USING ELEMENTARITY IN THE STUDY OF COMPACT SPACES

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CONSIDER

COMPACT HAUSDORFF SPACE X

ITS LATTICE OF CLOSED SETS 2^X

GENERAL IDEA:

TAKE ELEMENTARY SUBLATTICE

$$L \prec 2^X$$

LET wL BE THE WALLMAN SPACE OF L

[LIKE STONE SPACE

$$wL = \{ u : u \text{ IS ULTRAFILTER ON } L \}$$

$$a \in L \longmapsto a^* = \{ u \in wL : a \in u \}$$

THE FAMILY $\{ a^* : a \in L \}$ SERVES AS A BASE FOR THE CLOSED SETS]

- wL IS COMPACT HAUSDORFF
- $x \longmapsto u_x = \{ a : x \in a \}$ IS A CONTINUOUS SURJECTION FROM X ONTO wL.

THE SPACE wL INHERITS MANY PROPERTIES FROM X.

- CONNECTEDNESS
- DISCONNECTEDNESS

[IF $2^X \models (\exists x)(\exists y)(x>0 \wedge y>0 \wedge x \sqcap y=0 \wedge x \sqcup y=1)$

THEN $L \models$ ———— " ————]

- DECOMPOSABLE CONTINUUM
- INDECOMPOSABLE CONTINUUM
- UNICOHERENT CONTINUUM
- NOT UNICOHERENT CONTINUUM
- ..... MORE ... BERD VAN DER STEEG'S THESIS

THESE PROPERTIES ARE

- FIRST-ORDER (IN LATTICE TERMS)
- DETERMINED BY SOME/ANY BASE FOR THE CLOSED SETS.

DIMENSION

• COVERING DIMENSION

$\mathrm{DIM}\, X \leq n$: IF $x_1 \sqcap x_2 \sqcap \cdots \sqcap x_{n+2} = 0$

THEN THERE ARE

$y_1 \geq x_1,\ y_2 \geq x_2, \ldots,\ y_{n+2} \geq x_{n+2}$

WITH

$y_1 \sqcap y_2 \sqcap \cdots \sqcap y_{n+2} = 0$

AND

$y_1 \sqcup y_2 \sqcup \cdots \sqcup y_{n+2} = 1$

WE HAVE $\mathrm{DIM}\, wL = \mathrm{DIM}\, X$.

CLEARLY $2^X \models \mathrm{DIM} \leq n \leftrightarrow L \models \mathrm{DIM} \leq n$

IN ADDITION $\mathrm{DIM} \leq n$ DEPENDS ONLY ON SOME/ANY BASE.

. Large Inductive Dimension.

$\mathrm{Ind}\, X \le n$: if $x \sqcap y = 0$

then there are u and v

such that

$x \sqcap u = 0 \wedge y \sqcap v = 0 \wedge$

$u \sqcup v = 1 \wedge \mathrm{Ind}(u \sqcap v) \le n-1$

We have $\mathrm{Ind}\, wL \le \mathrm{Ind}\, X$

Again $2^X \models \mathrm{Ind} \le n \leftrightarrow L \models \mathrm{Ind} \le n$

But we only have

$L \models \mathrm{Ind} \le n \to \mathrm{Ind}\, wL \le n$

Example

Start with compact Hausdorff X

for which $\dim X = 1 < 2 = \mathrm{Ind}\, X$

Take countable L.

Then wL is compact metric, so

$1 = \dim wL = \mathrm{Ind}\, wL < \mathrm{Ind}\, X$

DIMENSIONSGRAD [BROUWER]

$Dg\,X \le n$ IF F AND G ARE CLOSED AND DISJOINT THEN THERE IS A CUT C BETWEEN F AND G WITH $Dg\,C \le n-1$

C IS A CUT BETWEEN F AND G MEANS: EVERY CONTINUUM THAT MEETS F AND G ALSO MEETS C.

THE FORMULA $Dg \le n$ IS FIRST-ORDER

SO $2^X \models Dg \le n \leftrightarrow L \models Dg \le n$

BUT: $L \models Dg \le n$ DOES NOT AUTOMATICALLY IMPLY $Dg\,wL \le n$

FOR EXAMPLE $[0,1]$ HAS A LATTICE-BASE Π WITHOUT CONNECTED ELEMENTS, SO $\Pi \models Dg \le 0$ (VACUOUSLY)

WE DO GET $Dg\,wL \le Dg\,X$

BUT WE DO NEED TO USE

$$L \prec 2^X$$

SKETCH OF PROOF.

MAIN POINT: IF $F, G, C \in L$ AND C IS A CUT BETWEEN F AND G IN X THEN C^* IS A CUT BETWEEN F^* AND G^* IN wL.

ASSUME A IS CLOSED IN wL SUCH THAT $F^* \cap A \neq \emptyset$, $G^* \cap A \neq \emptyset$, $C^* \cap A = \emptyset$.

PICK $B \in L$ WITH $A \subseteq B^*$, $B^* \cap C^* = \emptyset$.

[B^* IS NOT CONNECTED BUT WE'RE INTERESTED IN A]

NO CONTINUUM IN B MEETS BOTH F AND G. COMPACTNESS AND ELEMENTARITY CONSPIRE TO GIVE US P AND Q IN L SUCH THAT $P \cap Q = \emptyset$, $P \cup Q = B$, $F \cap B \subseteq P$, $G \cap B \subseteq Q$.

NOW $A \subseteq P^* \cup Q^*$ $\quad P^* \cap Q^* = \emptyset$

$$A \cap P^* \neq \emptyset \qquad A \cap Q^* = \emptyset$$

SO, YES, A IS NOT CONNECTED.

APPLICATIONS

VEDENISSOFF'S INEQUALITY

$\dim X \; (= \dim wL \underset{\uparrow}{=} \operatorname{Ind} wL) \leq \operatorname{Ind} X$

COUNTABLE L

FEDORCHUK'S INEQUALITY

$\dim X \; (= \dim wL \underset{\uparrow}{=} D_G\, wL) \leq D_G X$

COUNTABLE L

[$D_G X \leq \operatorname{Ind} X$ IS EASY]

VEDENISSOFF'S PROOF:

RELATIVELY EASY INDUCTION

FEDORCHUK'S PROOF:

CLOSING-OFF ARGUMENT

(SHADES OF LÖWENHEIM-SKOLEM)

STING IS IN $\operatorname{Cut}(F, C, G)$

$(\forall x)\big((\operatorname{CONN}(x) \wedge x \cap F \neq \emptyset \wedge x \cap G \neq \emptyset) \to$

$x \cap C \neq \emptyset\big)$

CHAINABILITY AND SPAN

A CONTINUUM X IS CHAINABLE
IF EVERY OPEN COVER HAS
A REFINEMENT $\{V_i : i < n\}$
SUCH THAT $V_i \cap V_j \neq \emptyset \leftrightarrow |i-j| \leq 1$

X HAS ---------- SPAN ZERO
IF EVERY CONTINUUM Z IN $X \times X$
WITH

$Z = Z^{-1}$	SYMMETRIC	s
$\pi_2[Z] = \pi_1[Z]$	....	σ
$\pi_2[Z] \subseteq \pi_1[Z]$	SEMI-	σ_0
—"— $\pi_1[Z] = X$	SURJECTIVE	

MEETS THE DIAGONAL
[THAT'S SIX SPANS]

LELEK

$$\begin{tikzcd}
\text{CHAINABLE} \arrow[r] & \sigma_0 = 0 \arrow[r] \arrow[d] & \sigma = 0 \arrow[r, "/" marking] \arrow[d] \arrow[l, bend right] & s = 0 \arrow[d] \\
& \sigma_0^* = 0 \arrow[r] & \sigma^* = 0 \arrow[r] & s^* = 0
\end{tikzcd}$$

BIG QUESTION

DO ANY ARROWS REVERSE?

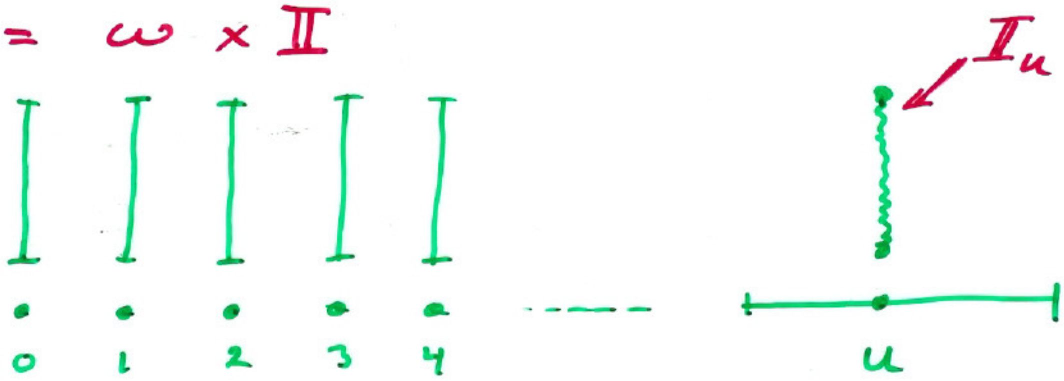
ESPECIALLY $\sigma_0 = 0 \rightarrow$ CHAINABLE!

OUR ATTEMPT: LOOK AT

$\mathbb{H}^*$ AND $\mathbb{I}_u$

$\mathbb{H} = [0, \infty)$ $\qquad \mathbb{H}^* = \beta\mathbb{H} \setminus \mathbb{H}$

$\mathbb{M} = \omega \times \mathbb{I}$



$f : \mathbb{H} \longrightarrow \mathbb{H}$ $\qquad f^* = \beta f \restriction \mathbb{H}^*$

$x \longmapsto x + 1$

$Z = f^*$ SHOWS $\sigma^*(\mathbb{H}^*) > 0$

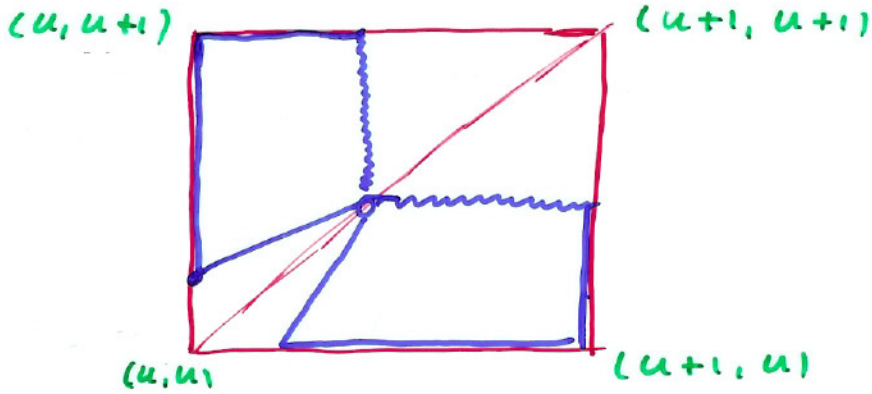
$\sigma(\mathbb{I}_u) > 0$: 

L HAS A FIXED-POINT FREE AUTOHOMEOMORPHISM, LIKE $\mathbb{H}^*$.

IN FACT $[0, L]$ HAS ONE, CALL IT h.

CH: L IS A RETRACT OF $[L, 1]$.

THIS GIVES F-PF AUTOHOMEOMOPPHISM AND SO $\sigma_0^*(\mathbb{I}_u) > 0$.



THE BLUE CONTINUUM WITNESSES

$$s^*(\mathbb{I}_u) > 0$$

CONNECT IT TO $f^* \cup (f^*)^{-1}$ TO GET

$$s^*(\mathbb{H}^*) > 0$$

THIS IS RATHER UNFORTUNATE

$$2^{\mathbb{I}} \prec 2^{\mathbb{I}}_{u} \quad \text{(ULTRAPOWER)}$$

↑ BASE FOR $\mathbb{I}_u$

CHAINABILITY CAN BE READ OFF FROM SOME / ANY BASE.

SO IT IS NOT FIRST-ORDER.

[BECAUSE $\mathbb{I}$ IS CHAINABLE]

CHAINABILITY IS $L_{\omega,\omega}$-EXPRESSIBLE.

SPAN IS

NOT FIRST-ORDER

OR

NOT BASE-DETERMINED.

QUESTION: WHICH?

THIS PUTS A DAMPER ON THE FOLLOWING STRATEGY FOR GETTING METRIC EXAMPLES.

CONSTRUCT A POSSIBLY NON-METRIC EXAMPLE X AND TAKE wL WHERE $L \prec 2^X$ IS COUNTABLE.

BUT NOT COMPLETELY.
OUR EXAMPLES DO NOT PRECLUDE THAT wL AND X SHARE CHAINABILITY, SPAN (NON)ZERO

IT WORKS PARTIALLY WHEN $L = M \cap 2^X$, WHERE $M \prec H(\theta)$ WITH $X \in M$.

CHAINABILITY: THANKS TO THE POSSIBILITY OF QUANTIFYING OVER ω.