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Dow, Alan [Dow, Alan Stewart] (1-NC3); **Hart, Klaas Pieter** (NL-DELFEM)

A zero-dimensional F-space that is not strongly zero-dimensional. (English summary)

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All hypothesized spaces are completely regular and T_1. We recall that a space X is: *zero-dimensional* (*strongly zero-dimensional*) iff it has a base consisting of clopen sets (iff βX is zero-dimensional, i.e., iff for every continuous function $f\colon X \to [-1,1]$ and subsets A, B of X such that $f[A] = \{-1\}$ and $f[B] = \{1\}$, there exists a continuous function $c\colon X \to \{-1,1\}$ such that $c[A] = \{-1\}$ and $c[B] = \{1\}$); and it is an *F-space* iff for every continuous function $f\colon X \to \mathbb{R}$ there is a continuous function $k\colon X \to \mathbb{R}$ such that $f = k \cdot |f|$. Numerous characterizations and examples of spaces illustrating these concepts and some implications among them can be found in Section 6.2 of [R. Engelking, *General topology*, translated from the Polish by the author, second edition, Sigma Ser. Pure Math., 6, Heldermann, Berlin, 1989; MR1039321] ([E]) and in 14.25 of [L. Gillman and M. Jerison, *Rings of continuous functions*, reprint of the 1960 edition, Graduate Texts in Mathematics, No. 43, Springer, New York, 1976; MR0407579] ([GJ]). The main result in the article being reviewed is a very nice construction which answers in the affirmative a question discussed in the 1980s, mentioned in a conference talk by A. Dow in the 1990s, and raised in print in 2016 by W. McGovern in ["Zero-dimensional F-space which is not strongly zero-dimensional", mathoverflow.net/questions/239324]: Is there a zero-dimensional F-space which is not strongly zero-dimensional? One might be inclined to think this question cannot have an affirmative answer since satisfaction of the last definition above implies $k[\{x : f(x) < 0\}] = \{-1\}$ and $k[\{x : f(x) > 0\}] = \{1\}$; however, compact connected F-spaces exist (see 14.27 of [GJ]).

The authors remark that their construction was inspired by a certain example (ii) which was inspired by another example (i). The example (i), of a subspace of $\omega_1 \times [0,1]$ by C. H. Dowker in 1955, also described in 6.2.20 of [E], proved that not every zero-dimensional space is strongly zero-dimensional. The construction (ii), a quotient of $\omega_1 \times \mathbb{A}$, where $\mathbb{A}$ denoted Alexandroff's split interval, was one example K. P. Hart developed and presented in ["Is Stone-Čech compactification of 0-dimensional space also 0-dimensional?", mathoverflow.net/questions/93719] to answer the following question raised there by F. Dashiell: What is an example of a locally compact zero-dimensional space which is not strongly zero-dimensional? Although the authors use the term "inspired", much work was required for them to develop their space and variations on it. In particular, since no F-space can contain a convergent sequence of distinct points (see [GJ, 14N]), the building space $\omega_1 \times \mathbb{A}$ needed to be replaced.

While the authors' main construction is too complicated to outline here, a brief (inadequate) indication of it is the following: Denoting the G_δ-modification of a space Y as $(Y)_\delta$, they replace ω_1 in (ii) with $(\omega_2)_\delta$ and replace $\mathbb{A}$ with the split interval over a suitable selected ordered continuum K having a dense subset which can be enumerated as $\langle d_\alpha : \alpha < \omega_2 \rangle$, where each tail $T_\alpha = \{d_\beta : \beta > \alpha\}$ is dense in K. They next define for each $\alpha \leq \omega_2$, $K_\alpha = \{\langle x, i \rangle \in K \times 2 : \text{if } x \notin T_\alpha \text{ then } i = 0\}$ and $X_\alpha = (\omega \times K_\alpha)^*$, where * denotes the Čech-Stone remainder, and each K_α is topologized so that more and more neighboring points are identified as α increases. Then the authors present a well organized and involved proof that $X = \bigcup \{\{\alpha\} \times X_\alpha : \alpha < \omega_2\}$ is a zero-dimensional

F-space which is not strongly zero-dimensional.

In the final section of the article, the authors present some variations of their construction which produce spaces with additional properties, such as local compactness, and they conclude by raising some questions.

For a more detailed (and entertaining) source of information about their construction and what led them to discover it, the reviewer recommends readers access the second author's F-space talk available at webpages.charlotte.edu/adow/CarolinaSeminar.html. (The YouTube link for this particular talk is www.youtube.com/watch?v=rGoJxhqL7rY.) *R. M. Stephenson, Jr.*

References

1. Fred Dashiell, Is Stone-Čech compactification of 0-dimensional space also 0-dimensional?, https://mathoverflow.net/questions/93719 (version: 2018-10-17).
2. C.H. Dowker, Local dimension of normal spaces, Q. J. Math. Oxford Ser. (2) 6 (1955) 101–120, https://doi.org/10.1093/qmath/6.1.101, MR86286. MR0086286
3. Ryszard Engelking, General Topology, 2nd ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989. Translated from the Polish by the author, MR1039321. MR1039321
4. Leonard Gillman, Meyer Jerison, Rings of Continuous Functions, Graduate Texts in Mathematics, vol. 43, Springer-Verlag, New York, Heidelberg, 1976. Reprint of the 1960 edition, MR0407579. MR0407579
5. Klaas Pieter Hart, The Čech-Stone compactification of the real line, in: Recent Progress in General Topology, Prague, 1991, North-Holland, Amsterdam, 1992, pp. 317–352, MR1229130. MR1229130
6. Klaas Pieter Hart, Jan van Mill, Covering dimension and finite-to-one maps, Topol. Appl. 158 (18) (2011) 2512–2519, https://doi.org/10.1016/j.topol.2011.08.017, MR2847324. MR2847324
7. F. Hausdorff, Untersuchungen über Ordnungstypen, Leipz. Ber. 58 (1906) 106–169, zbMATH 37.0070.03.
8. Felix Hausdorff, Grundzüge der Mengenlehre, Verlag von Veit & Comp., Leipzig, 1914 (German). Mit 53 Figuren im Text, zbMATH 45.0123.01. MR0031025
9. Kenneth Kunen, Set Theory. An Introduction to Independence Proofs, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam, New York, 1980, MR597342. MR0597342
10. I.K. Lifanov, The dimension of a product of ordered continua, Dokl. Akad. Nauk SSSR 177 (1967) 778–781 (Russian), MR0221480. MR0221480
11. W. McGovern, Zero-dimensional F-space which is not strongly zero-dimensional, https://mathoverflow.net/questions/239324 (version: 2016-05-20).
12. Roy Prabir, Nonequality of dimensions for metric spaces, Trans. Am. Math. Soc. 134 (1968) 117–132, https://doi.org/10.2307/1994832, MR227960. MR0227960

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.