EMBEDDABILITY OF THE MEASURE ALGEBRA.

A BOOLEAN ALGEBRA IS A BOOLEAN RING WITH UNIT:

A RING $(B, +, \cdot, 1)$ IN WHICH

$$x \cdot x = x$$

FOR ALL x.

DEFINE
$$x \wedge y = x \cdot y$$
$$x \vee y = x + y + x \cdot y$$
$$x' = 1 - x$$

THEN $(B, \wedge, \vee, 0, 1, \cdot')$ IS A BOOLEAN ALGEBRA.

SUCH A STRUCTURE SATISFIES ALL LAWS THAT $\cap, \cup, \emptyset, \mathcal{U}, \cdot^{c}$ SATISFY BECAUSE ......

-------- OF STONE'S REPRESENTATION THEOREM FOR BOOLEAN ALGEBRAS:

GIVEN $(B, \wedge, \vee, 0, 1, \cdot')$ THERE ARE A SET X AND AN ALGEBRA OF SETS $\mathcal{B}$ [SUBSETS OF X] SUCH THAT

$$(B, \wedge, \vee, 0, 1, \cdot') \simeq (\mathcal{B}, \cap, \cup, \emptyset, X, X\setminus\cdot)$$

IN FACT X IS A COMPACT HAUSDORFF ZERO-DIMENSIONAL SPACE AND $\mathcal{B}$ IS EXACTLY THE FAMILY OF CLOSED-AND-OPEN SUBSETS OF X.

CONSTRUCTION:

X: SPACE OF MAXIMAL IDEALS WITH HULL-KERNEL TOPOLOGY.

$$a \longmapsto a^+ = \{ m \in X : a \notin m \}$$

DEFINES THE ISOMORPHISM.

MEASURE ALGEBRA

$\mathbf{Bor}$: BOREL SUBSETS OF $[0,1]$

$\mathcal{N}$: IDEAL OF NULL SETS

(SETS OF MEASURE ZERO).

$\mathbb{M} = \mathbf{Bor}/\mathcal{N}$

IS <u>THE</u> MEASURE ALGEBRA

ITS STONE SPACE IS A COMPACT, ZERO-DIMENSIONAL HAUSDORFF SPACE THAT

- IS NOT SEPARABLE
- SATISFIES THE CCC (PAIRWISE DISJOINT FAMILIES OF OPEN SETS ARE COUNTABLE)
- EVEN: FOR EACH n ONE CAN WRITE $\mathbb{M} = \bigcup_i \mathbb{M}_i^n$ WITH $\mathbb{M}_i^n$ AN n-LINKED SUBSET, I.E.,

  $F \subseteq \mathbb{M}_i^n \wedge |F| = n \longrightarrow \bigwedge F > 0$

POWER SET OF $\mathbb{N}$ MOD FIN.

THE FAMILY FIN IS AN IDEAL IN THE BOOLEAN ALGEBRA $\mathcal{P}(\mathbb{N})$.

OUR SECOND ALGEBRA IS THE QUOTIENT $\mathcal{P}(\mathbb{N})/\mathrm{FIN}$.

STANDARD NOTATION:

$A =^* B$ FOR $A = B$ (MOD FIN)

$A \subseteq^* B$ FOR $A \subseteq B$ (MOD FIN)

ETC.

THUS, E.G.,

$\{1, 2, 3, \text{---------}, 10^{100}\} =^* \emptyset$

IS THERE A DIFFERENCE BETWEEN $\mathbb{M}$ AND $\mathcal{P}(\mathbb{N})/\mathrm{FIN}$?

FINITARY: NO.

TARSKI: ANY TWO ATOMLESS BOOLEAN ALGEBRAS ARE ELEMENTARILY EQUIVALENT. I.E. THEY SATISFY THE SAME BOOLEAN IDENTITIES.

ATOM: $x > 0$ plus

$$x \geq y \geq 0 \rightarrow x = y \text{ or } y = 0$$

ASIDE: FROM THIS IT FOLLOWS THAT THERE IS 'ONLY ONE' COUNTABLE ATOMLESS BOOLEAN ALGEBRA AND HENCE 'ONLY ONE' CANTOR SET.

IS THERE A DIFFERENCE BETWEEN $\mathbb{M}$ AND $\mathcal{P}(\mathbb{N})/\mathrm{FIN}$

INFINITARY: OH YES!

TAKE BIJECTION $\mathbb{N} \leftrightarrow \mathbb{Q}$.
FOR $x \in (1, \infty) \setminus \mathbb{Q}$ SET

$$A_x = \left\{ \tfrac{1}{n}[nx] : n \in \mathbb{N} \right\}$$

NOTE: IF $1 < x < y$ AND $n(y-x) > 1$ THEN $\frac{1}{n}[ny] > x$ AND SO $A_x \cap A_y =^* \emptyset$.

THUS WE FIND AN UNCOUNTABLE PAIRWISE DISJOINT FAMILY IN $\mathcal{P}(\mathbb{N})/\mathrm{FIN}$: $\{A_x : x \in (1, \infty) \setminus \mathbb{Q}\}$

CLEARLY THEN

$$\mathbb{M} \not\cong \mathcal{P}(\mathbb{N})/\mathrm{FIN}$$

OR EVEN

$$\mathcal{P}(\mathbb{N})/\mathrm{FIN} \not\hookrightarrow \mathbb{M}$$

ALSO

$\mathbb{M}$ IS COMPLETE

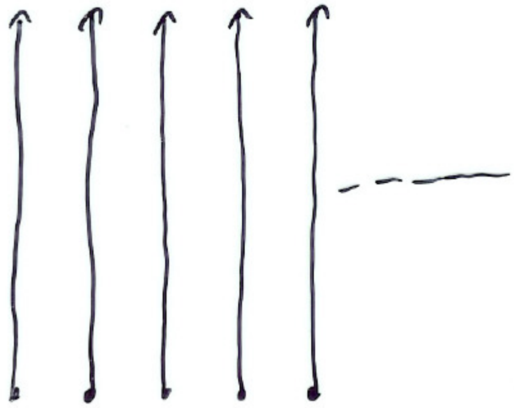
SIMPLY OBSERVE

$$\bigvee_n [B_n]_n = \left[\bigcup_n B_n \right]_n$$

$\mathcal{P}(\mathbb{N})/\mathrm{FIN}$ IS NOT

WORK WITH $\mathbb{N} \times \mathbb{N}$;

$$V_n = \{(n, m) : m \in \mathbb{N}\}$$



$V_1 \ V_2 \ V_3 \ V_4 \ V_5 \ \cdots$

$\sup\{V_n : n \in \mathbb{N}\}$ DOES NOT EXIST.

(MOD <u>FINITE</u>)

CAN WE EMBED $\mathbb{M}$ INTO $\mathcal{P}(\mathbb{N})/\mathrm{fin}$?

EQUIVALENT QUESTION:

IS THERE A COMPACTIFICATION $\gamma\mathbb{N}$ OF $\mathbb{N}$ WITH $\gamma\mathbb{N}\setminus\mathbb{N}$ HOMEMEMORPHIC TO THE STONE SPACE OF $\mathbb{M}$?

EQUIVALENT QUESTION:

CAN WE EMBED THE BANACH-ALGEBRA $L^\infty[0,1]$ INTO ℓ^∞/c_0?

ANSWER: WE CAN'T PROVE EITHER 'YES' OR 'NO'.

MEANING: WE CAN PROVE THAT WE CAN'T PROVE EITHER 'YES' OR 'NO'.

WE CANNOT PROVE 'NO'.

THE CONTINUUM HYPOTHESIS IMPLIES:

EVERY BOOLEAN ALGEBRA OF CARDINALITY $\mathfrak{c}$ CAN BE EMBEDDED INTO $\mathcal{P}(\mathbb{N})/\mathrm{FIN}$.

OUR $\mathbb{M}$ IS SUCH AN ALGEBRA, SO THERE.

EVERYBODY KNOWS:

CH DOES NOT LEAD TO CONTRADICTIONS, HENCE

"$\mathbb{M}$ CAN BE EMBEDDED INTO $\mathcal{P}(\mathbb{N})/\mathrm{FIN}$"

DOES NOT LEAD TO CONTRADICTIONS EITHER

WE CANNOT PROVE 'YES'.

FIRST:

WHY IT IS PLAUSIBLE THAT WE CANNOT PROVE 'YES'.

REPLACE $[0,1]$ BY $\mathbb{R}$.

LET $\mathcal{Q}$ BE THE (COUNTABLE) SUBALGEBRA OF $\mathbb{M}$ GENERATED BY $\{[p,q) : p,q \in \mathbb{Q};\ p \leq q\}$.

ASSUME $\varphi : \mathbb{M} \hookrightarrow \mathcal{P}(\mathbb{N})/\mathrm{FIN}$ IS AN EMBEDDING.

LET $\Phi : \mathcal{B}or \longrightarrow \mathcal{P}(\mathbb{N})$ BE A LIFTING, I.E.,

$$[\Phi(B)]_{\mathrm{FIN}} = \varphi([B]_n)$$

$$\begin{tikzcd} B \arrow[r] \arrow[d] & \Phi(B) \arrow[d] \\ {[B]_n} \arrow[r] & \varphi([B]_n) = [\Phi(B)]_{\mathrm{FIN}} \end{tikzcd}$$

WE CAN MAKE IT SO THAT $\Phi \restriction \mathcal{Q}$ IS AN EMBEDDING INTO $\mathcal{P}(\mathbb{N})$, SO $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$, ETC, WHENEVER $A, B \in \mathcal{Q}$.

FOR $n \in \mathbb{N}$ ENUMERATE $\Phi([n, n+1))$ AS $\{k(n,i) : i \in \mathbb{N}\}$ AND CHOOSE $m(n,i)$ SUCH THAT

$$k(n,i) \in \Phi\left(\left[\frac{m(n,i)}{2^{i+1}}, \frac{m(n,i)+1}{2^{i+1}}\right) + n\right)$$

WRITE

$$U_n = \bigcup_{i=1}^{\infty} \left[\frac{m(n,i)}{2^{i+1}}, \frac{m(n,i)+1}{2^{i+1}}\right) + n$$

NOTE: $\lambda(U_n) \leq \sum_{i=1}^{\infty} 2^{-i-1} = 1/2$.

FINALLY, LET $U = \bigcup_{n=1}^{\infty} U_n$

AND $F = \mathbb{R} \setminus U$

NOTE $\lambda([n, n+1) \cap F) \geq 1/2$

IT FOLLOWS THAT

$$\Phi(F) \cap \Phi([n, n+1))$$

IS INFINITE.

TAKE FIRST INDEX i_n SUCH THAT $k(n, i_n) \in \Phi(F)$.

PUT $I_n = \left[\frac{m(n, i_n)}{2^{i_n+1}}, \frac{m(n, i_n)+1}{2^{i_n+1}} \right) + n$

NOW

- $\bigcup_{n=1}^{\infty} I_n \cap F = \emptyset$
- SO $\Phi(\bigcup_{n=1}^{\infty} I_n) \cap \Phi(F) =^* \emptyset$
- BUT $\{k(m, i_n) : n \in \mathbb{N}\} \subseteq$

$$\bigcup_{n=1}^{\infty} \Phi(I_n) \cap \Phi(F)$$

- AND SO $\Phi(\bigcup_{n=1}^{\infty} I_n) \neq^* \bigcup_{n=1}^{\infty} \Phi(I_n)$.

CONCLUSION:

<u>IF</u> $\Phi \restriction \mathbb{Q}$ IS AN EMBEDDING THEN Φ DOES NOT RESPECT INFINITE UNIONS OF THE ABOVE TYPE.

AN EMBEDDING $\varphi : \mathbb{M} \hookrightarrow \mathcal{P}(\mathbb{N})/\mathrm{FIN}$

CANNOT HAVE A MODERATELY

DECENT LIFTING.

THE OPEN COLOURING AXIOM

IF X IS SEPARABLE METRIC

AND $$[X]^2 = K_0 \cup K_1$$

WITH K_0 OPEN

THEN EITHER THERE IS

AN UNCOUNTABLE $Y \subseteq X$ SUCH

THAT $[Y]^2 \subseteq K_0$

OR $X = \bigcup_{n=1}^{\infty} X_n$ SUCH THAT

$[X_n]^2 \subseteq K_1$ FOR ALL n.

OCA CONTRADICTS CH

BUT BY ITSELF DOES NOT

LEAD TO CONTRADICTIONS.

OCA IS USED TO SHOW:

<u>IF</u> THERE IS AN EMBEDDING

$\varphi : \mathbb{M} \hookrightarrow \mathcal{P}(\mathbb{N})/\mathrm{fin}$

<u>THEN</u> THERE ARE AN EMBEDDING

$\psi : \mathbb{M} \hookrightarrow \mathcal{P}(\mathbb{N})/\mathrm{fin}$

AND A LIFTING $\Psi : \mathcal{B}or \longrightarrow \mathcal{P}(\mathbb{N})$

SUCH THAT
- $\Psi \restriction \mathcal{Q}$ IS AN EMBEDDING
- Ψ DOES RESPECT INFINITE UNIONS OF THE TYPE DESCRIBED EARLIER.

THUS, OCA IMPLIES THERE IS NO EMBEDDING AND SO "$\mathbb{M}$ CANNOT BE EMBEDDED INTO $\mathcal{P}(\mathbb{N})/\mathrm{fin}$" DOES NOT LEAD TO CONTRADICTIONS.

CREDITS:

CH IMPLIES EMBEDDABILITY:

PAROVIČENKO

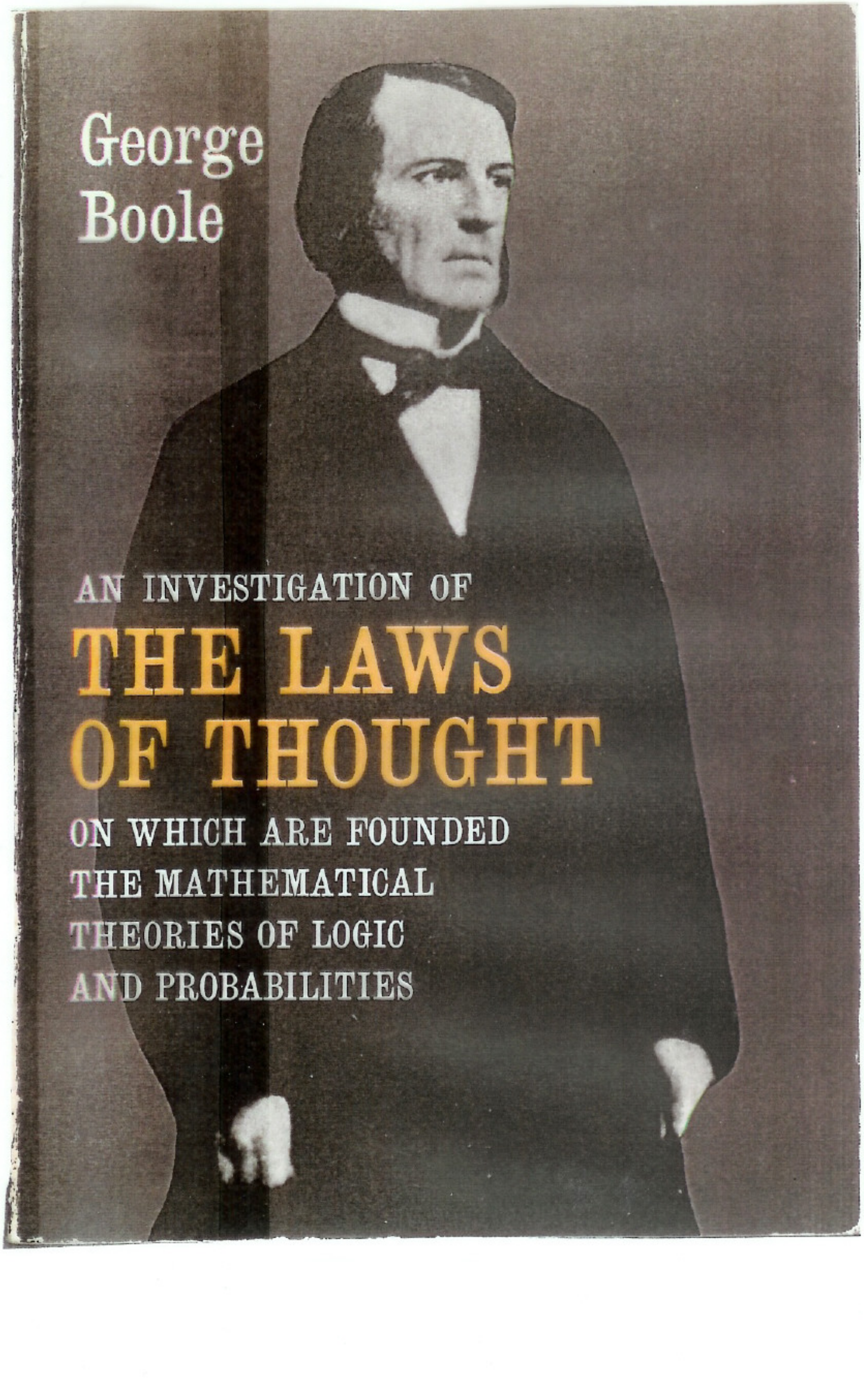
FORMULATION OF OCA AND ITS CONSISTENCY

TODORČEVIĆ

OCA IMPLIES NON-EMBEDDABILITY

ALAN DOW & KPH.

George Boole



AN INVESTIGATION OF

THE LAWS OF THOUGHT

ON WHICH ARE FOUNDED THE MATHEMATICAL THEORIES OF LOGIC AND PROBABILITIES