa}(z)$ *exists for every* $z \in \mathrm{R}_\omega$ *uniformly in* $\kappa \in K$.

Then $f_\kappa \in \mathrm{H}^\infty(\mathrm{R}_\omega)$, $f_\kappa(A) \in \mathcal{L}(X)$, $f_{\iota,\kappa}(A)x \to f(A)x$ for each $x \in X$ uniformly in $\kappa \in K$, and $\|f_\kappa(A)\| \le \limsup_\iota \|f_{\iota,\kappa}(A)\|$.

Proof. The proof is analogous to the proof of [19, Proposition 5.1.4]. By [2, Proposition A.3], the function $F_\iota := (f_{\iota,\kappa})_\kappa : \mathrm{R}_\omega \to \ell^\infty(K)$ is bounded and holomorphic. Vitali's theorem [2, Theorem A.5] for nets implies that $F := (f_\kappa)_\kappa$ is holomorphic and that the convergence $F_\iota \to F$ is uniform on compact subsets of R_ω. Moreover, condition 1) clearly implies that F is bounded.

Let $\mu < \omega < \delta < s_0(A)$. Then

$$\left(f_{\iota,\kappa}(z)(\mu - z)^{-2}\right)(A) = \lim_{n\to\infty} \frac{-1}{2\pi} \int_{-n}^{n} \frac{f_{\iota,\kappa}(\delta + is)}{(\mu - \delta - is)^2} R(\delta + is, A)\,\mathrm{d}s,$$

where the limit is uniform in ι and κ. Together with the uniform convergence of the integrand on $[-n, n]$, it follows that $(f_{\iota,\kappa}(z)(\mu - z)^{-2})(A) \to (f_\kappa(z)(\mu - z)^{-2})(A)$ in norm, uniformly in $\kappa \in K$. Hence for $x \in \mathrm{dom}(A^2)$,

$$f_{\iota,\kappa}(A)x = \Big(\frac{f_{\iota,\kappa}(z)}{(\mu - z)^2}\Big)(A)(\mu - A)^2 x \to \Big(\frac{f_\kappa(z)}{(\mu - z)^2}\Big)(A)(\mu - A)^2 x = f_\kappa(A)x$$

uniformly in $\kappa \in K$. Clearly $\|f_\kappa(A)x\| \le \limsup_\iota \|f_{\iota,\kappa}(A)\|\,\|x\|$. Since $f(A)$ is a closed operator with $\mathrm{dom}(f(A)) \supseteq \mathrm{dom}(A^2)$, which is dense, $f_\kappa(A)$ is bounded and $\|f_\kappa(A)\| \le \limsup_\iota \|f_{\iota,\kappa}(A)\|$. Again by the density of $\mathrm{dom}(A^2)$, $f_{\iota,\kappa}(A) \to f_\kappa(A)$ strongly, uniformly in $\kappa \in K$. □

Remark 3.2. In a similar way one can extend the classical instances of convergence lemmas for sectorial and strip-type operators [19, Prop. 5.1.4, 5.1.7] to parametrized families of functions.

Theorem 3.3. (Hille–Yosida)
Let A be a densely defined operator on the Banach space X such that $(-\infty, 0) \subseteq \varrho(A)$ and $M := \sup_{n\in\mathbb{N},\lambda>0} \|\lambda^n(\lambda + A)^{-n}\| < \infty$. Then A is of strong half-plane type 0 *and $\left\|e^{-tA}\right\| \le M$ for all $t \ge 0$.*

Proof. First we show that A is of half-plane type. Fix μ such that $\operatorname{Re}\mu > 0$. For large $\lambda > 0$, more precisely for $\lambda > |\mu|^2/(2\operatorname{Re}\mu)$, one has $|\lambda - \mu| < \lambda$. By the Taylor series expansion of the resolvent it follows that $-\mu \in \varrho(A)$ and

$$(\mu + A)^{-1} = \sum_{n=0}^{\infty} (\lambda - \mu)^n (\lambda + A)^{-(n+1)}.$$

Estimating norms we obtain

$$\left\|(\mu + A)^{-1}\right\| \le M \sum_{n=0}^{\infty} \frac{|\lambda - \mu|^n}{\lambda^{n+1}} = \frac{M}{\lambda - |\lambda - \mu|}.$$

and with $\lambda \to \infty$ we conclude that $\left\|(\mu + A)^{-1}\right\| \le M/\operatorname{Re}\mu$. It follows that A is of strong half-plane type 0.

Define $r_{n,t}(z) := (1 + (tz)/n)^{-n}$. For fixed $\omega < 0$ and large $n \in \mathbb{N}$ we have

$$\sup_{\operatorname{Re} z \ge \omega} |r_{n,t}(z)| = \left(\inf_{\operatorname{Re} z \ge \omega} \left|1 + \frac{tz}{n}\right|\right)^{-n} = \left(1 + \frac{t\omega}{n}\right)^{-n}.$$

Since $(1 + t\omega/n)^{-n} \to e^{-t\omega}$ as $n \to \infty$, we have $\sup_n \|r_{n,t}\|_{\mathrm{H}^\infty(\mathrm{R}_\omega)} < \infty$. Also, by hypothesis,

$$\|r_{n,t}(A)\| = \left\|(1 + {}^{t}\!/\!_{n}\, A)^{-n}\right\| = \left\|({}^{n}\!/\!_{t})^n ({}^{n}\!/\!_{t} + A)^{-n}\right\| \le M \qquad \text{for all } n \in \mathbb{N}.$$

Applying the Convergence Lemma yields $\left\|e^{-tA}\right\| \le M$, as desired. $\square$

If we re-examine the proof in view of the dependence on $t \ge 0$, we obtain the following.

Corollary 3.4. (Euler approximation)
Let $-A$ be the generator of a uniformly bounded C_0-semigroup $(T(t))_{t \ge 0}$ on a Banach space X. Then for each $x \in X$

$$\left[\left(1 + {}^{t}\!/\!_{n}A\right)^{-1}\right]^n x \to T(t)x \quad \text{as } n \to \infty$$

uniformly in t from compact subintervals of $[0, \infty)$.

Proof. Standard semigroup theory yields that A satisfies the hypotheses of the Hille–Yosida Theorem 3.3. The proof of that theorem shows that $r_{n,t}(A)x \to e^{-tA}x = T(t)x$ for each $x \in X$ and for each $t \ge 0$. However, by employing the full force of the Convergence Lemma 3.1, one can adapt the proof in order to show that $r_{n,t}(A)x \to T(t)x$ uniformly in t from compact subintervals of $\mathbb{R}_+$, for each $x \in X$. $\square$

Of course, one can use the Convergence Lemma to establish the Euler approximation for any (not necessarily bounded) C_0-semigroup on a Banach space.

Remark 3.5. Note that it follows from Proposition A.7 that "higher-order" Hille–Yosida estimates imply the lower-order ones, i.e.,

$$\sup_{n \in \mathbb{N}, \lambda > 0} \left\|\lambda^n (\lambda + A)^{-n}\right\| = \sup_{n \ge N, \lambda > 0} \left\|\lambda^n (\lambda + A)^{-n}\right\|$$

for each $N \ge 1$.

4. The Trotter–Kato Theorem

While in the Convergence Lemma the function is approximated and the operator is fixed, in the following we fix the function and approximate the operator. The correct set-up requires that the approximants A_n are "of the same type" as the operator, with the relevant constants being uniformly bounded.

More precisely, a family of operators $(A_\iota)_\iota$ is called *uniformly of half-plane type* $\omega \in \mathbb{R} \cup \{-\infty\}$ if each A_ι is of half-plane type ω and $\sup_\iota M_\alpha(A_\iota) < \infty$ for each $\alpha < \omega$.

Example 4.1. Let A be of half-plane type 0, and let $A_\lambda := \lambda A(\lambda + A)^{-1}$ for $\lambda \geq 1$ be the *Yosida approximants*. Then the family $(A_\lambda)_{\lambda\geq 1}$ is uniformly of half-plane type 0. Indeed, for $\operatorname{Re}\mu < 0$ a little computation shows that $\mu \in \varrho(A_\lambda)$ with

$$R(\mu, A_\lambda) = \frac{\lambda^2}{(\lambda-\mu)^2} R\Big(\frac{\lambda\mu}{\lambda-\mu}, A\Big) - \frac{1}{\lambda-\mu}. \tag{4.1}$$

Note that

$$\operatorname{Re}\left(\frac{\lambda\mu}{\lambda-\mu}\right) = \lambda^2 \operatorname{Re}\frac{1}{\lambda-\mu} - \lambda = \frac{\lambda^2 \operatorname{Re}\mu - \lambda\,|\mu|^2}{|\lambda-\mu|^2} < 0.$$

The second term in (4.1) can be estimated by

$$\frac{1}{|\lambda-\mu|} \leq \min\Big(\frac{1}{\lambda}, \frac{1}{|\operatorname{Re}\mu|}\Big).$$

If A satisfies an estimate $\|R(z, A)\| \leq M/|\operatorname{Re} z|$ for $\operatorname{Re} z < 0$ (e.g., in the case that $-A$ generates a bounded C_0-semigroup) then

$$\|R(\mu, A_\lambda)\| \leq \frac{\lambda^2}{|\lambda-\mu|^2}\frac{M\,|\lambda-\mu|^2}{\lambda^2|\operatorname{Re}\mu| + \lambda\,|\mu|^2} + \frac{1}{|\lambda-\mu|} \leq \frac{M+1}{|\operatorname{Re}\mu|},$$

independently of $\lambda > 0$. In the general case, fix $\alpha > 0$ and define $\epsilon := \alpha/(\alpha+1)$. Then $\epsilon\alpha = \alpha - \epsilon \leq \lambda(\alpha - \epsilon)$ since $\epsilon < \alpha$ and $\lambda \geq 1$. Hence, if $\operatorname{Re}\mu \leq -\alpha$,

$$\operatorname{Re}\left(\frac{\lambda\mu}{\lambda-\mu}\right) = \operatorname{Re}\frac{\lambda^2}{\lambda-\mu} - \lambda \leq \frac{\lambda^2}{|\lambda-\mu|} - \lambda \leq \frac{\lambda^2}{\lambda+\alpha} - \lambda \leq -\epsilon.$$

It follows that

$$\|R(\mu, A_\lambda)\| \leq \frac{\lambda^2}{(\lambda - \operatorname{Re}\mu)^2} M_{-\epsilon}(A) + \frac{1}{\lambda - \operatorname{Re}\mu} \leq M_{-\epsilon}(A) + 1$$

whenever $\lambda \geq 1$ and $\operatorname{Re}\mu \leq -\alpha$.

In the previous example we clearly have $\lim_{\lambda\to\infty} R(\mu, A_\lambda) = R(\mu, A)$ in norm uniformly in μ from compact subsets of the open half-plane $\{\operatorname{Re} z < 0\}$.

Proposition 4.2. *Let $(A_\iota)_{\iota\in I}$ be a net of operators, uniformly of half-plane type τ, and let A be an operator such that $R(\mu, A_\iota) \to R(\mu, A)$ in norm/strongly whenever $\operatorname{Re}\mu < \tau$. Then A is also of half-plane type τ. Moreover, for $\omega < \tau$ and $f \in \mathcal{E}(\mathrm{R}_\omega)$ one has $f(A_\iota) \to f(A)$ in norm/strongly.*

Suppose furthermore that A is densely defined. If $f \in \mathrm{H}^\infty(\mathrm{R}_\omega)$ and $f(A_\iota) \in \mathcal{L}(X)$ for all $\iota \in I$ with $C := \sup_\iota \|f(A_\iota)\| < \infty$, then also $f(A) \in \mathcal{L}(X)$, and $f(A_\iota) \to f(A)$ strongly.

Proof. The first assertion is straightforward. For the second we employ once again Vitali's theorem [2, Theorem A.5] for nets to conclude that the convergence of the resolvents is uniform in μ from compact subsets of $\{\operatorname{Re} z < \tau\}$. Then a standard argument similar to part of the proof of Theorem 3.1 shows that $f(A_\iota) \to f(A)$ in norm/strongly.

Now suppose that A is densely defined, so $\operatorname{dom}(A^2)$ is dense in X. Take $x \in \operatorname{dom}(A^2)$, $f \in \mathrm{H}^\infty(L_\omega)$ and $e(z) := f(z)(\mu - z)^{-2} \in \mathcal{E}(\mathrm{R}_\omega)$, where $\mu < \omega$ is fixed. Then

$$\left\|e(A_\iota)(\mu - A)^2 x\right\| = \left\|f(A_\iota)R(\mu, A_\iota)^2(\mu - A)^2 x\right\| \leq C\left\|R(\mu, A_\iota)^2(\mu - A)^2 x\right\|.$$

Since we know already that $e(A_\iota) \to e(A)$ and $R(\mu, A_\iota) \to R(\mu, A)$, we conclude that

$$\|f(A)x\| = \left\|e(A)(\mu - A)^2 x\right\| \leq C\left\|R(\mu, A)^2(\mu - A)^2 x\right\| = C\,\|x\|.$$

Since $\operatorname{dom}(A^2)$ is dense, it follows that $f(A) \in \mathcal{L}(X)$ with $\|f(A)\| \leq C$. To prove that $f(A_\iota) \to f(A)$ (strongly), we need only to show $f(A_\iota)x \to f(A)x$ for all

$x \in \mathrm{dom}(A^2)$. So take $x \in \mathrm{dom}(A^2)$ and let $y := (\mu - A)^2 x$. We have seen above that $f(A_\iota)R(\mu, A_\iota)^2 y \to f(A)x$, so we estimate the difference

$$\begin{aligned} \left\| f(A_\iota)x - f(A_\iota)R(\mu, A_\iota)^2 y \right\| &= \left\| f(A_\iota)R(\mu, A)^2 y - f(A_\iota)R(\mu, A_\iota)^2 y \right\| \\ &\leq C \left\| R(\mu, A)^2 y - R(\mu, A_\iota)^2 y \right\| \to 0 \end{aligned}$$

by hypothesis. □

Remark 4.3. As in the Convergence Lemma, there is a version of Proposition 4.2 that yields some uniformity: suppose that $(f_\kappa)_{\kappa \in K} \subseteq \mathrm{H}^\infty(\mathrm{R}_\omega)$ is uniformly bounded and $\sup_{\iota,\kappa} \|f_\kappa(A_\iota)\| < \infty$. Then the convergence $f_\kappa(A_\iota)x \to f_\kappa(A)x$ is uniform in κ, for every $x \in X$.

Another variant of Proposition 4.2 considers parametrized nets $(A_{\iota,\kappa})_{\iota,\kappa}$. We leave the details to the reader.

Theorem 4.4. (Trotter–Kato)
Suppose that, for each $n \in \mathbb{N}$, A_n is the generator of a C_0-semigroup, and that $\left\| e^{-tA_n} \right\| \leq M$ for all $t \geq 0, n \in \mathbb{N}$. Suppose further that A is a densely defined operator and for some $\lambda_0 < 0$ one has $\lambda_0 \in \varrho(A)$ and $R(\lambda_0, A_n) \to R(\lambda_0, A)$ strongly. Then A generates a C_0-semigroup and one has $e^{-tA_n}x \to e^{-tA}x$ uniformly in $t \in [0, \tau]$, for each $x \in X, \tau > 0$.

Proof. The theorem is a consequence of Proposition 4.2 and Remark 4.3, as soon as we show that actually $\{\mathrm{Re}\, z < 0\} \subseteq \varrho(A)$ and $R(\mu, A_n) \to R(\mu, A)$ strongly *whenever* $\mathrm{Re}\,\mu < 0$. This is done as in [11, Proposition III.4.4]. □

Remark 4.5. A common assumption on A_n and A implying that $R(\lambda_0, A_n) \to R(\lambda_0, A)$ strongly is the following: the operator A is densely defined, $\lambda_0 - A$ has dense range, and there exists a core D of A such that $A_n x \to Ax$ for all $x \in D$. See [11, Theorem III.4.9].

However, one might not always be given the operator A. Instead one may know that $R(\lambda_0, A_n) \to Q \in \mathcal{L}(X)$ strongly, and Q has dense range. By general arguments as in [19, Appendix A.5] one has $Q = R(\lambda_0, A)$ for some possibly multi-valued operator A, which is densely defined by the range assumption on Q. It then remains to show that A is in fact single-valued, i.e., Q is injective.

5. Weak Bounded Variation Conditions and m-Bounded Calculus

Let A be of half-plane type ω and $\alpha < \omega$. Then A is said to have *bounded* H^∞*-calculus* on R_α if there is $M \geq 0$ such that

$$\|f(A)\| \leq M \|f\|_{\infty,\mathrm{R}_\alpha} \tag{5.1}$$

for all $f \in \mathrm{H}^\infty(\mathrm{R}_\alpha)$. If A is densely defined, it suffices to have (5.1) for all $f \in \mathcal{E}(\mathrm{R}_\alpha)$. Indeed, one can apply the convergence lemma to the functions

$$f_n(z) := f(z) \left(\frac{n}{n - \alpha + z} \right)^2 \qquad (\mathrm{Re}\, z \geq \alpha, n \in \mathbb{N}).$$

(Note that $|f_n| \leq |f|$ for all $n \in \mathbb{N}$.)

Remark 5.1. This notion of bounded H^∞-calculus for operators of half-plane type is completely analogous to the notion of bounded H^∞-calculus on sectors and strips for sectorial and strip-type operators, respectively [19, Chapter 5]. For sectorial operators this notion plays an essential role in applications to evolution equations, in particular to the problem of maximal regularity [23]. Via the log / exp-correspondence, bounded H^∞-calculus on strips and sectors can be reduced to one another, see [19, Prop.5.3.3] or [15].

The theory of bounded H^∞-calculus on half-planes has been unexplored up to now. In fact, there have been only two classes of examples: operators similar to generators of (quasi-)contraction semigroups on Hilbert spaces, and the sectorial operators with a bounded sectorial calculus of angle $< \pi/2$. A genuine class of operators of half-plane type (generators of non-holomorphic bounded semigroups, say) with a bounded H^∞-calculus on a half-plane is missing, let alone a characterization of that notion.

Now, for a *sectorial* operator A of angle ω_0, the boundedness of the H^∞-calculus on a sector S_ω can — under certain additional conditions on the Banach space and the resolvent — be characterized by so-called "square function" or "quadratic" estimates on A and A^*. In the simplest case and if $X = H$ is a Hilbert space, these are of the form

$$\int_0^\infty \|\varphi(tA)x\|^2 \, \frac{\mathrm{d}t}{t} \le M \|x\|^2 \qquad (x \in X),$$

where φ is a non-zero holomorphic function on the sector S_ω satisfying $|\varphi(z)| \le \min(|z|^s, |z|^{-s})$ for some $s > 0$. In the non-Hilbertian case, the form of the quadratic estimates is different, and it leads to the so-called γ-spaces [22, 13]. Alternatively, building on [4] Cowling et al. in [7] introduced *weak quadratic estimates* of the form

$$\int_0^\infty |\langle \varphi(tA)x, x'\rangle| \le M \|x\| \|x'\| \qquad (x \in X,\ y \in X') \tag{5.2}$$

and showed that under certain conditions on the function φ these indeed characterize the boundedness of a H^∞-calculus on a sector. (We are oversimplifying here, please consult [7] for the precise statements.) If one uses the function $\varphi(z) = \frac{z}{(e^{\pm i\theta}-z)^2}$ for $\pi > \theta > \omega_0$, then (5.2) takes the form

$$\int_{\partial \mathrm{S}_\theta} \left|\left\langle AR(\lambda, A)^2 x, x'\right\rangle\right| \, |\mathrm{d}\lambda| \le M \|x\| \|x'\| \qquad (x \in X, x' \in X'),$$

a condition put forward by Kunstmann and Weis in [23]. It was shown in [3] that for *strip-type operators* A one has an analogous condition, namely

$$\int_{\partial \mathrm{St}_\theta} \left|\left\langle R(\lambda, A)^2 x, x'\right\rangle\right| \, |\mathrm{d}\lambda| \le M \|x\| \|x'\| \qquad (x \in X, x' \in X'). \tag{5.3}$$

Here St_θ is the vertical strip $\{z \in \mathbb{C} \mid -\theta < \operatorname{Re} z < \theta\}$, so the integral is over two vertical lines. These conditions represent *weak bounded variation* of the functions $AR(\lambda, A)$ and $R(\lambda, A)$ respectively. Our aim in the present section is to explore the condition (5.3) when we replace the strip by a half-plane. Let us begin with an auxiliary result.

Lemma 5.2. *Let $\alpha \in \mathbb{R}$ and let A be an operator such that the vertical line $\alpha + i\mathbb{R}$ is contained in $\varrho(A)$. Suppose further that for some $n \ge 1$ and $C \ge 0$ one has*

$$\int_{\mathbb{R}} \left|\left\langle R(\alpha + it, A)^{n+1} x, x'\right\rangle\right| \, \mathrm{d}t \le C \|x\| \, \|x'\| \qquad (x \in X, x' \in X'). \tag{5.4}$$

Then $\sup_{t \in \mathbb{R}} \|R(\alpha + it, A)^n\| < \infty$. *If A is densely defined or X is reflexive then*

$$\langle R(\alpha{+}it, A)^n x, x'\rangle = -\int_{-\infty}^t in \left\langle R(\alpha{+}is, A)^{n+1} x, x'\right\rangle \, \mathrm{d}s \qquad (x \in X,\ x' \in X')$$

and $\|R(\alpha + it, A)^n\| \le nC$ *for each* $t \in \mathbb{R}$.

Proof. We have $\frac{\mathrm{d}}{\mathrm{d}t} R(\alpha{+}it, A)^n = -niR(\alpha{+}it, A)^{n+1}$, hence (5.4) yields an operator $Q : X \to X''$ such that

$$\langle R(\alpha + it, A)^n x, x'\rangle - \langle Qx, x'\rangle = -\int_{-\infty}^t ni \left\langle R(\alpha + is, A)^{n+1} x, x'\right\rangle \, \mathrm{d}s$$

for all $t \in \mathbb{R}$ and $x \in X, x' \in X'$. It follows that

$$\sup_{t\in\mathbb{R}} \|R(\alpha + it, A)^n\| \le nC + \|Q\| < \infty.$$

Moreover, we have $R(\alpha + it, A)^n x \to Qx$ as $t \to -\infty$ in the weak*-sense on X'' for each $x \in X$. By Lemma A.1 we have $R(\alpha, A)^n R(\alpha + it, A)^n x \to 0$ in norm. It follows that

$$(R(\alpha, A)'')^n \, Qx = QR(\alpha, A)^n x = 0 \qquad (x \in X).$$

If $\mathrm{dom}(A)$ is dense, then Q vanishes on the dense subspace $\mathrm{dom}(A^n)$, hence $Q = 0$. If X is reflexive then $X'' = X$, and $R(\alpha, A)'' = R(\alpha, A)$ is injective, whence also $Q = 0$. □

We now investigate the consequences of a weak bounded variation condition (5.4) for the functional calculus. We need to introduce the notion of m-bounded calculus. The corresponding notion on strips is equivalent to bounded H^∞-calculus [3, Proposition 2.7]. We shall see in Section 7 below that (strong) 1-bounded calculus on half-planes does not imply bounded H^∞-calculus even for operators on Hilbert space.

Definition 5.3. Let A be an operator of half-plane type ω on a Banach space X, and let $\beta < \omega$ and $m \in \mathbb{N}_0$. Then A is said to have *m-bounded calculus* on R_β if there is $K \ge 0$ such that $f^{(m)}(A) \in \mathcal{L}(X)$ and

$$\|f^{(m)}(A)\| \le K \, \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\beta)} \qquad \text{for all } f \in \mathrm{H}^\infty(\mathrm{R}_\beta).$$

Let $K(A, \beta, m)$ be the least such K.

It may not be immediately clear that this definition is meaningful. The problem is resolved by the Cauchy inequalities [27, Corollary 4.3, p.48].

Lemma 5.4. *Let $f \in \mathrm{H}^\infty(\mathrm{R}_\alpha)$ and $m \in \mathbb{N}$. For $a \in \mathrm{R}_\alpha$ and $0 < r < \mathrm{Re}(a) - \alpha$,*

$$|f^{(m)}(a)| \le \frac{m!}{r^m} \sup\{|f(z)| \mid |z - a| = r\}.$$

Hence $f^{(m)} \in \mathrm{H}^\infty(\mathrm{R}_\beta)$ with

$$\|f^{(m)}\|_{\mathrm{H}^\infty(\mathrm{R}_\beta)} \le \frac{m!}{(\beta - \alpha)^m} \, \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\alpha)} \, . \tag{5.5}$$

Examples 5.5. For $\lambda \in \mathbb{C}$ and $t \ge 0$, define

$$f_\lambda(z) = \frac{1}{\lambda - z} \, , \qquad e_t(z) = e^{-tz}.$$

Then $f_\lambda \in \mathrm{H}^\infty(\mathrm{R}_\beta)$ if $\beta > \mathrm{Re}\,\lambda$ and $e_t \in \mathrm{H}^\infty(\mathrm{R}_\beta)$ for all $\beta \in \mathbb{R}$. If A has m-bounded calculus on R_β, then

$$\begin{aligned} \left\|R(\lambda, A)^{m+1}\right\| &\le \frac{K(A, \beta, m)}{m!(\beta - \mathrm{Re}\,\lambda)} && (\mathrm{Re}\,\lambda < \beta), \\ \|\exp(-tA)\| &\le \frac{K(A, \beta, m)}{t^m} \, e^{-t\beta} && (t > 0). \end{aligned}$$

We can now establish the connection between the weak bounded variation condition (5.4) and the new notion of m-bounded calculus.

Theorem 5.6. *Let A be a densely defined operator of half-plane type on a Banach space X, and let $m \ge 1$. Then the following assertions hold.*

a) *If $\alpha < s_0(A)$ and there is a constant C such that*

$$\int_{\mathbb{R}} \left|\langle R(\alpha + it, A)^{m+1} x, x' \rangle\right| \, \mathrm{d}t \le C \, \|x\| \, \|x'\| \qquad (x \in X, x' \in X'), \tag{5.6}$$

then A has m-bounded calculus on R_α, and

$$K(A,\alpha,m) \le \frac{m!}{2\pi} C(A,\alpha,m).$$

b) *If $\beta < s_0(A)$ and A has $(m-1)$-bounded calculus on R_β, then (5.6) holds for each $\alpha < \beta$, with*

$$C = \frac{\pi}{m!\,(\beta-\alpha)} K(A,\beta,m-1).$$

When (5.6) holds for some C, we shall denote the least such C by $C(A,\alpha,m)$.

Proof. a) Let $\beta < \alpha$ and $f \in \mathcal{E}(\mathrm{R}_\beta)$. It is easily derived from Lemma 5.4 that $f^{(m)} \in \mathcal{E}(\mathrm{R}_{\beta'})$ for each $\beta' > \beta$. Hence we can compute $f^{(m)}(A)$ as

$$f^{(m)}(A) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Re}\, z = \alpha} f^{(m)}(z) R(z,A)\,\mathrm{d}z = \frac{-1}{2\pi} \int_{\mathbb{R}} f^{(m)}(\alpha + it) R(\alpha+it, A)\,\mathrm{d}t.$$

Since $R(\cdot, A)$ is uniformly bounded on $\alpha + i\mathbb{R}$ and $f^{(k)}(\alpha+it) \to 0$ as $|t| \to \infty$ for each $k \ge 0$, we can integrate by parts m times to obtain

$$\begin{aligned} f^{(m)}(A) &= \frac{-1}{2\pi} \int_{\mathbb{R}} f^{(m-1)}(\alpha+it) R(\alpha+it,A)^2\,\mathrm{d}t \\ &= \cdots = \frac{-m!}{2\pi} \int_{\mathbb{R}} f(\alpha+it) R(\alpha+it,A)^{m+1}\,\mathrm{d}t. \end{aligned}$$

Now it follows from (5.6) that

$$\|f^{(m)}(A)\| \le \frac{C\, m!}{2\pi} \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\alpha)}\,.$$

Next, we employ an approximation argument of a standard type. Let $f \in \mathrm{H}^\infty(\mathrm{R}_\beta)$ and define $\varphi_k(z) := k^2(k-\beta+z)^{-2}$ for $k \ge 1$. Then

1) $f\varphi_k \in \mathcal{E}(\mathrm{R}_\beta)$,
2) $(f\varphi_k)^{(m)}(z) \to f^{(m)}(z)$ as $k \to \infty$,
3) $\|f\varphi_k\|_{\mathrm{H}^\infty(\mathrm{R}_\alpha)} \le \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\alpha)}$,
4) $\sup_k \|(f\varphi_k)^{(m)}\|_{\mathrm{H}^\infty(\mathrm{R}_\alpha)} < \infty$.

By the Convergence Lemma (Theorem 3.1),

$$\|f^{(m)}(A)\| \le \sup_k \|(f\varphi_k)^{(m)}(A)\| \le \frac{C\, m!}{2\pi} \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\alpha)}.$$

In the last step we start with $f \in \mathrm{H}^\infty(\mathrm{R}_\alpha)$ and let $f_k(z) := f(z + 1/k)$. By what we have proved already,

$$\|f_k^{(m)}(A)\| \le \frac{C\, m!}{2\pi} \|f_k\|_{\mathrm{H}^\infty(\mathrm{R}_\alpha)} \le \frac{C\, m!}{2\pi} \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\alpha)} < \infty.$$

The Convergence Lemma, again, yields that $f^{(m)}(A) \in \mathcal{L}(X)$ with

$$\|f^{(m)}(A)\| \le \frac{C\, m!}{2\pi} \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\alpha)} < \infty,$$

as desired.

b) We fix $\alpha < \beta$, $x \in X$, $x' \in X'$ and $R > 0$. Let ϵ be a measurable function such that $|\epsilon(t)| = 1$ and

$$\big|\langle R(\alpha+it,A)^{m+1}x, x'\rangle\big| = \langle R(\alpha+it,A)^{m+1}x, x'\rangle \cdot \epsilon(t)$$

for all $t \in \mathbb{R}$. Define

$$f(z) := \frac{1}{m!} \int_{-R}^{R} \frac{\epsilon(t)}{(\alpha+it-z)^2}\,\mathrm{d}t \qquad (\mathrm{Re}\, z > \beta).$$

Then f is an elementary function with

$$f^{(m-1)}(z) = \int_{-R}^{R} \frac{\epsilon(t)}{(\alpha + it - z)^{m+1}} \, \mathrm{d}t \qquad (\operatorname{Re} z > \beta).$$

Moreover,

$$|f(z)| \le \frac{1}{m!} \int_{\mathbb{R}} \frac{\mathrm{d}t}{(\operatorname{Re} z - \alpha)^2 + t^2} = \frac{\pi}{m!\,(\operatorname{Re} z - \alpha)} \le \frac{\pi}{m!\,(\beta - \alpha)} \qquad (\operatorname{Re} z > \beta).$$

Fubini's theorem yields

$$f^{(m-1)}(A) = \int_{-R}^{R} \epsilon(t) R(\alpha + it, A)^{m+1} \, \mathrm{d}t,$$

from which it follows that

$$\begin{aligned}\int_{-R}^{R} \left|\langle R(\alpha + it, A)^{m+1} x, x' \rangle\right| \, \mathrm{d}t &= \int_{-R}^{R} \epsilon(t) \left\langle R(\alpha + it, A)^{m+1} x, x' \right\rangle \, \mathrm{d}t \\ &= \left|\left\langle f^{(m-1)}(A)x, x' \right\rangle\right| \le K(A, \beta, m-1) \, \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\beta)} \, \|x\| \, \|x'\| \\ &\le \frac{\pi}{m!\,(\beta - \alpha)} K(A, \beta, m-1) \;\; \|x\| \, \|x'\| \, .\end{aligned}$$

As $R > 0$ was arbitrary, b) is proved. □

Remark 5.7. We note that the assumption of dense domain in Theorem 5.6 is needed only in part a), to pass from elementary functions to all H^∞-functions. Moreover, the proof shows that under the assumptions of a) one has $\|f^{(m)}(A)\| \le \frac{Cm!}{2\pi} \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\alpha)}$ if $f^{(m)}$ is elementary on a larger half-plane and $f^{(j)}$ vanishes at $\alpha \pm i\infty$ for all $0 \le j \le m-1$. An example of a function f that is not elementary but satisfies these requirements is $f(z) = (\lambda - z)^{-1}$ with $\operatorname{Re} \lambda < \alpha$.

6. Strong m-Bounded Calculus

Theorem 5.6 does not establish an equivalence of m-bounded calculus and weak bounded variation estimates of the form (5.6), due to the "loss of order" in part b). We shall see below that this phenomenon can be avoided when one considers m-bounded calculus and estimates (5.6) with specified dependence of $C(A, \alpha, m)$ and $K(A, \alpha, m)$ on α. This leads to the following definition.

Definition 6.1. An operator A has *strong m-bounded calculus of type* ω if A has m-bounded calculus on R_β for each $\beta < \omega$, and there is $C \ge 0$ such that

$$\left\| f^{(m)}(A) \right\| \le \frac{C}{(\omega - \beta)^m} \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\beta)} \qquad (f \in \mathrm{H}^\infty(\mathrm{R}_\beta), \beta < \omega).$$

Let us revisit Examples 5.5.

Examples 6.2. If A has strong m-bounded calculus of type ω, then one has

$$\left\| R(\lambda, A)^{m+1} \right\| \le \frac{C}{m!\,(\omega - \beta)^m (\beta - \operatorname{Re} \lambda)} \qquad (\operatorname{Re} \lambda < \beta < \omega).$$

The minimum on the right-hand side is attained at $\beta = (m \operatorname{Re} \lambda + \omega)/(m+1)$ and hence we obtain the Hille–Yosida estimate

$$\left\| R(\lambda, A)^{m+1} \right\| \le \frac{Ce(m+1)}{m!(\omega - \operatorname{Re} \lambda)^{m+1}} \qquad (\operatorname{Re} \lambda < \omega). \tag{6.1}$$

By Corollary A.9 it follows that

$$\left\| R(\lambda, A)^k \right\| \le \frac{Ce(m+1)}{m!(\omega - \operatorname{Re} \lambda)^k} \qquad (\operatorname{Re} \lambda < \omega)$$

for all $1 \le k \le n$; in particular, A is of strong half-plane type ω.

Similarly, we have

$$\|\exp(-tA)\| \le \frac{C}{t^m(\omega-\beta)^m} e^{-t\beta} \qquad (t>0,\ \beta<\omega),$$

and minimizing the right-hand side over $\beta < \omega$ yields $\beta = \omega - \frac{m}{t}$ and

$$\|\exp(-tA)\| \le C\frac{e^m}{m^m} e^{-t\omega} \qquad (t>0). \tag{6.2}$$

In particular, if A is densely defined, then by Proposition 2.5 $-A$ generates a C_0-semigroup.

For $K(A,\beta,m)$ to behave like $(\omega-\beta)^{-m}$, the inequalities in Theorem 5.6 and simple examples indicate that we expect $C(A,\alpha,m)$ to behave like $(\omega-\alpha)^{-m}$. We shall see in the next proposition that such an estimate is actually independent of m (with varying constants, of course) and this will lead to a characterisation in Theorem 6.4 below.

Proposition 6.3. *Let A be an operator on a Banach space X and $\omega \in \mathbb{R}$ such that $\sigma(A) \subseteq \overline{\mathrm{R}_\omega}$. In addition, let $m \ge 1$ and $C \ge 0$ such that*

$$\int_{\mathbb{R}} \left|\langle R(\alpha+it,A)^{m+1}x, x'\rangle\right| \,\mathrm{d}t \le \frac{C}{(\omega-\alpha)^m}\,\|x\|\,\|x'\| \qquad (\alpha<\omega,\ x\in X, x'\in X'). \tag{6.3}$$

Then the following assertions hold.

a) *If A is of half-plane type ω, then for each $1 \le k \le m{+}1$*

$$\left\|R(\lambda,A)^k\right\| \le \frac{Ce(m{+}1)}{2\pi(\omega-\operatorname{Re}\lambda)^k} \qquad (\operatorname{Re}\lambda<\omega). \tag{6.4}$$

In particular, A is of strong half-plane type ω.

b) *If X is reflexive then A is densely defined.*

Moreover, under the additional hypothesis that A is densely defined the following assertions hold:

c) *For each $1 \le k \le m$*

$$\left\|R(\lambda,A)^k\right\| \le \frac{Cm}{(\omega-\operatorname{Re}\lambda)^k} \qquad (\operatorname{Re}\lambda<\omega). \tag{6.5}$$

In particular, A is of strong half-plane type ω.

d) *For each $1 \le k \le m$ and $\alpha < \omega$*

$$\int_{\mathbb{R}} \left|\langle R(\alpha+it,A)^{k+1}x, x'\rangle\right| \,\mathrm{d}t \le \frac{Cm}{k(\omega-\alpha)^k}\|x\|\,\|x'\| \quad (x\in X, x'\in X').$$

e) *A has a strong k-bounded calculus of type ω for each $1 \le k \le m$. More precisely, one has*

$$\left\|f^{(k)}(A)\right\| \le \frac{Cm}{2\pi}\frac{(k{-}1)!}{(\omega-\alpha)^k}\,\|f\|_{\mathrm{H}^\infty(\mathrm{R}_\alpha)} \qquad (f\in \mathrm{H}^\infty(\mathrm{R}_\alpha), \alpha<\omega). \tag{6.6}$$

Proof. a) By Remark 5.7 it follows from (6.3) that $\left\|f^{(m)}(A)\right\| \le (Cm!/2\pi)\,\|f\|_{\mathrm{H}^\infty(\mathrm{R}_\alpha)}$ for $f(z) := (\lambda-z)^{-1}$ and $\operatorname{Re}\lambda < \alpha < \omega$. This yields

$$\left\|R(\lambda,A)^{m+1}\right\| \le \frac{C}{2\pi}\frac{1}{(\omega-\alpha)^m(\alpha-\operatorname{Re}\lambda)} \qquad (\operatorname{Re}\lambda<\alpha<\omega).$$

As in (6.1), varying α here leads to

$$\left\|R(\lambda,A)^{m+1}\right\| \le \frac{C}{2\pi}\frac{(m+1)^{m+1}}{m^m}\frac{1}{(\omega-\operatorname{Re}\lambda)^{m+1}} \le \frac{Ce(m+1)}{2\pi(\omega-\operatorname{Re}\lambda)^{m+1}}.$$

Now a) follows from Corollary A.9.
b) and c) If X is reflexive or A is densely defined then (6.5) follows from Lemma 5.2

for $k = m$ and then from Corollary A.9 for other values of k. In particular we have that $A - \omega$ is sectorial, i.e., $\sup_{\lambda<0} \|\lambda R(\lambda, A-\omega)\| \le Cm < \infty$. It is a standard result [19, Prop.2.1.1] that A must be densely defined if X is reflexive.
d) We employ downward induction, the case $k = m$ being the hypothesis (6.3). For the step from k to $k-1$ let $x \in X$ and $x' \in X'$; then a) together with Lemma 5.2 and Fubini's theorem yield

$$\begin{aligned}\int_{\mathbb{R}} \left|\langle R(\alpha+it, A)^k x, x'\rangle\right| \,\mathrm{d}t &= k \int_{\mathbb{R}} \left|\langle \int_{-\infty}^{\alpha} R(u+it, A)^{k+1} x, x'\rangle \,\mathrm{d}u\right| \mathrm{d}t \\ &\le k \int_{-\infty}^{\alpha} \int_{\mathbb{R}} \left|\langle R(u+it, A)^{k+1} x, x'\rangle\right| \mathrm{d}t\, \mathrm{d}u \le k \frac{mC}{k} \left(\int_{-\infty}^{a} \frac{\mathrm{d}u}{(\omega-u)^k} \right) \|x\|\, \|x'\| \\ &= \frac{mC}{(k-1)(\omega-\alpha)^{k-1}} \|x\|\, \|x'\|.\end{aligned}$$

e) follows directly from d) and Theorem 5.6. □

Proposition 6.3 enables us to prove the following major result. The fact that the condition (i) (for $m = 1$) implies that $\omega - A$ generates a bounded C_0-semigroup was proved in [12, 26] and it is known as the Gomilko-Shi-Feng theorem. An early version can be found in [21, Theorem 12.6.1].

Theorem 6.4. *Let $\omega \in \mathbb{R}$ and let A be a densely defined operator on a Banach space X such that $\mathrm{L}_\omega \subseteq \varrho(A)$. The following assertions are equivalent for $m \ge 1$:*

(i) *There exists a constant C_m such that*

$$\int_{\mathbb{R}} \left|\langle R(\alpha+it, A)^{m+1} x, x'\rangle\right| \mathrm{d}t \le \frac{C_m}{(\omega-\alpha)^m} \|x\|\, \|x'\| \qquad (x \in X,\ x' \in X', \alpha < \omega);$$

(ii) *A is of half-plane type ω and has strong m-bounded calculus of type ω;*
(iii) *A is of half-plane type ω and has strong 1-bounded calculus of type ω.*

In particular, properties (i) *and* (ii) *are independent of $m \ge 1$. They imply that $-A$ generates a C_0-semigroup T with $\|T(t)\| \le Me^{-\omega t}$ for some M.*

Proof. (i)⇒(ii),(iii): This follows from Proposition 6.3.
(iii)⇒(ii): Let $\alpha < \beta < \omega$ and $f \in \mathrm{H}^\infty(\mathrm{R}_\alpha)$. We apply the strong 1-bounded calculus to $f^{(m-1)} \in \mathrm{H}^\infty(\mathrm{R}_\beta)$ and employ (5.5) to obtain

$$\|f^{(m)}(A)\| \le \frac{K}{\omega-\beta} \|f^{(m-1)}\|_{\mathrm{H}^\infty(\mathrm{R}_\beta)} \le \frac{KC_{m-1}(m-1)!}{(\omega-\beta)(\beta-\alpha)^{m-1}} \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\alpha)}.$$

Optimising with respect to β yields

$$\|f^{(m)}(A)\| \le \frac{KC_{m-1}e\, m!}{(\omega-\alpha)^m} \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\alpha)}.$$

(ii)⇒(i): Suppose we have

$$\|f^m(A)\| \le \frac{C}{(\omega-\beta)^m} \|f\|_{\mathrm{H}^\infty(\mathrm{R}_\beta)} \qquad (\beta < \omega,\ f \in \mathrm{H}^\infty(\mathrm{R}_\beta)).$$

Then by Theorem 5.6.b) with m replaced by $m+1$,

$$\int_{\mathbb{R}} \left|\left\langle R(\alpha+it, A)^{m+2} x, x'\right\rangle\right| \mathrm{d}t \le \frac{\pi C}{(m+1)!\, (\beta-\alpha)(\omega-\beta)^m} \|x\|\, \|x'\|$$

for $\alpha < \beta < \omega$. Optimising with respect to β yields

$$\int_{\mathbb{R}} \left|\left\langle R(\alpha+it, A)^{m+2} x, x'\right\rangle\right| \mathrm{d}t \le \frac{\pi Ce}{m!\, (\omega-\alpha)^{m+1}} \|x\|\, \|x'\|\,.$$

Now it follows from Proposition 6.3.d) with m replaced by $(m+1)$ and $k = m$ that

$$\int_{\mathbb{R}} \left|\left\langle R(\alpha+it, A)^{m+1} x, x'\right\rangle\right| \mathrm{d}t \le \frac{\pi Ce\left(1+\frac{1}{m}\right)}{m!\, (\omega-\alpha)^m} \|x\|\, \|x'\|\,.$$

Finally, suppose that (i)-(iii) hold. Then by Example 6.2 and the density of $\operatorname{dom}(A)$ in X we obtain that $-A$ generates a C_0-semigroup T with $\|T(t)\| \le Me^{-\omega t}$ for some $M \ge 1$. □

Corollary 6.5. *Let $\omega \in \mathbb{R}$ and let A be a densely defined operator on a Banach space X such that $\mathrm{L}_\omega \subseteq \varrho(A)$. Assume that there is a constant C such that, for all $\alpha < \omega$,*

$$\begin{aligned} (\omega-\alpha)\int_{\mathbb{R}} \|R(\alpha+it,A)x\|^2\,\mathrm{d}t &\le C\|x\|^2 \qquad (x\in X),\\ (\omega-\alpha)\int_{\mathbb{R}} \|R(\alpha+it,A)'x'\|^2\,\mathrm{d}t &\le C\|x'\|^2 \qquad (x'\in X'). \end{aligned}$$

Then A has strong 1-bounded calculus of type ω.

Proof. It is a standard application of the Cauchy–Schwarz inequality that the assumptions imply (i) of Theorem 6.4 for $m = 1$. □

The following generalisation is proved in the same way.

Theorem 6.6. *Let $\omega \in \mathbb{R}$ and let A be a densely defined operator on a Banach space X such that $\mathrm{L}_\omega \subseteq \varrho(A)$. Let in addition $m \ge 1$ and $g : (-\infty, \omega) \to [0, \infty)$ be a function satisfying $g(\alpha) = O(|\alpha|^{-m})$ as $\alpha \to -\infty$ and*

$$\int_{\mathbb{R}} \left|\left\langle R(\alpha+it,A)^{m+1}x, x'\right\rangle\right|\,\mathrm{d}t \le g(\alpha)\|x\|\,\|x'\| \qquad (x\in X, x'\in X', \alpha<\omega).$$

Then $-A$ generates a C_0-semigroup T with

$$\|T(t)\| \le \frac{m!e}{2\pi} g(\omega - t^{-1}) t^{-m} e^{-\omega t} \qquad (t > 0).$$

Proof. From Lemma 5.2 we obtain

$$\|R(\alpha+it,A)^m\| \le m g(\alpha) = O\big(\,|\alpha|^{-m}\,\big).$$

Hence Corollary A.9 implies that A is of (strong) half-plane type ω. Now we can apply Theorem 5.6 to obtain

$$\|f^{(m)}(A)\| \le \frac{m!}{2\pi} g(\alpha) \qquad (\alpha<\omega,\ f\in \mathrm{H}^\infty(\mathrm{R}_\alpha)).$$

Inserting $f(z) = e^{-tz}$ and optimizing over $\alpha < \omega$ yields

$$\big\|e^{-tA}\big\| \le \frac{m!}{2\pi} t^{-m} \inf_{\alpha<\omega} g(\alpha) e^{-t\alpha} \le \frac{m!e}{2\pi} g\big(\omega - t^{-1}\big) t^{-m} e^{-\omega t}.$$

(The last inequality comes from specializing $\alpha := \omega - t^{-1}$.) □

Remarks 6.7. 1) For $\omega = 0$, $m = 1$ and $g(\alpha) = M|\alpha|^{-1}(1 + |\alpha|^{-d})$, Theorem 6.6 recovers a result of Eisner [10] on semigroups with polynomial growth.

2) The converse of the Gomilko-Shi-Feng theorem holds in Hilbert spaces (see Theorem 7.1), but it was observed in [12, p.296] that it is not true in general Banach spaces or even in reflexive spaces. Thus there exist operators A such that $-A$ generates a C_0-semigroup but A does not have 1-bounded calculus on any half-plane.

3) The property (i) of Theorem 6.4 for $m = 1$ is a natural form of weak bounded variation for half-plane operators. Actually, it is somewhat stronger than the corresponding properties for sectors and strips, because of the dependence on α. For sectors and strips, weak bounded variation for a single value of the corresponding parameter implies bounded H^∞-calculus [7, 23, 3]. We shall see in Example 7.2 that strong 1-bounded caclulus on half-planes does not imply

bounded $^\infty$-calculus even for operators on Hilbert space. Thus bounded H^∞-calculus on half-planes cannot be characterised by weak bounded variation of the resolvent in the way that it can for strips and sectors.

7. Operators of Half-Plane Type on Hilbert Spaces

It was observed in [12] and [26] that the converse of the Gomilko-Shi-Feng theorem is true in Hilbert spaces. We present a reformulation of this fact in which the implication (i) $\implies$ (iii), together with Examples 5.5, recovers the Gearhart–Prüss theorem. This says that the exponential growth bound of a C_0-semigroup with generator $-A$, on a Hilbert space, equals $-s_0(A)$ [2, Theorem 5.2.1].

Theorem 7.1. *Let A be a densely defined operator of half-plane type on a Hilbert space H. The following properties are equivalent:*

(i) *$-A$ generates a C_0-semigroup on H;*
(ii) *A has strong 1-bounded calculus on a half-plane;*
(iii) *A has strong 1-bounded calculus of type ω for each $\omega < s_0(A)$.*

Proof. The implication (iii)⇒(ii) is trivial, and the implication (ii)⇒(i) follows from Theorem 6.4. For the proof of the implication (i)⇒(iii) suppose that $-A$ generates T. Then there exist $M \geq 1$ and $\omega_0 \in \mathbb{R}$ with $\|T(s)\| \leq Me^{-\omega_0 s}$ for $s \geq 0$. For $\alpha < \omega_0$ and $x \in X$ we have

$$R(\alpha + it, A)x = -\int_0^\infty e^{-its}e^{\alpha s}T(s)x \,\mathrm{d}s$$

and hence by Plancherel's theorem

(7.1)
$$\|R(\alpha+it,A)x\|_{\mathrm{L}_2(\mathbb{R};\mathrm{d}t)} = \sqrt{2\pi}\,\|e^{\alpha s}T(s)x\|_{\mathrm{L}_2(\mathbb{R}_+;\mathrm{d}s)} \leq M\|x\|\sqrt{\frac{\pi}{\omega_0-\alpha}} \qquad (x \in H).$$

Now let $\omega < s_0(A)$ and take $\alpha < \min(\omega, \omega_0)$. By definition of $s_0(A)$, $R(\cdot, A)$ is uniformly bounded on $\omega + i\mathbb{R}$. Hence, from the resolvent identity

$$R(\omega+it,A)x = (I + (\alpha-\omega)R(\omega+it,A))R(\alpha+it,A)x \qquad (t \in \mathbb{R})$$

it follows that $\|R(\omega+it,A)x\|_{\mathrm{L}_2(\mathbb{R},\mathrm{d}t)} \leq C\,\|x\|$ for some constant C. Plancherel's theorem again yields

$$\|e^{\omega s}T(s)x\|_{\mathrm{L}_2(\mathbb{R}_+,\mathrm{d}s)} \leq \sqrt{2\pi}C\,\|x\|.$$

Convolving with $e^{\alpha s}T(s)$, using $\|e^{\alpha s}T(s)\| \in \mathrm{L}_2(\mathbb{R}_+)$ yields

$$\frac{e^{\omega t}-e^{\alpha t}}{\omega-\alpha}\,\|T(t)x\| = \left\|\int_0^t e^{\alpha(t-s)}T(t-s)e^{\omega s}T(s)x\mathrm{d}s\right\| \leq C'\,\|x\| \qquad (t \geq 0).$$

Consequently, $\|T(t)\| \leq M_\omega e^{-\omega t}$ (cf. Datko's theorem [11, Theorem V.1.8]). It follows that we can replace ω_0 by ω in (7.1) (with a different M, of course). Passing to adjoint operators, we obtain similarly

$$\begin{aligned}\|R(\alpha+it,A)^*y\|_{\mathrm{L}_2(\mathbb{R};\mathrm{d}t)} &= \sqrt{2\pi}\,\|e^{\alpha s}T(s)^*y\|_{\mathrm{L}_2(\mathbb{R}_+;\mathrm{d}s)} \\ &\leq M_\omega\|y\|\sqrt{\frac{\pi}{\omega-\alpha}} \qquad (y \in H, \alpha < \omega).\end{aligned}$$

Now Corollary 6.5 applies, and A has strong 1-bounded calculus of type ω. □

By the Boyadzhiev–deLaubenfels theorem [5, 14, 16, 20] a densely defined operator on a Hilbert space generates a C_0-group if and only if it has a bounded H^∞-calculus on a vertical strip. The following example shows that the analogue involving semigroups and H^∞-calculus on half-planes does not hold. In particular, strong 1-bounded calculus does not imply bounded H^∞-calculus, in the case of half-planes.

Example 7.2. Let X be a separable Hilbert space. Using the theory of conditional Schauder bases, Le Merdy [24] has shown that there exists an operator A on X such that

1) A is densely defined, invertible and sectorial of angle 0, and
2) A does not have BIP (bounded imaginary powers).

(See also [19, Cor.9.1.8] for this construction.) Then $-A$ generates a bounded holomorphic C_0-semigroup. For $\alpha < 0$, because A is invertible, $A - \alpha$ does not have BIP [19, Proposition 3.5.5]. Consequently A does not have a bounded $H^\infty(\mathrm{R}_\alpha)$-calculus for any $\alpha < 0$.

A sectorial operator on a Hilbert space that has bounded H^∞-calculus on some sector has it on each sector of angle larger than the angle of sectoriality. The "reason" is a theorem of Cowling, Doust, McIntosh and Yagi based on an approximation result for analytic functions on strips [19, Thm.5.4.1], together with the Gearhart-Prüss theorem. In a sense, Theorem 7.1 is the analogue for operators of half-plane type. We conjecture that the corresponding result involving H^∞-calculus (in place of strong 1-bounded calculus) is false in general, but we cannot prove that at this point. The best we can achieve here is the following positive result.

Proposition 7.3. *Let A be a densely defined operator of half-plane type on a Hilbert space H such that A has a bounded H^∞-calculus on some right half-plane. If the semigroup generated by $-A$ is holomorphic then A has bounded H^∞-calculus R_α for each $\alpha < s_0(A)$.*

Proof. For $\beta < s_0(A)$, $A - \beta$ is invertible and sectorial of angle less than $\pi/2$. Moreover, A has bounded H^∞-calculus on R_β if and only if $A - \beta$ has bounded H^∞-calculus on $\mathrm{R}_0 = \mathrm{S}_{\pi/2}$. (Note that by Proposition 2.8 the half-plane calculus coincides with the sectorial calculus.) Now, by McIntosh's theorem [19, Thm.7.3.1], $A-\beta$ has bounded H^∞-calculus on $\mathrm{S}_{\pi/2}$ if and only if $A-\beta$ has bounded imaginary powers. But by [19, Proposition 3.5.5], this is independent of β. □

Suppose that $-A$ generates a C_0-semigroup of *contractions* on H. Then A has contractive $\mathrm{H}^\infty(\mathrm{R}_\alpha)$-calculus for each $\alpha < 0$. This is basically equivalent to the classical von Neumann inequality for contractions on Hilbert spaces and can be proved (via the convergence lemma) as in [19, p.179].

It follows that if $-A$ generates T, and T is *similar* to a quasi-contraction semigroup (contractive after shifting), then A has bounded H^∞-calculus on a right half-plane. The converse, however, is false.

Example 7.4. In [25, Prop.4.8] Le Merdy modifies the famous counterexample of Pisier to Halmos' problem to give an example of an operator A on a Hilbert space H with the following properties:

1) $-A$ generates a bounded C_0-semigroup T on H and is injective;
2) There is $c \geq 0$ such that $\|r(A)\| \leq c\,\|r\|_\infty$ for each rational function r that is bounded on R_0;
3) T is not similar to a contraction semigroup.

By approximating holomorphic functions by rational functions [19, Prop.F.3] it follows from the convergence lemma that A has a bounded H^∞-calculus on R_α for each $\alpha < 0$. Finally, one can employ Chernoff's trick as in [19, Lemma 7.3.14] to obtain an example where T is not even similar to a quasi-contraction semigroup.

Appendix A. Asymptotics of Resolvents

In this appendix we prove some results on the asymptotic behaviour of the resolvent mapping $z \mapsto R(z, A)$ for some (in general unbounded) operator A on a

Banach space X. More precisely, we suppose that $D \subseteq \varrho(A)$ is a subset of the resolvent set of A such that the point at infinity is an accumulation point of D, i.e.,

$$\{z \in D \mid |z| \geq r\} \neq \emptyset \quad \text{for each } r > 0;$$

and we shall look at the asymptotic behaviour of R_z as $z \to \infty$, by which we mean $z \in D$ and $|z| \to \infty$.

For simplicity and because the operator A is rather unimportant in this context, we abbreviate $R_z := R(z, A)$. The operator family $(R_z)_{z \in D}$ satisfies the *resolvent identity*

$$(z - w) R_z R_w = R_w - R_z \qquad (z, w \in D), \tag{A.1}$$

the basis of all following arguments.

We begin with a technical result that contains all the essence. Let Y be a second Banach space and fix a vector space topology on $\mathcal{L}(Y; X)$ such that for each $w \in D$ the operator $S \mapsto R_w S$ on $\mathcal{L}(Y; X)$ is continuous. (For example, the norm topology and the weak and strong operator topology satisfy this.) By $\Phi : D \to \mathcal{L}(Y; X)$ we denote any operator-valued function on D.

Lemma A.1. *With the terminological conventions from above, let $n \in \mathbb{N}_0$, and suppose that $R_z^n \Phi_z / z \to 0$ and $\Phi_z / z^n \to 0$ as $|z| \to \infty$, $z \in D$. Then for any fixed $w \in D$*

$$R_w^n R_z^n \Phi_z \to 0 \quad \textit{as } |z| \to \infty,\ z \in D.$$

Proof. The proof is by induction on $n \in \mathbb{N}_0$. If $n = 0$ then there is nothing to prove. So suppose $n \geq 1$ and the claim to be true for $n - 1$ in place of n. Then by the resolvent identity

$$R_w R_z^n \Phi_z = R_z^{n-1} \frac{R_w \Phi_z}{z - w} - \frac{R_z^n \Phi_z}{z - w} \tag{A.2}$$

By hypothesis, the second summand tends to 0 as $z \to \infty$. If we define $\Psi_z := R_w \Phi_z / (z - w)$, then $\Psi_z / z^{n-1} \to 0$ as $z \to \infty$. Moreover,

$$R_z^{n-1} \Psi_z / z = \left(R_w R_z^n \Phi_z + \frac{R_z^n \Phi_z}{z - w} \right) / z \to 0 \quad \text{as} \quad z \to \infty$$

by (A.2) and the hypothesis. So we can apply the induction hypothesis with Ψ_z in place of Φ_z and conclude that

$$R_w^{n-1} R_z^{n-1} \frac{R_w \Phi_z}{z - w} = R_w^{n-1} R_z^{n-1} \Psi_z \to 0 \qquad (z \to \infty).$$

Now we multiply (A.2) by R_w^{n-1} from the left, and the claim follows. $\square$

We are now in position to prove our main result.

Theorem A.2. *With the terminological conventions from above, let $n, j \in \mathbb{N}_0$, and suppose that $R_z^n \Phi_z / z^j \to 0$ and $\Phi_z / z^n \to 0$ as $|z| \to \infty$, $z \in D$. Then*

$$R_w^{jn} R_z^n \Phi_z \to 0 \quad \textit{as } |z| \to \infty,\ z \in D.$$

Proof. We proceed by induction over $j \geq 0$. For $j = 0$ there is nothing to prove. Suppose that $R_z^n \Phi_z / z^{j+1} \to 0$. Then by the induction hypothesis (with Φ_z / z in place of Φ) $R_w^{nj} R_z^n \Phi_z / z \to 0$. Now we apply Lemma A.1 with $R_w^{nj} \Phi_z$ in place of Φ_z to obtain $R_w^n R_w^{nj} R_z^n \Phi_z \to 0$. $\square$

Remark A.3. Theorem A.2 can be interpreted in terms of *extrapolation spaces.* Namely, for $w \in D$ fixed, the n-th extrapolation norm on X is given by $\|x\|_{-n} := \|R_w^n x\|$, $x \in X$. (A different choice of w leads to an equivalent norm.) Theorem A.2 roughly says that if $R_z^n \Phi_z$ has only polynomial growth as $|z| \to \infty$ along D, then it must tend to 0 in some (sufficiently high) extrapolation norm.

Corollary A.4. *Let $Q(z) = \sum_{j=0}^{m} z^j Q_j$ be a polynomial in z with coefficients $Q_j \in \mathcal{L}(Y;X)$. If $R_z^n \Phi_z - Q(z) \to 0$ and $\Phi_z/z^n \to 0$ as $|z| \to \infty$, $z \in D$, then $Q = 0$.*

Proof. By hypothesis, $R_z^n \Phi_z / z^m \to Q_m$. And $\Phi_z/z^{n+m} \to 0$ since $m \geq 0$. By Theorem A.2 it follows that $R_w^n Q_m = 0$, whence $Q_m = 0$, since R_w is injective. Inductively, one obtains $Q_j = 0$ for $j = m-1, m-2, \ldots, 0$, i.e., $Q = 0$. □

Remark A.5. Let us stress the fact that our formulation covers a wide range of individual results, by varying Y and the topology on $\mathcal{L}(X)$. Clearly the norm, strong and weak operator topology are admissible choices. If one takes $Y = \mathbb{C}$ to be 1-dimensional, then $\mathcal{L}(Y;X)$ is essentially equal to X, and one can interpret Φ_z as an element of X. In particular, for the case $\Phi_z := \varphi(z)x$ for a fixed vector $x \in X$ and a scalar function φ, we obtain results about the asymptotic behaviour of the "individual" orbit $z \mapsto R_z^n x$.

Corollary A.6. *If $\sup_{z \in D} \|R_z^n\| < \infty$, then for each $w \in D$ $R_w^n R_z^n \to 0$ in norm as $z \in D, |z| \to \infty$. If in addition $\operatorname{dom}(A)$ is dense, then $R_z^n \to 0$ strongly.*

Proof. We apply Lemma A.1 with $\Phi = I$ and conclude that $R_z^n R_w^n \to 0$ in norm. Note that $\operatorname{ran}(R_w) = \operatorname{dom}(A)$. If this is dense, then by induction $\operatorname{ran}(R_w^n) = \operatorname{dom}(A^n)$ is dense as well. Then the claim follows by approximation. □

Let us apply our results to sectorial operators and operators of (strong) half-plane type. An operator A is *sectorial* if $(-\infty, 0) \subseteq \varrho(A)$ and $\sup_{z<\infty} \|zR(z,A)\| < \infty$. The next result shows that "higher-order" sectoriality is equivalent to sectoriality.

Proposition A.7. *Let $M \geq 0$ and $n \in \mathbb{N}$, and let A be a closed operator on a Banach space X such that $(-\infty, 0) \subseteq \varrho(A)$ and*

$$\left\|R(z,A)^k\right\| \leq \frac{M}{|z|^k} \qquad (z < 0) \tag{A.3}$$

for $k = n$. Then (A.3) *holds for all $1 \leq k \leq n$. In particular, A is sectorial.*

Proof. It suffices to prove (A.3) for $k = n$ under the assumption that it holds for $k = n+1$. Since $-nR_z^{n+1}$ is the derivative of R_z^n, the estimate (A.3) for $k = n+1$ shows that R_z^n has integrable derivative on $(-\infty, 0)$. In particular, $Q := \lim_{z \to -\infty} R_z^n$ exists by Cauchy's criterion, and

$$R_z^n - Q = -\int_{-\infty}^{z} nR_t^{n+1} \, \mathrm{d}t \qquad (-\infty < z < 0).$$

It follows from Corollary A.4 that $Q = 0$; taking norms yields

$$\|R_z^n\| \leq -\int_{-\infty}^{z} n \left\|R_t^{n+1}\right\| \mathrm{d}t \leq Mn \int_{-\infty}^{z} |t|^{n+1} \, \mathrm{d}t = \frac{M}{|z|^n}$$

for $\infty < z < 0$. □

By rotating and shifting, we immediately obtain the following corollaries.

Corollary A.8. *Let $\theta_1, \theta_2 \in \mathbb{R}$ and let $S := \{0 \neq z \in \mathbb{C} \mid \theta_1 \leq \arg z \leq \theta_2\}$ be a corresponding sector in the complex plane. Let $M \geq 0$ and $n \in \mathbb{N}$, and let A be an operator such that $S \subseteq \varrho(A)$ and*

$$\sup_{z \in S} \|z^n R(z,A)^n\| \leq M.$$

Then $\sup_{z \in S} \left\|[z^k R(z,A)^k\right\| \leq M$ for each $1 \leq k \leq n$.

Proof. Fix $\theta \in [\theta_1, \theta_2]$ and apply Proposition A.7 to $-e^{-i\theta}A$ in place of A. □

Corollary A.9. *Let $M \geq 0$, $n \in \mathbb{N}$ and $\omega \in \mathbb{R}$, and let A be an operator such that $\{z \in \mathbb{C} \mid \operatorname{Re} z < \omega\} \subseteq \varrho(A)$ with*

$$\|R(z,A)^n\| \leq \frac{M}{(\omega - \operatorname{Re} z)^n} \qquad (\operatorname{Re} z < \omega).$$

Then

$$\left\|R(z,A)^k\right\| \leq \frac{M}{(\omega - \operatorname{Re} z)^k} \qquad (\operatorname{Re} z < \omega).$$

for each $1 \leq k \leq n$.

Proof. Fix $\theta \in \mathbb{R}$ and apply Proposition A.7 to $A + (\omega + i\theta)$. □

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