A CONNECTED F- SPACE.

THERE ARE A COMPACT AND CONNECTED F-SPACE K AND A CONTINUOUS FUNCTION $f : K \longrightarrow [0,1]$ SUCH THAT $\Omega_f = \bigcup_t \operatorname{Int} f^{\leftarrow}(t)$ IS NOT DENSE.

NOTE: LET $g : K \longrightarrow [0,1]$ BE CONTINUOUS [NOT CONSTANT]

TAKE t IN THE INTERIOR OF $g[K]$ (AN INTERVAL)

F-SPACE $\vdash$ $\operatorname{cl} g^{\leftarrow}[[0,t)] \cap \operatorname{cl} g^{\leftarrow}[(t,1]] = \emptyset$

CONNECTED $\vdash$ $\operatorname{cl} g^{\leftarrow}[[0,t)] \cup \operatorname{cl} g^{\leftarrow}[(t,1]] \neq K$

So: $\operatorname{Int} g^{\leftarrow}(t) \neq \emptyset$.

WHY?

WHY NOT!

THEORY OF PROJECTIONS IN VECTOR LATTICES.

WORK IN $C(X)$

$F \subseteq C(X)$ IS *d-INDEPENDENT* IF FOR EVERY (NON-EMPTY) OPEN SET $\mathcal{U}$ THE FAMILY

$$\{f \restriction \mathcal{U} : f \in F\} \setminus \{0\}$$

IS LINEARLY INDEPENDENT.

$\mathcal{B} \subseteq C(X)$ IS A *d-BASIS* IF IT IS d-INDEPENDENT AND FOR EVERY $g \in C(X)$ THERE IS A MAXIMAL DISJOINT FAMILY $\mathcal{U}_g$ OF OPEN SETS SUCH THAT FOR EVERY $\mathcal{U} \in \mathcal{U}_g$ THE RESTRICTION $g \restriction \mathcal{U}$ IS A (FINITE) LINEAR COMBINATION OF $\{f \restriction \mathcal{U} : f \in \mathcal{B}\}$

FOR OUR SPACE K THE FAMILY $\{1\}$ IS MAXIMALLY d-INDEPENDENT

[$\{g, 1\}$ IS NOT d-INDEPENDENT BECAUSE OF $\mathrm{Int}\, g^{\leftarrow}(t)$]

IT IS NOT A d-BASIS BECAUSE OUR f IS NOT A "d-LINEAR COMBINATION" OF $\{1\}$

[IF h IS SUCH A COMBINATION OF 1 THEN Ω_h IS DENSE]

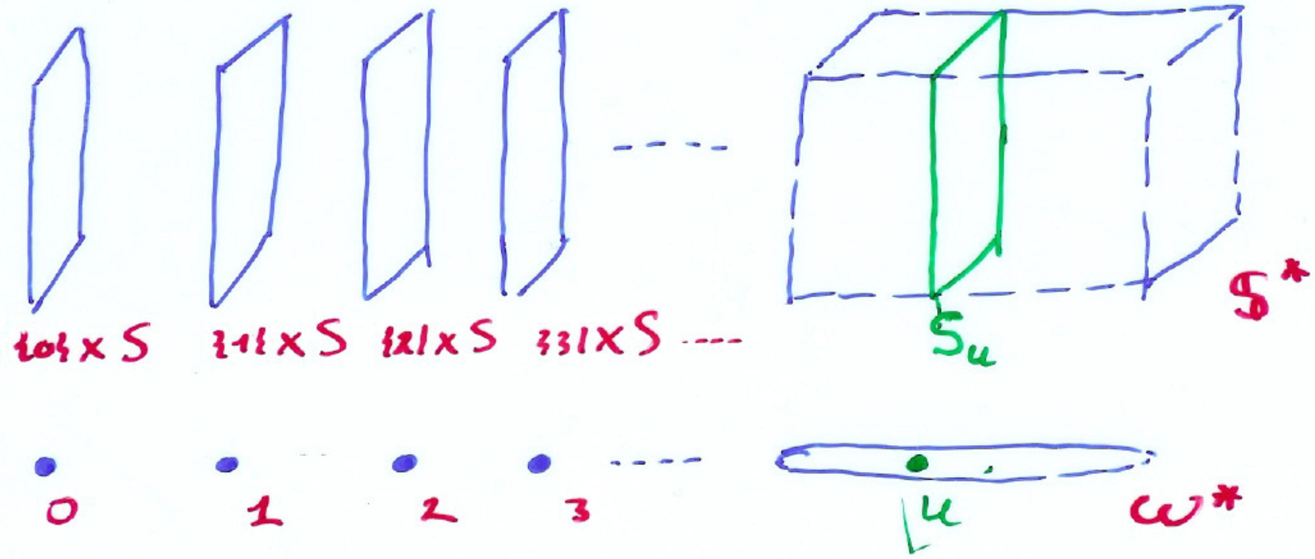
IF X HAS A CLOPEN π-BASE THEN IN $C(X)$:

MAX. d-INDEPENDENT $\longrightarrow$ d-BASIS.

CONSTRUCTION

START WITH $\mathbb{S} = \omega \times S$, WHERE S IS THE UNIT SQUARE $[0,1]^2$.

CONSIDER $\beta\mathbb{S}$ (ČECH-STONE COMP'N)



- $\pi : \mathbb{S} \longrightarrow \omega \quad \langle n, x, y \rangle \longmapsto n$
- $\beta\pi : \beta\mathbb{S} \longrightarrow \beta\omega$
- TAKE $u \in \omega^*$ AND $S_u = \beta\pi^{\leftarrow}(u)$
- $q : \mathbb{S} \longrightarrow [0,1] \quad \langle n, x, y \rangle \longmapsto x$
- $q_u = \beta q \restriction S_u$
- q_u IS NOT IT; WE FIND $K \subseteq S_u$ SUCH THAT $f = q_u \restriction K$ IS AS REQUIRED.

- RECURSIVELY THIN OUT S_u:

$$X_0 = S_u \qquad q_0 = q_u$$

$$X_1 = S_u \setminus \bigcup_t \mathrm{Int}\, q_0^{\leftarrow}(t) \qquad q_1 = q_0 \restriction X_1$$

$\vdots$

$$X_{\alpha+1} = X_\alpha \setminus \bigcup_t \mathrm{Int}_\alpha\, q_\alpha^{\leftarrow}(t) \qquad q_{\alpha+1} = q_\alpha \restriction X_{\alpha+1}$$

α LIMIT: $X_\alpha = \bigcap_{\beta<\alpha} X_\beta \qquad q_\alpha = \bigcap_{\beta<\alpha} q_\beta$

THERE IS AN ORDINAL δ ($< \mathfrak{c}^+$) SUCH THAT $X_{\delta+1} = X_\delta$

NOTE: $\forall t \quad \mathrm{Int}_\delta\, q_\delta^{\leftarrow}(t) = \emptyset$

[Int_α IS INTERIOR-IN-X_α]

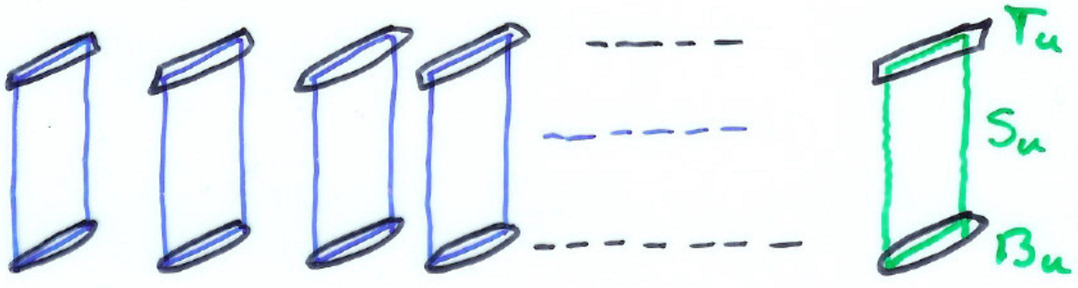
- IN $\beta\mathbb{S}$ TAKE CLOSURES OF

$\mathbb{B} = \omega \times [0,1] \times \{0\}$ AND $\mathbb{T} = \omega \times [0,1] \times \{1\}$

PUT $B_u = \mathrm{cl}\, \mathbb{B} \cap S_u$ AND

$T_u = \mathrm{cl}\, \mathbb{T} \cap S_u$

[BOTTOM LINE AND TOP LINE]



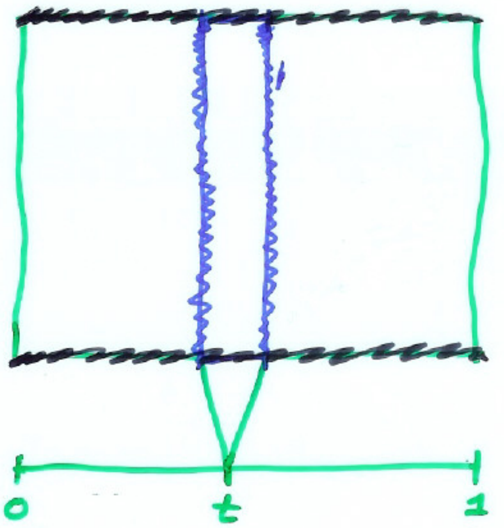
SOME OBSERVATIONS.

- $\operatorname{Fr} q_u^{\leftarrow}(t) = L_t \cup R_t$

  $L_t = q_u^{\leftarrow}(t) \cap \operatorname{cl} q_u^{\leftarrow}[[0,t)]$

  $R_t = q_u^{\leftarrow}(t) \cap \operatorname{cl} q_u^{\leftarrow}[(t,1]]$

  BOTH CONNECTED



NOTE
L_t AND R_t
INTERSECT
B_u AND T_u

- INDUCTIVELY ON α:

  EACH COMPONENT OF X_α MEETS B_u AND T_u.

- CONCLUDE: $K = B_u \cup X_\delta$

  IS
  - COMPACT
  - CONNECTED
  - F-SPACE

- BY CONSTRUCTION FOR $f = q_u \restriction K$

  WE HAVE $\operatorname{Int} f^{\leftarrow}(t) \subseteq B_u$ (ALL t)

  SO $\bigcup_t \operatorname{Int} f^{\leftarrow}(t)$ IS NOT DENSE

ABOUT THE PROJECTIONS.

LET X BE COMPACT AND EXTREMALLY DISCONNECTED

LET $\mathcal{B}$ BE A d-BASIS WITH $1 \in \mathcal{B}$ (FOR $C(X)$).

FOR $f \in C(X)$ FIND $\mathcal{U}_f$

DEFINE $\bar{f} \in \bar{C}(X)$ [EXTENDED REAL VALUED]

FIRST ON $\cup\mathcal{U}_f$:

IF $U \in \mathcal{U}_f$ THEN $\bar{f} \upharpoonright U \equiv$ COEFF. OF 1.

THEN ON X: $\cup\mathcal{U}_f$ IS C^*-EMBEDDED

THE MAP $f \longmapsto \bar{f}$ IS A PROJECTION OF $C(X)$ (EVEN $\bar{C}(X)$) ONTO

$E\bar{C}(X) = \{ f : \Omega_f \text{ is dense} \}$.

THIS IS A "BAD" PROJECTION.

CREDITS:

d-BASES AND d-INDEPENDENCE:
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βX: E. ČECH AND M. H. STONE

F-SPACE: GILLMAN AND HENRIKSEN

CONNECTEDNESS: HAUSDORFF

THE SPACE WILL APPEAR
IN POSITIVITY.