9. Combinatorial Set Theory

In this chapter we discuss topics in infinitary combinatorics such as trees and partition properties.

Partition Properties

Let us consider the following argument (the *pigeonhole principle*): If seven pigeons occupy three pigeonholes, then at least one pigenhole is occupied by three pigeons. More generally: If an infinite set is partitioned into finitely many pieces, then at least one piece is infinite.

Recall that a *partition* of a set S is a pairwise disjoint family $P = \{X_i : i \in I\}$ such that $\bigcup_{i\in I} X_i = S$. With the partition P we can associate a function $F : S \to I$ such that $F(x) = F(y)$ if and only if x and y are in the same $X \in P$. Conversely, any function $F : S \to I$ determines a partition of S. (We shall sometimes say that F is a partition of S.)

For any set A and any natural number $n > 0$,

$$[A]^n = \{X \subset A : |X| = n\} \tag{9.1}$$

is the set of all subsets of A that have exactly n elements. It is sometimes convenient, when A is a set of ordinals, to identify $[A]^n$ with the set of all sequences $\langle\alpha_1, \ldots, \alpha_n\rangle$ in A such that $\alpha_1 < \ldots < \alpha_n$. We shall consider partitions of sets $[A]^n$ for various infinite sets A and natural numbers n. Our starting point is the theorem of Ramsey dealing with finite partitions of $[\omega]^n$.

If $\{X_i : i \in I\}$ is a partition of $[A]^n$, then a set $H \subset A$ is *homogeneous* for the partition if for some i, $[H]^n$ is included in X_i; that is, if all the n-element subsets of H are in the same piece of the partition.

Theorem 9.1 (Ramsey). *Let n and k be natural numbers. Every partition $\{X_1, \ldots, X_k\}$ of $[\omega]^n$ into k pieces has an infinite homogeneous set.*

Equivalently, for every $F : [\omega]^n \to \{1, \ldots, k\}$ there exists an infinite $H \subset \omega$ such that F is constant on $[H]^n$.

Proof. By induction on n. If $n = 1$, the theorem is trivial, so we assume that it holds for n and prove for $n + 1$. Let F be a function from $[\omega]^{n+1}$ into

$\{1, \ldots, k\}$. For each $a \in \omega$, let F_a be the function on $[\omega - \{a\}]^n$ defined as follows:

$$F_a(X) = F(\{a\} \cup X).$$

By the induction hypothesis, there exists for each $a \in \omega$ and each infinite $S \subset \omega$ an infinite set $H_a^S \subset S - \{a\}$ such that F_a is constant on $[H_a^S]^n$. We construct an infinite sequence $\langle a_i : i = 0, 1, 2, \ldots \rangle$: We let $S_0 = \omega$ and $a_0 = 0$, and

$$S_{i+1} = H_{a_i}^{S_i}, \qquad a_{i+1} = \text{the least element of } S_{i+1} \text{ greater than } a_i.$$

It is clear that for each $i \in \omega$, the function F_{a_i} is constant on $[\{a_m : m > i\}]^n$; let $G(a_i)$ be its value. Now there is an infinite subset $H \subset \{a_i : i \in \omega\}$ such that G is constant on H. It follows that F is constant on $[H]^{n+1}$; this is because for $x_1 < \ldots < x_{n+1}$ in H we have $F(\{x_1, \ldots, x_{n+1}\}) = F_{x_1}(\{x_2, \ldots, x_{n+1}\})$. $\square$

The following lemma explains the terminology introduced in Chapter 7 where Ramsey ultrafilters were defined:

Lemma 9.2. *Let D be a nonprincipal ultrafilter on ω. D is Ramsey if and only if for all natural numbers n and k, every partition $F : [\omega]^n \to \{1, \ldots, k\}$ has a homogeneous set $H \in D$.*

Proof. First assume that D has the partition property stated in the lemma. Let $\mathcal{A}$ be a partition of ω such that $A \notin D$ for all $A \in \mathcal{A}$; we shall find $X \in D$ such that $|X \cap A| \leq 1$ for all $A \in \mathcal{A}$. Let $F : [\omega]^2 \to \{0, 1\}$ be as follows: $F(x, y) = 1$ if x and y are in different members of $\mathcal{A}$. If $H \in D$ is homogeneous for F, then clearly H has at most one element common with each $A \in \mathcal{A}$.

Now let us assume that D is a Ramsey ultrafilter. We shall first prove that D has the following property:

(9.2) if $X_0 \supset X_1 \supset X_2 \supset \ldots$ are sets in D, then there is a sequence $a_0 < a_1 < a_2 < \ldots$ such that $\{a_n\}_{n=0}^\infty \in D$, $a_0 \in X_0$ and $a_{n+1} \in X_{a_n}$ for all n.

Thus let $X_0 \supset X_1 \supset \ldots$ be sets in D. Since D is a p-point, there exists $Y \in D$ such that each $Y - X_n$ is finite. Let us define a sequence $y_0 < y_1 < \ldots$ in Y as follows:

$$\begin{aligned}
y_0 &= \text{the least } y_0 \in Y \text{ such that } \{y \in Y : y > y_0\} \subset X_0, \\
y_1 &= \text{the least } y_1 \in Y \text{ such that } y_1 > y_0 \text{ and } \{y \in Y : y > y_1\} \subset X_{y_0}, \\
&\ldots \\
y_n &= \text{the least } y_n \in Y \text{ such that } y_n > y_{n-1} \text{ and } \{y \in Y : y > y_n\} \subset X_{y_{n-1}}.
\end{aligned}$$

For each n, let $A_n = \{y \in Y : y_n < y \leq y_{n+1}\}$. Since D is Ramsey, there exists a set $\{z_n\}_{n=0}^\infty \in D$ such that $z_n \in A_n$ for all n.

We observe that for each n, $z_{n+2} \in X_{z_n}$: Since $z_{n+2} > y_{n+2}$, we have $z_{n+2} \in X_{y_{n+1}}$, and since $y_{n+1} \geq z_n$, we have $X_{y_{n+1}} \subset X_{z_n}$ and hence $z_{n+2} \in X_{z_n}$.

Thus if we let $a_n = z_{2n}$ and $b_n = z_{2n+1}$, for all n, then either $\{a_n\}_{n=0}^{\infty} \in D$ or $\{b_n\}_{n=0}^{\infty} \in D$; and in either case we get a sequence that satisfies (9.2).

Now we use the property (9.2) to prove the partition property; we proceed by induction on n and follow closely the proof of Ramsey's Theorem. Let F be a function from $[\omega]^{n+1}$ into $\{1, \ldots, k\}$. For each $a \in \omega$, let F_a be the function on $[\omega - \{a\}]^n$ defined by $F_a(x) = F(x \cup \{a\})$. By the induction hypothesis, there exists for each $a \in \omega$ a set $H_a \in D$ such that F_a is constant on $[H_a]^n$. There exists $X \in D$ such that the constant value of F_a is the same for all $a \in X$; say $F_a(x) = r$ for all $a \in X$ and all $x \in [H_a]^n$.

For each n, let $X_n = X \cap H_0 \cap H_1 \cap \ldots \cap H_n$. By (9.2) there exists a sequence $a_0 < a_1 < a_2 < \ldots$ such that $a_0 \in X_0$ and $a_{n+1} \in X_{a_n}$ for each n, and that $\{a_n\}_{n=0}^{\infty} \in D$. Let $H = \{a_n\}_{n=0}^{\infty}$. It is clear that for each $i \in \omega$, $a_i \in X$ and $\{a_m : m > i\} \subset H_{a_i}$. Hence $F_{a_i}(x) = r$ for all $x \in [\{a_m : m > i\}]^n$, and it follows that F is constant on $[H]^{n+1}$. $\square$

To facilitate our investigation of generalizations of Ramsey's Theorem, we shall now introduce the *arrow notation*. Let κ and λ be infinite cardinal numbers, let n be a natural number and let m be a (finite or infinite) cardinal. The symbol

$$\kappa \to (\lambda)^n_m \tag{9.3}$$

(read: κ *arrows* λ) denotes the following *partition property*: Every partition of $[\kappa]^n$ into m pieces has a homogeneous set of size λ. In other words, every $F : [\kappa]^n \to m$ is constant on $[H]^n$ for some $H \subset \kappa$ such that $|H| = \lambda$. Using the arrow notation, Ramsey's Theorem is expressed as follows:

$$\aleph_0 \to (\aleph_0)^n_k \qquad (n, k \in \omega). \tag{9.4}$$

The subscript m in (9.3) is usually deleted when $m = 2$, and so

$$\kappa \to (\lambda)^n$$

is the same as $\kappa \to (\lambda)^n_2$.

The relation $\kappa \to (\lambda)^n_m$ remains true if κ is made larger or if λ or m are made smaller. A moment's reflection is sufficient to see that the relation also remains true when n is made smaller.

Obviously, the relation (9.3) makes sense only if $\kappa \geq \lambda$ and $\kappa > m$; if $m = \kappa$, then it is clearly false. Thus we always assume $2 \leq m < \kappa$ and $\lambda \leq \kappa$. If $n = 1$, then (9.3) holds just in case either $\kappa > \lambda$, or $\kappa = \lambda$ and $\operatorname{cf} \kappa > m$. We shall concentrate on the nontrivial case: $n \geq 2$.

We start with two negative partition relations.

Lemma 9.3. *For all* κ,

$$2^\kappa \not\to (\omega)^2_\kappa.$$

In other words, there is a partition of 2^κ *into* κ *pieces that does not have an infinite homogeneous set.*

Proof. In fact, we find a partition that has no homogeneous set of size 3. Let $S = \{0,1\}^\kappa$ and let $F : [S]^2 \to \kappa$ be defined by $F(\{f,g\}) =$ the least $\alpha < \kappa$ such that $f(\alpha) \neq g(\alpha)$. If f, g, h are distinct elements of S, it is impossible to have $F(\{f,g\}) = F(\{f,h\}) = F(\{g,h\})$. □

Lemma 9.4. *For every* κ,

$$2^\kappa \not\to (\kappa^+)^2.$$

(Thus the obvious generalization of Ramsey's Theorem, namely $\aleph_1 \to (\aleph_1)^2_2$, is false.)

To construct a partition of $[2^\kappa]^2$ that violates the partition property, let us consider the linearly ordered set $(P,<)$ where $P = \{0,1\}^\kappa$, and $f < g$ if and only if $f(\alpha) < g(\alpha)$ where α is the least α such that $f(\alpha) \neq g(\alpha)$ (the lexicographic ordering of P).

Lemma 9.5. *The lexicographically ordered set* $\{0,1\}^\kappa$ *has no increasing or decreasing* κ^+*-sequence.*

Proof. Assume that $W = \{f_\alpha : \alpha < \kappa^+\} \subset \{0,1\}^\kappa$ is such that $f_\alpha < f_\beta$ whenever $\alpha < \beta$ (the decreasing case is similar). Let $\gamma \leq \kappa$ be the least γ such that the set $\{f_\alpha{\restriction}\gamma : \alpha < \kappa^+\}$ has size κ^+, and let $Z \subset W$ be such that $|Z| = \kappa^+$ and $f{\restriction}\gamma \neq g{\restriction}\gamma$ for $f, g \in Z$. We may as well assume that $Z = W$, so let us do so.

For each $\alpha < \kappa^+$, let ξ_α be such that $f_\alpha{\restriction}\xi_\alpha = f_{\alpha+1}{\restriction}\xi_\alpha$ and $f_\alpha(\xi_\alpha) = 0$, $f_{\alpha+1}(\xi_\alpha) = 1$; clearly $\xi_\alpha < \gamma$. Hence there exists $\xi < \gamma$ such that $\xi = \xi_\alpha$ for κ^+ elements f_α of W. However, if $\xi = \xi_\alpha = \xi_\beta$ and $f_\alpha{\restriction}\xi = f_\beta{\restriction}\xi$, then $f_\beta < f_{\alpha+1}$ and $f_\alpha < f_{\beta+1}$; hence $f_\alpha = f_\beta$. Thus the set $\{f_\alpha{\restriction}\xi : \alpha < \kappa^+\}$ has size κ^+, contrary to the minimality assumption on γ. □

Proof of Lemma 9.4. Let $2^\kappa = \lambda$ and let $\{f_\alpha : \alpha < \lambda\}$ be an enumeration of the set $P = \{0,1\}^\kappa$. Let $\prec$ be a linear ordering of λ induced by the lexicographic ordering of P: $\alpha \prec \beta$ if $f_\alpha < f_\beta$.

Now we define a partition $F : [\lambda]^2 \to \{0,1\}$ by letting $F(\{\alpha,\beta\}) = 1$ when the ordering $\prec$ of $\{\alpha,\beta\}$ agrees with the natural ordering; and letting $F(\{\alpha,\beta\}) = 0$ otherwise. If $H \subset \lambda$ is a homogeneous set of order-type κ^+, then $\{f_\alpha : \alpha \in H\}$ constitutes an increasing or decreasing κ^+-sequence in $(P,<)$; a contradiction. □

By Lemma 9.4, the relation $\kappa \to (\aleph_1)^2$ is false if $\kappa \leq 2^{\aleph_0}$. On the other hand, if $\kappa > 2^{\aleph_0}$, then $\kappa \to (\aleph_1)^2$ is true, as follows from this more general theorem:

Theorem 9.6 (Erdős-Rado).

$$\beth_n^+ \to (\aleph_1)_{\aleph_0}^{n+1}.$$

In particular, $(2^{\aleph_0})^+ \to (\aleph_1)_{\aleph_0}^2$.

Proof. We shall first prove the case $n = 1$ since the induction step parallels closely this case. Thus let $\kappa = (2^{\aleph_0})^+$ and let $F : [\kappa]^2 \to \omega$ be a partition of $[\kappa]^2$ into $\aleph_0$ pieces. We want to find a homogeneous $H \subset \kappa$ of size $\aleph_1$.

For each $a \in \kappa$, let F_a be a function on $\kappa - \{a\}$ defined by $F_a(x) = F(\{a, x\})$. We shall first prove the following claim: There exists a set $A \subset \kappa$ such that $|A| = 2^{\aleph_0}$ and such that for every countable $C \subset A$ and every $u \in \kappa - C$ there exists $v \in A - C$ such that F_v agrees with F_u on C.

To prove the claim, we construct an ω_1-sequence $A_0 \subset A_1 \subset \ldots \subset A_\alpha \subset \ldots$, $\alpha < \omega_1$, of subsets of κ, each of size $2^{\aleph_0}$, as follows: Let A_0 be arbitrary, and for each limit α, let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Given A_α, there exists a set $A_{\alpha+1} \supset A_\alpha$ of size $2^{\aleph_0}$ such that for each countable $C \subset A_\alpha$ and every $u \in \kappa - C$ there exists $v \in A_{\alpha+1} - C$ such that F_v agrees with F_u on C (because the number of such functions is $\le 2^{\aleph_0}$). Then we let $A = \bigcup_{\alpha<\omega_1} A_\alpha$, and clearly A has the required property.

Next we choose some $a \in \kappa - A$, and construct a sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ in A as follows: Let x_0 be arbitrary, and given $\{x_\beta : \beta < \alpha\} = C$, let x_α be some $v \in A - C$ such that F_v agrees with F_a on C. Let $X = \{x_\alpha : \alpha < \omega_1\}$.

Now we consider the function $G : X \to \omega$ defined by $G(x) = F_a(x)$. It is clear that if $\alpha < \beta$, then $F(\{x_\alpha, x_\beta\}) = F_{x_\beta}(x_\alpha) = F_a(x_\alpha) = G(x_\alpha)$. Since the range of G is countable, there exists $H \subset X$ of size $\aleph_1$ such that G is constant on H. It follows that F is constant on $[H]^2$.

Thus we have proved the theorem for $n = 1$. The general case is proved by induction. Let us assume that $\beth_{n-1}^+ \to (\aleph_1)_{\aleph_0}^n$ and let $F : [\kappa]^{n+1} \to \omega$, where $\kappa = \beth_n^+$. For each $a \in \kappa$, let $F_a : [\kappa - \{a\}]^n \to \omega$ be defined by $F_a(x) = F(x \cup \{a\})$. As in the case $n = 1$, there exists a set $A \subset \kappa$ of size $\beth_n$ such that for every $C \subset A$ of size $|C| \le \beth_{n-1}$ and every $u \in \kappa - C$ there exists $v \in A - C$ such that F_v agrees with F_u on $[C]^n$.

Next we choose $a \in \kappa - A$ and construct a set $X = \{x_\alpha : \alpha < \beth_{n-1}^+\} \subset A$ such that for each α, F_{x_α} agrees with F_a on $[\{x_\beta : \beta < \alpha\}]^n$.

Then we consider $G : [X]^n \to \omega$ where $G(x) = F_a(x)$. As before, if $\alpha_1 < \ldots < \alpha_{n+1}$, then $F(\{x_{\alpha_1}, \ldots, x_{\alpha_{n+1}}\}) = G(\{x_{\alpha_1}, \ldots, x_{\alpha_n}\})$. By the induction hypothesis, there exists $H \subset X$ of size $\aleph_1$ such that G is constant on $[H]^n$. It follows that F is constant on $[H]^{n+1}$. □

Erdős and Rado proved that for each n, the partition property $\beth_n^+ \to (\aleph_1)_{\aleph_0}^{n+1}$ is best possible. The property also generalizes easily to larger cardinals.

A natural generalization of the partition property (9.3) is when we allow λ to be a limit ordinal, not just a cardinal. Let κ, n and m be as in (9.3) and let $\alpha > 0$ be a limit ordinal. The symbol

$$\kappa \to (\alpha)^n_m \tag{9.5}$$

stands for: For every $F : [\kappa]^n \to m$ there exists an $H \subset \kappa$ of order-type α such that F is constant on $[H]^n$.

There are various results about the partition relation (9.5). For instance, Baumgartner and Hajnal proved in [1973] that $\aleph_1 \to (\alpha)^2$ for all $\alpha < \omega_1$. The analogous case for $\aleph_2$ is different: If $2^{\aleph_0} = \aleph_1$, then $\aleph_2 \to (\omega_1)^2$ (by Erdős-Rado), but it is consistent (with $2^{\aleph_0} = \aleph_1$) that $\aleph_2 \not\to (\omega_1 + \omega)^2$.

Among other generalizations of (9.3), we mention the following:

$$\kappa \to (\alpha, \beta)^n \tag{9.6}$$

means that for every $F : [\kappa]^n \to \{0, 1\}$, either there exists an $H_1 \subset \kappa$ of order-type α such that $F = 0$ on $[H_1]^n$ or there exists and $H_2 \subset \kappa$ of order-type β such that $F = 1$ on $[H_2]^n$.

Theorem 9.7 (Dushnik-Miller). *For every infinite cardinal κ,*

$$\kappa \to (\kappa, \omega)^2.$$

Proof. Let $\{A, B\}$ be a partition of $[\kappa]^2$. For every $x \in \kappa$, let $B_x = \{y \in \kappa : x < y \text{ and } \{x, y\} \in B\}$. First let us assume that in every set $X \subset \kappa$ of cardinality κ there exists an $x \in X$ such that $|B_x \cap X| = \kappa$. In this case, we construct an infinite H with $[H]^2 \subset B$ as follows:

Let $X_0 = \kappa$ and $x_0 \in X_0$ such that $|B_{x_0} \cap X_0| = \kappa$. For each n, let $X_{n+1} = B_{x_n} \cap X_n$ and let $x_{n+1} \in X_{n+1}$ be such that $x_{n+1} > x_n$ and $|B_{x_{n+1}} \cap X_{n+1}| = \kappa$. Then let $H = \{x_n\}_{n=0}^\infty$; it is clear that $[H]^2 \subset B$.

Thus let us assume, for the rest of the proof, that there exists a set $S \subset \kappa$ of cardinality κ such that

$$\text{for every } x \in S,\ |B_x \cap S| < \kappa. \tag{9.7}$$

If κ is regular, then we construct (by induction) an increasing κ-sequence $\langle x_\alpha : \alpha < \kappa \rangle$ in S such that $\{x_\alpha, x_\beta\} \in A$ for all $\alpha < \beta$; this is possible by (9.7).

Thus let us assume that κ is singular, let $\lambda = \operatorname{cf} \kappa$ and let $\langle \kappa_\xi : \xi < \lambda \rangle$ be an increasing sequence of regular cardinals $> \lambda$ with limit κ. Furthermore, we assume that there is no infinite H with $[H]^2 \subset B$, and that $\kappa_\xi \to (\kappa_\xi, \omega)^2$ holds for every $\xi < \lambda$. We shall find a set $H \subset \kappa$ of cardinality κ such that $[H]^2 \subset A$.

Let $\{S_\xi : \xi < \lambda\}$ be a partition of S into disjoint sets such that $S_\xi = \kappa_\xi$. It follows from our assumptions that there exist sets $K_\xi \subset S_\xi$, $|K_\xi| = \kappa_\xi$, such that $[K_\xi]^2 \subset A$.

For every $x \in K_\xi$ there exists, by (9.7), some $\alpha < \lambda$ such that $|B_x \cap S| < \kappa_\alpha$; since $\lambda < \kappa_\xi$, there exists an $\alpha(\xi)$ such that the set $Z_\xi = \{x \in K_\xi : |B_x \cap S| < \kappa_{\alpha(\xi)}\}$ has cardinality κ_ξ.

Let $\langle \xi_\nu : \nu < \lambda \rangle$ be an increasing sequence of ordinals $< \lambda$ such that if $\nu_1 < \nu_2$ then $\alpha(\xi_{\nu_1}) < \xi_{\nu_2}$. We define, by induction on ν,

$$H_\nu = Z_{\xi(\nu)} - \bigcup\{B_x : x \in \bigcup_{\eta<\nu} Z_{\xi(\eta)}\}.$$

Clearly, $|H_\nu| = \kappa_{\xi(\nu)}$, and $[H_\nu]^2 \subset A$.

Finally, we let $H = \bigcup_{\nu<\lambda} H_\nu$. It follows from the construction of H that $[H]^2 \subset A$. □

Weakly Compact Cardinals

In the positive results given by the Erdős-Rado Theorem, the size of the homogeneous set is smaller than the size of the set being partitioned. A natural question arises, whether the relation $\kappa \to (\kappa)^2$ can hold for cardinals other than $\kappa = \omega$.

Definition 9.8. A cardinal κ is *weakly compact* if it is uncountable and satisfies the partition property $\kappa \to (\kappa)^2$.

The reason for the name "weakly compact" is that these cardinals satisfy a certain compactness theorem for infinitary languages; we shall investigate weakly compact cardinals further in Part II.

Lemma 9.9. *Every weakly compact cardinal is inaccessible.*

Proof. Let κ be a weakly compact cardinal. To show that κ is regular, let us assume that κ is the disjoint union $\bigcup\{A_\gamma : \gamma < \lambda\}$ such that $\lambda < \kappa$ and $|A_\gamma| < \kappa$ for each $\gamma < \lambda$. We define a partition $F : [\kappa]^2 \to \{0, 1\}$ as follows: $F(\{\alpha, \beta\}) = 0$ just in case α and β are in the same A_γ. Obviously, this partition does not have a homogeneous set $H \subset \kappa$ of size κ.

That κ is a strong limit cardinal follows from Lemma 9.4: If $\kappa \le 2^\lambda$ for some $\lambda < \kappa$, then because $2^\lambda \not\to (\lambda^+)^2$, we have $\kappa \not\to (\lambda^+)^2$ and hence $\kappa \not\to (\kappa)^2$. □

We shall prove in Chapter 17 that every weakly compact cardinal κ is the κth inaccessible cardinal.

Trees

Many problems in combinatorial set theory can be formulated as problems about trees.

In this chapter we discuss Suslin's Problem as well as the use of trees in partition calculus and large cardinals.

Definition 9.10. A *tree* is a partially ordered set $(T, <)$ with the property that for each $x \in T$, the set $\{y : y < x\}$ of all predecessors of x is well-ordered by $<$.

The αth *level* of T consists of all $x \in T$ such that $\{y : y < x\}$ has order-type α. The height of T is the least α such that the αth level of T is empty; in other words, it is the height of the well-founded relation $<$:

$$\begin{aligned} o(x) &= \text{the order-type of } \{y : y < x\}, \\ \alpha\text{th level} &= \{x : o(x) = \alpha\}, \\ \text{height}(T) &= \sup\{o(x) + 1 : x \in T\}. \end{aligned} \tag{9.8}$$

A *branch* in T is a maximal linearly ordered subset of T. The *length* of a branch b is the order-type of b. An α*-branch* is a branch of length α.

We shall now turn our attention to Suslin's Problem introduced in Chapter 4. In Lemma 9.14 below we show that the problem can be restated as a question about the existence of certain trees of height ω_1.

Suslin's Problem asks whether the real line is the only complete dense unbounded linearly ordered set that satisfies the countable chain condition. An equivalent question is whether every dense linear ordering that satisfies the countable chain condition is *separable*, i.e., has a countable dense subset.

Definition 9.11. A *Suslin line* is a dense linearly ordered set that satisfies the countable chain condition and is not separable.

Thus Suslin's Problem asks whether a Suslin line exists. We shall show that the existence of a Suslin line is equivalent to the existence of a Suslin tree.

Let T be a tree. An *antichain* in T is a set $A \subset T$ such that any two distinct elements x, y of A are incomparable, i.e., neither $x < y$ nor $y < x$.

Definition 9.12. A tree T is a *Suslin tree* if

(i) the height of T is ω_1;
(ii) every branch in T is at most countable;
(iii) every antichain in T is at most countable.

For the formulation of Suslin's Problem in terms of trees it is useful to consider Suslin trees that are called *normal.*

Let α be an ordinal number, $\alpha \le \omega_1$. A *normal α-tree* is a tree T with the following properties:

(9.9)
(i) $\text{height}(T) = \alpha$;
(ii) T has a unique least point (the root);
(iii) each level of T is at most countable;
(iv) if x is not maximal in T, then there are infinitely many $y > x$ at the next level (immediate successors of x);

(v) for each $x \in T$ there is some $y > x$ at each higher level less than α;
(vi) if $\beta < \alpha$ is a limit ordinal and x, y are both at level β and if $\{z : z < x\} = \{z : z < y\}$, then $x = y$.

See Exercise 9.6 for a representation of normal trees.

Lemma 9.13. *If there exists a Suslin tree then there exists a normal Suslin tree.*

Proof. Let T be a Suslin tree. T has height ω_1, and each level of T is countable. We first discard all points $x \in T$ such that $T_x = \{y \in T : y \geq x\}$ is at most countable, and let $T_1 = \{x \in T : T_x \text{ is uncountable}\}$. Note that if $x \in T_1$ and $\alpha > o(x)$, then $|T_y| = \aleph_1$ for some $y > x$ at level α. Hence T_1 satisfies condition (v). Next, we satisfy property (vi): For every chain $C = \{z : z < y\}$ in T_1 of limit length we add an extra node a_C and stipulate that $z < a_C$ for all $z \in C$, and $a_C < x$ for every x such that $x > z$ for all $z \in C$. The resulting tree T_2 satisfies (iii), (v) and (vi). For each $x \in T_2$ there are uncountably many branching points $z > x$, i.e., points that have at least two immediate successors (because there is no uncountable chain and T_2 satisfies (v)). The tree $T_3 = \{\text{the branching points of } T_2\}$ satisfies (iii), (v) and (vi) and each $x \in T_3$ is a branching point. To get property (iv), let T_4 consists of all $z \in T_3$ at limit levels of T_3. The tree T_4 satisfies (i), (iii), (iv), and (v); and then $T_5 \subset T_4$ satisfying (ii) as well is easily obtained. □

Lemma 9.14. *There exists a Suslin line if and only if there exists a Suslin tree.*

Proof. (a) Let S be a Suslin line. We shall construct a Suslin tree. The tree will consist of closed (nondegenerate) intervals on the Suslin line S. The partial ordering of T is by inverse inclusion: If $I, J \in T$, then $I \leq J$ if and only if $I \supset J$.

The collection T of intervals is constructed by induction on $\alpha < \omega_1$. We let $I_0 = [a_0, b_0]$ be arbitrary (such that $a_0 < b_0$). Having constructed I_β, $\beta < \alpha$, we consider the countable set $C = \{a_\beta : \beta < \alpha\} \cup \{b_\beta : \beta < \alpha\}$ of endpoints of the intervals I_β, $\beta < \alpha$. Since S is a Suslin line, C is not dense in S and so there exists an interval $[a, b]$ disjoint from C; we pick some such $[a_\alpha, b_\alpha] = I_\alpha$. The set $T = \{I_\alpha : \alpha < \omega_1\}$ is uncountable and partially ordered by $\supset$. If $\alpha < \beta$, then either $I_\alpha \supset I_\beta$ or I_α is disjoint from I_β. It follows that for each α, $\{I \in T : I \supset I_\alpha\}$ is well-ordered by $\supset$ and thus T is a tree.

We shall show that T has no uncountable branches and no uncountable antichains. Then it is immediate that the height of T is at most ω_1; and since every level is an antichain and T is uncountable, we have $\text{height}(T) = \omega_1$.

If $I, J \in T$ are incomparable, then they are disjoint intervals of S; and since S satisfies the countable chain condition, every antichain in T is at most countable. To show that T has no uncountable branch, we note first that if

b is a branch of length ω_1, then the left endpoints of the intervals $I \in B$ form an increasing sequence $\{x_\alpha : \alpha < \omega_1\}$ of points of S. It is clear that the intervals $(x_\alpha, x_{\alpha+1})$, $\alpha < \omega_1$, form a disjoint uncountable collection of open intervals in S, contrary to the assumption that S satisfies the countable chain condition.

(b) Let T be a normal Suslin tree. The line S will consist of all branches of T (which are all countable). Each $x \in T$ has countably many immediate successors, and we order these successors as rational numbers. Then we order the elements of S lexicographically: If α is the least level where two branches $a, b \in S$ differ, then α is a successor ordinal and the points $a_\alpha \in a$ and $b_\alpha \in b$ are both successors of the same point at level $\alpha - 1$; we let $a < b$ or $b < a$ according to whether $a_\alpha < b_\alpha$ or $b_\alpha < a_\alpha$.

It is easy to see that S is linearly ordered and dense. If (a, b) is an open interval in S, then one can find $x \in T$ such that $I_x \subset (a, b)$, where I_x is the interval $I_x = \{c \in S : x \in c\}$. And if I_x and I_y are disjoint, then x and y are incomparable points of T. Thus every disjoint collection of open intervals of S must be at most countable, and so S satisfies the countable chain condition.

The line S is not separable: If C is a countable set of branches of T, let α be a countable ordinal bigger than the length of any branches $b \in C$. Then if x is any point at level α, the interval I_x does not contain any $b \in C$, and so C is not dense in S. □

Lemma 9.14 reduces Suslin's Problem to a purely combinatorial problem. In Part II we shall return to it and show that the problem is independent of the axioms of set theory.

We now turn our attention to the following problem, related to Suslin trees.

Definition 9.15. An *Aronszajn tree* is a tree of height ω_1 all of whose levels are at most countable and which has no uncountable branches.

Theorem 9.16 (Aronszajn). *There exists an Aronszajn tree.*

Proof. We construct a tree T whose elements are some bounded increasing transfinite sequences of rational numbers. If $x, y \in T$ are two such sequences, then we let $x \le y$ just in case y extends x, i.e., $x \subset y$. Also, if $y \in T$ and x is an initial segment of y, then we let $x \in T$; thus the αth level of T will consist of all those $x \in T$ whose length is α.

It is clear that an uncountable branch in T would yield an increasing ω_1-sequence of rational numbers, which is impossible. Thus T will be an Aronszajn tree, provided we arrange that T has $\aleph_1$ levels, all of them at most countable. We construct T by induction on levels. For each $\alpha < \omega_1$ we construct a set U_α of increasing α-sequences of rationals; U_α will be the αth level of T. We construct the U_α so that for each α, $|U_\alpha| \le \aleph_0$, and that:

(9.10) For each $\beta < \alpha$, each $x \in U_\beta$ and each $q > \sup x$ there is $y \in U_\alpha$ such that $x \subset y$ and $q \ge \sup y$.

Condition (9.10) enables us to continue the construction at limit steps.

To start, we let $U_0 = \{\emptyset\}$. The successor steps of the construction are also fairly easy. Given U_α, we let $U_{\alpha+1}$ be the set of all extensions $x^\frown r$ of sequences in U_α such that $r > \sup x$. It is clear that since U_α satisfies condition (9.10), $U_{\alpha+1}$ satisfies it also (for $\alpha + 1$), and it is equally clear that $U_{\alpha+1}$ is at most countable.

Thus let α be a limit ordinal ($\alpha < \omega_1$) and assume that we have constructed all levels U_γ, $\gamma < \alpha$, of T below α, and that all the U_γ satisfy (9.10); we shall construct U_α. The points $x \in T$ below level α form a (normal) tree T_α of length α. We claim that T_α has the following property:

(9.11) For each $x \in T_\alpha$ and each $q > \sup x$ there is an increasing α-sequence of rationals y such that $x \subset y$ and $q \geq \sup y$ and that $y{\restriction}\beta \in T_\alpha$ for all $\beta < \alpha$.

The last condition means that $\{y{\restriction}\beta : \beta < \alpha\}$ is a branch in T_α. To prove the claim, we let α_n, $n = 0, 1, \ldots$, be an increasing sequence of ordinals such that $x \in U_{\alpha_0}$ and $\lim_n \alpha_n = \alpha$, and let $\{q_n\}_{n=0}^\infty$ be an increasing sequence of rational numbers such that $q_0 > \sup x$ and $\lim_n q_n \leq q$. Using repeatedly condition (9.10), for all α_n ($n = 0, 1, \ldots$), we can construct a sequence $y_0 \subset y_1 \subset \ldots \subset y_n \ldots$ such that $y_0 = x$, $y_n \in U_{\alpha_n}$, and $\sup y_n \leq q_n$ for each n. Then we let $y = \bigcup_{n=0}^\infty y_n$; clearly, y satisfies (9.11).

Now we construct U_α as follows: For each $x \in T_\alpha$ and each rational number q such that $q > \sup x$, we choose a branch y in T_α that satisfies (9.11), and let U_α consists of all these $y : \alpha \to \boldsymbol{Q}$. The set U_α so constructed is countable and satisfies condition (9.10).

Then $T = \bigcup_{\alpha<\omega_1} U_\alpha$ is an Aronszajn tree. □

The Aronszajn tree constructed in Theorem 9.16 has the property that there exists a function $f : T \to \boldsymbol{R}$ such that $f(x) < f(y)$ whenever $x < y$ (Exercise 9.8). With a little more care, one can construct T so that there is a function $f : T \to \boldsymbol{Q}$ such that $f(x) < f(y)$ if $x < y$. Such trees are called *special Aronszajn trees*. In Part II we'll show that it is consistent that all Aronszajn trees are special.

Almost Disjoint Sets and Functions

In combinatorial set theory one often consider families of sets that are as much different as possible; a typical example is an almost disjoint family of infinite sets—any two intersect in a finite set. Here we present a sample of results and problems of this kind.

Definition 9.17. A collection of finite sets Z is called a Δ-*system* if there exists a finite set S such that $X \cap Y = S$ for any two distinct sets $X, Y \in Z$.

The following theorem is often referred to as the Δ-*Lemma*:

Theorem 9.18 (Shanin). *Let W be an uncountable collection of finite sets. Then there exists an uncountable $Z \subset W$ that is a Δ-system.*

Proof. Since W is uncountable, it is clear that uncountably many $X \in W$ have the same size; thus we may assume that for some n, $|X| = n$ for all $X \in W$. We prove the theorem by induction on n. If $n = 1$, the theorem is trivial. Thus assume that the theorem holds for n, and let W be such that $|X| = n + 1$ for all $X \in W$.

If there is some element a that belongs to uncountably many $X \in W$, we apply the induction hypothesis to the collection $\{X - \{a\} : X \in W$ and $a \in X\}$, and obtain $Z \subset W$ with the required properties.

Otherwise, each a belongs to at most countably many $X \in W$, and we construct a disjoint collection $Z = \{X_\alpha : \alpha < \omega_1\}$ as follows, by induction on α. Given X_ξ, $\xi < \alpha$, we find $X = X_\alpha \in W$ that is disjoint from all X_ξ, $\xi < \alpha$. ☐

For an alternative proof, using Fodor's Theorem, see Exercise 9.10. Theorem 9.18 generalizes to greater cardinals, under the assumption of GCH:

Theorem 9.19. *Assume $\kappa^{<\kappa} = \kappa$. Let W be a collection of sets of cardinality less than κ such that $|W| = \kappa^+$. Then there exist a collection $Z \subset W$ such that $|Z| = \kappa^+$ and a set A such that $X \cap Y = A$ for any two distinct elements $X, Y \in Z$.* ☐

Definition 9.20. If X and Y are infinite subsets of ω then X and Y are *almost disjoint* if $X \cap Y$ is finite.

Let κ be a regular cardinal. If $X \cap Y$ are subsets of κ of cardinality κ then X and Y are *almost disjoint* if $|X \cap Y| < \kappa$.

An *almost disjoint family* of sets is a family of pairwise almost disjoint sets.

Lemma 9.21. *There exists an almost disjoint family of $2^{\aleph_0}$ subsets of ω.*

Proof. Let S be the set of all finite 0–1 sequences: $S = \bigcup_{n=0}^{\infty} \{0,1\}^n$. For every $f : \omega \to \{0, 1\}$, let $A_f \subset S$ be the set $A_f = \{s \in S : s \subset f\} = \{f{\restriction}n : n \in \omega\}$. Clearly, $A_f \cap A_g$ is finite if $f \neq g$; thus $\{A_f : f \in \{0,1\}^\omega\}$ is a family of $2^{\aleph_0}$ almost disjoint subsets of the countable set S, and the lemma follows. ☐

A generalization from ω to arbitrary regular κ is not provable in ZFC (although under GCH the generalization is straightforward; see Exercise 9.11). Without assuming the GCH, the best one can do is to find an almost disjoint family of κ^+ subsets of κ; this follows from Lemma 9.23 below.

Definition 9.22. Let κ be a regular cardinal. Two functions f and g on κ are *almost disjoint* if $|\{\alpha : f(\alpha) = g(\alpha)\}| < \kappa$.

Lemma 9.23. *For every regular cardinal κ, there exists an almost disjoint family of κ^+ functions from κ to κ.*

Proof. It suffices to show that given κ almost disjoint functions $\{f_\nu : \nu < \kappa\}$, then there exists $f : \kappa \to \kappa$ almost disjoint from all f_ν, $\nu < \kappa$; this we do by diagonalization: Let $f(\alpha) \neq f_\nu(\alpha)$ for all $\nu < \alpha$. □

Let us consider the special case when $\kappa = \omega_1$.

Definition 9.24. A tree $(T, <)$ is a *Kurepa tree* if:

(i) $\text{height}(T) = \omega_1$;
(ii) each level of T is at most countable;
(iii) T has at least $\aleph_2$ uncountable branches.

If T is a Kurepa tree, then the family of all ω_1-branches is an almost disjoint family of uncountable subsets of T. In fact, since the levels of T are countable, we can identify the ω_1-branches with the functions from ω_1 into ω and get the following result: *There exists an almost disjoint family of $\aleph_2$ functions $f : \omega_1 \to \omega$.*

Lemma 9.25. *A Kurepa tree exists if and only if there exists a family $\mathcal{F}$ of subsets of ω_1 such that*:

(9.12) (i) $|\mathcal{F}| \geq \aleph_2$;
(ii) *for each* $\alpha < \omega_1$, $|\{X \cap \alpha : X \in \mathcal{F}\}| \leq \aleph_0$.

Proof. (a) Let $(T, <_T)$ be a Kurepa tree. Since T has size $\aleph_1$, we may assume that $T = \omega_1$, and moreover that $\alpha < \beta$ whenever $\alpha <_T \beta$. If we let $\mathcal{F}$ be the family of all ω_1-branches of T, then $\mathcal{F}$ satisfies (9.12).

(b) Let $\mathcal{F}$ be a family of subsets of ω_1 such that (9.12) holds. For each $X \in \mathcal{F}$, let f_X be the functions on ω_1 defined by

$$f_X(\alpha) = X \cap \alpha \qquad (\alpha < \omega_1).$$

For each $\alpha < \omega_1$, let $U_\alpha = \{f_X{\restriction}\alpha : X \in \mathcal{F}\}$ and let $T = \bigcup_{\alpha<\omega_1} U_\alpha$. Then $(T, \subset)$ is a tree, the U_α are the levels of T and the functions f_X correspond to branches of T. By (9.12)(ii), every U_α is countable, and it follows that T is a Kurepa tree. □

The existence of a Kurepa tree is independent of the axioms of set theory. In fact, the nonexistence of Kurepa trees is equiconsistent with an inaccessible cardinal.

The Tree Property and Weakly Compact Cardinals

Generalizing the concept of Aronszajn tree to cardinals $> \omega_1$ we say that a regular uncountable cardinal κ has the *tree property* if every tree of height κ whose levels have cardinality $< \kappa$ has a branch of cardinality κ.

Lemma 9.26.

(i) *If κ is weakly compact, then κ has the tree property.*
(ii) *If κ is inaccessible and has the tree property, then κ is weakly compact, and in fact $\kappa \to (\kappa)^2_m$ for every $m < \kappa$.*

Proof. (i) Let κ be weakly compact and let $(T, <_T)$ be a tree of height κ such that each level of T has size $< \kappa$. Since κ is inaccessible, $|T| = \kappa$ and we may assume that $T = \kappa$. We extend the partial ordering $<_T$ of κ to a linear ordering $\prec$: If $\alpha <_T \beta$, then we let $\alpha \prec \beta$; if α and β are incomparable and if ξ is the first level where the predecessors α_ξ, β_ξ of α and β are distinct, we let $\alpha \prec \beta$ if and only if $\alpha_\xi < \beta_\xi$.

Let $F : [\kappa]^2 \to \{0, 1\}$ be the partition defined by $F(\{\alpha, \beta\}) = 1$ if and only if $\prec$ agrees with $<$ on $\{\alpha, \beta\}$. By weak compactness, let $H \subset \kappa$ be homogeneous for F, $|H| = \kappa$.

We now consider the set $B \subset \kappa$ of all $x \in \kappa$ such that $\{\alpha \in H : x <_T \alpha\}$ has size κ. Since every level has size $< \kappa$, it is clear that at each level there is at least one x in B. Thus if we show that any two elements of B are $<_T$-comparable, we shall have proved that B is a branch in T of size κ.

Thus assume that x, y are incomparable elements of B; let $x \prec y$. Since both x and y have κ successors in H, there exist $\alpha < \beta < \gamma$ in H such that $x <_T \alpha$, $y <_T \beta$, and $x <_T \gamma$. By the definition of $\prec$, we have $\alpha \prec \beta$ and $\gamma \prec \beta$. Thus $F(\{\alpha, \beta\}) = 1$ and $F(\{\gamma, \beta\}) = 0$, contrary to the homogeneity of H.

(ii) Let κ be an inaccessible cardinal with the tree property, and let $F : [\kappa]^2 \to I$ be a partition such that $|I| < \kappa$. We shall find a homogeneous $H \subset \kappa$ of size κ.

We construct a tree $(T, \subset)$ whose elements are some functions $t : \gamma \to I$, $\gamma < \kappa$. We construct T by induction: At step $\alpha < \kappa$, we put into T one more element t, calling it t_α. Let $t_0 = \emptyset$. Having constructed $t_0, \ldots, t_\beta, \ldots, \beta < \alpha$, let us construct t_α as follows, by induction on ξ. Having constructed $t_\alpha{\restriction}\xi$, we look first whether $t_\alpha{\restriction}\xi$ is among the t_β, $\beta < \alpha$ (note that for $\xi = 0$ we have $t_\alpha{\restriction}0 = t_0$). If not, then we consider t_α constructed: $t_\alpha = t_\alpha{\restriction}\xi$. If $t_\alpha{\restriction}\xi = t_\beta$ for some $\beta < \alpha$, then we let $t_\alpha(\xi) = i$ where $i = F(\{\beta, \alpha\})$.

$(T, \subset)$ is a tree of size κ; and since κ is inaccessible, each level of T has size $< \kappa$ and the height of T is κ. It follows from the construction that if $t_\beta \subset t_\alpha$, then $\beta < \alpha$ and $F(\{\beta, \alpha\}) = t_\alpha(\text{length}(t_\beta))$. By the assumption, T has a branch B of size κ. If we now let, for each $i \in I$,

$$H_i = \{\alpha : t_\alpha \in B \text{ and } t_\alpha{}^\frown i \in B\}, \tag{9.13}$$

then each H_i is homogeneous for the partition F, and at least one H_i has size κ. $\square$

It should be mentioned that an argument similar to the one above, only more complicated, shows that if κ is inaccessible and has the tree property, then $\kappa \to (\kappa)^n_m$ for all $n \in \omega$, $m < \kappa$.

Ramsey Cardinals

Let us consider one more generalization of Ramsey's Theorem. Let κ be an infinite cardinal, let α be an infinite limit ordinal, $\alpha \le \kappa$, and let m be a cardinal, $2 \le m < \kappa$. The symbol

$$\kappa \to (\alpha)^{<\omega}_m \tag{9.14}$$

denotes the property that for every partition F of the set $[\kappa]^{<\omega} = \bigcup_{n=0}^{\infty}[\kappa]^n$ into m pieces, there exists a set $H \subset \kappa$ of order-type α such that for each $n \in \omega$, F is constant on $[H]^n$. (Again, the subscript m is deleted when $m = 2$.)

It is not difficult to see that the partition property $\omega \to (\omega)^{<\omega}$ is false (see Exercise 9.13).

A cardinal κ is a *Ramsey cardinal* if $\kappa \to (\kappa)^{<\omega}$. Clearly, every Ramsey cardinal is weakly compact. We shall investigate Ramsey cardinals and property (9.14) in general in Part II.

Exercises

9.1. (i) Every infinite partially ordered set either has an infinite chain or has an infinite set of mutually incomparable elements.

(ii) Every infinite linearly ordered set either has an infinite increasing sequence of elements or has an infinite decreasing sequence of elements.

[Use Ramsey's Theorem.]

For each κ, let $\exp_0(\kappa) = \kappa$ and $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$.

9.2. For every κ, $(\exp_n(\kappa))^+ \to (\kappa^+)^{n+1}_\kappa$. In particular, we have $(2^\kappa)^+ \to (\kappa^+)^2$.

9.3. $\omega_1 \to (\omega_1, \omega + 1)^2$.

[Let $\{A, B\}$ be a partition of $[\omega_1]^2$. For every limit ordinal α let K_α be a maximal subset of α such that $[K_\alpha \cup \{\alpha\}]^2 \subset B$. If K_α is finite for each α, use Fodor's Theorem to find a stationary set S such that all K_α, $\alpha \in S$, are the same. Then $[S]^2 \subset A$.]

If A is an infinite set of ordinals and α an ordinal, let $[A]^\alpha$ denote the set of all increasing α-sequences in A. The symbol

$$\kappa \to (\lambda)^\alpha$$

stands for: For every partition $F : [\kappa]^\alpha \to \{0, 1\}$ of $[\kappa]^\alpha$ into two pieces, there exists a set H of order-type λ such that F is constant on $[H]^\alpha$.

9.4. For all infinite cardinals κ, $\kappa \not\to (\omega)^\omega$.

[For $s, t \in [\kappa]^\omega$ let $s \equiv t$ if and only if $\{n : s(n) \neq t(n)\}$ is finite. Pick a representative in each equivalence class. Let $F(s) = 0$ if s differs from the representative of its class at an even number of places; let $F(s) = 1$ otherwise. F has no infinite homogeneous set.]

9.5 (König's Lemma). If T is a tree of height ω such that each level of T is finite, then T has an infinite branch.

[To construct a branch $\{x_0, x_1, \ldots, x_n, \ldots\}$ in T, pick x_0 at level 0 such that $\{y : y > x_0\}$ is infinite. Then pick x_1, x_2, ... similarly.]

9.6. If T is a normal α-tree, then T is isomorphic to a tree $\overline{T}$ whose elements are β-sequences ($\beta < \alpha$), ordered by extension; if $t \subset s$ and $s \in \overline{T}$, then $t \in \overline{T}$, and the βth level of $\overline{T}$ is the set $\{t \in \overline{T} : \operatorname{dom} t = \beta\}$.

9.7. If T is a normal ω_1-tree and if T has uncountable branch, then T has an uncountable antichain.

[For each x in the branch B pick a successor z_x of x such that $z_x \notin B$. Let $A = \{z_x : x \in B\}$.]

9.8. Show that if T is the tree in Theorem 9.16 then there exists some $f : T \to \boldsymbol{R}$ such that $f(x) < f(y)$ whenever $x < y$.

9.9. An Aronszajn tree is special if and only if T is the union of ω antichains.

[If $T = \bigcup_{n=0}^\infty A_n$, where each A_n is an antichain, define $\pi : T \to \boldsymbol{Q}$ by induction on n, constructing $\pi \restriction A_n$ at stage n, so that the range of π remains finite.]

9.10. Prove Theorem 9.18 using Fodor's Theorem.

[Let $W = \{X_\alpha : \alpha < \omega_1\}$ with $X_\alpha \subset \omega_1$. For each α, let $f(\alpha) = X_\alpha \cap \alpha$. By Fodor's Theorem, f is constant on a stationary set S; by induction construct a Δ-system $W \subset \{X_\alpha : \alpha \in S\}$.]

9.11. If $2^{<\kappa} = \kappa$, then there exists an almost disjoint family of 2^κ subsets of κ.

[As in Lemma 9.21, let $S = \bigcup_{\alpha<\kappa} \{0,1\}^\alpha$; $|S| = \kappa$.]

9.12. Given a family $\mathcal{F}$ of $\aleph_2$ almost disjoint functions $f : \omega_1 \to \omega$, there exists a collection $\mathcal{S}$ of $\aleph_2$ pairwise disjoint *stationary* subsets of ω_1.

[Each $f \in \mathcal{F}$ is constant on a stationary set S_f with value n_f. There is $\mathcal{G} \subset \mathcal{F}$ of size $\aleph_2$ such that n_f is the same for all $f \in \mathcal{G}$. Let $\mathcal{S} = \{S_f : f \in \mathcal{G}\}$.]

9.13. $\omega \not\to (\omega)^{<\omega}$.

[For $x \in [\omega]^{<\omega}$, let $F(x) = 1$ if $|x| \in x$, and $F(x) = 0$ otherwise. If $H \subset \omega$ is infinite, pick $n \in H$ and show that F is not constant on $[H]^n$.]

Historical Notes

Theorem 9.1 is due to Ramsey [1929/30]. Ramsey ultrafilters are investigated in Booth [1970/71]. The theory of partition relations has been developed by Erdős, who has written a number of papers on the subject, some coauthored by Rado, Hajnal, and others. The arrow notation is introduced in Erdős and Rado [1956]. Other major comprehensive articles on partition relations are Erdős, Hajnal, and Rado [1965] and Erdős and Hajnal [1971].

Theorem 9.6 appears in Erdős and Rado [1956]. Lemma 9.4 is due to Sierpiński [1933]. Theorem 9.7 is in Dushnik-Miller [1941].

Weakly compact cardinals (as in Definition 9.8 as well as the tree property) were introduced by Erdős and Tarski in [1961].

The equivalence of Suslin's Problem with the tree formulation (Lemma 9.14) is due to Kurepa [1935]; this paper also presents Aronszajn's construction and Kurepa trees, with Lemma 9.25.

Theorems 9.18 and 9.19: Shanin [1946] and Erdős-Rado [1960].

Ramsey cardinals were first studied by Erdős and Hajnal in [1962].

Exercise 9.2: Erdős-Rado [1956], Exercises 9.4 and 9.13: Erdős-Rado [1952].

Exercise 9.5: D. König [1927].

Exercise 9.9: Galvin.