COSMIC DIMENSIONS

ALAN DOW + K.P.H.

NON
IMPEDITUS
AB ULLA
SCIENTIA

SOME HISTORY

DIM = IND = IND

SEPARABLE METRIZABLE SPACES

IND = IND

PERFECTLY NORMAL + LINDELÖF

DIM ≤ IND

NORMAL

ARKHANGEL'SKIĬ:

DIM = IND FOR COSMIC SPACES?

COSMIC = CONTINUOUS IMAGE OF A SEPARABLE METRIC SPACE (+ REGULAR)

IND - SMALL INDUCTIVE DIMENSION

Ind - LARGE ——— " ———

$\operatorname{ind} X \leq n$ IFF

X HAS A BASE CONSISTING OF SETS WHOSE BOUNDARIES SATISFY $\operatorname{ind} \leq n-1$

[$\operatorname{ind} = -1 \leftrightarrow$ SPACE IS EMPTY]

$\operatorname{Ind} X \leq n$ IFF

X HAS A 'LARGE' BASE CONSISTING OF SETS WHOSE BOUNDARIES SATISFY $\operatorname{Ind} \leq n-1$

[$\operatorname{Ind} = -1 \leftrightarrow$ SPACE IS EMPTY]

'LARGE' BASE: IF F IS CLOSED AND $U \supseteq F$ IS OPEN THEN $F \subseteq B \subseteq U$ FOR SOME B.

DIM - COVERING DIMENSION

EVERY FINITE OPEN COVER $\mathcal{U}$ HAS A FINITE OPEN REFINEMENT $\mathcal{V}$ SUCH THAT $\bigcap \mathcal{V}' = \emptyset$ WHENEVER $\mathcal{V}' \in [\mathcal{V}]^{n+1}$ IFF $\dim X \leq n$.

DELISTATHIS & WATSON

A COSMIC X SUCH THAT

$\dim X = 1$

$\operatorname{ind} X = \operatorname{Ind} X \geq 2$

- FROM CH
- NOW FROM $MA_{\sigma\text{-CENTERED}}$

NOTATION

$\mathcal{Q}$: ALL SEGMENTS IN $\mathbb{R}^2$ WITH RATIONAL END POINTS

$A = \{\langle p+\sqrt{2}, q\rangle : \langle p, q\rangle \in \mathbb{Q}^2\}$

$X = \mathbb{R}^2 \setminus A \qquad [\, \bigcup \mathcal{Q} \subseteq X \,]$

τ_e : EUCLIDEAN TOPOLOGY

τ : NEW TOPOLOGY ON X

DEMANDS

COSMIC: τ AND τ_e COINCIDE ON $X \setminus \bigcup \mathcal{Q}$ AND EACH $Q \in \mathcal{Q}$.
$\langle X, \tau \rangle$ IS A CONTINUOUS IMAGE OF $X \setminus \bigcup \mathcal{Q} \oplus \bigoplus \{Q : Q \in \mathcal{Q}\}$.

$\dim(X, \tau) \leq 1$: ALMOST AUTOMATIC

$\operatorname{Ind}(X, \tau) \geq 2$: EVERY OPEN SET U WITH $\emptyset \neq U$ AND $\overline{U} \neq X$ HAS A SPECIAL ONE-DIMENSIONAL SET IN ITS BOUNDARY

A COPY OF THE CANTOR SET WITH A VERY NICE ONE-DIMENSIONAL TOPOLOGY.

NOTE $\dim(X, \tau_e) = \operatorname{Ind}(X, \tau_e) = 1$

THIS MUST GO UP

THIS MUST REMAIN THE SAME

KURATOWSKI'S FUNCTION [1932]

FOR $x \in 2^{\mathbb{N}}$ LET C_x BE THE COUNTING FUNCTION OF $\operatorname{supp} x = \{ i : x(i) = 1 \}$

$$f : 2^{\mathbb{N}} \longrightarrow [-1, 1]$$

$$f(x) = \sum_{j \in \operatorname{dom} C_x} (-1)^{C_x(j)} \cdot 2^{-j}$$

[PARITY OF $C_x(j)$ DECIDES +/−]

GRAPH $f \subseteq 2^{\mathbb{N}} \times [-1, 1]$

IS ONE-DIMENSIONAL AT ALL POINTS $\langle x, f(x) \rangle$ WITH $\operatorname{supp} x$ FINITE.

τ_f : THE TOPOLOGY ON $2^{\mathbb{N}}$ OBTAINED BY IDENTIFYING x AND $\langle x, f(x) \rangle$.

$D = \{ x : \operatorname{supp} x \text{ FINITE} \}$

WORK EXERCISE 1.2.E IN ENGELKING'S
THEORY OF DIMENSIONS
FINITE AND INFINITE

MAIN STEP:

LET $\mathcal{U}$ BE GIVEN.

CONSTRUCT A COPY OF $2^{\mathbb{N}}$ IN $\operatorname{Fr}\mathcal{U} \cap X$
BY WAY OF A WELL-CHOSEN MAP
$\sigma : \mathbb{D} \longrightarrow \operatorname{Fr}\mathcal{U}$.
THE COPY IS $K = \overline{\sigma[D]}$ AND σ EXTENDS
TO A HOMEOMORPHISM $\bar{\sigma} : 2^{\mathbb{N}} \longrightarrow K$

- TRANSPLANT f TO K: $f_K = f \circ \bar{\sigma}^{-1}$

- EXTEND f_K TO ALL OF X (OR $\mathbb{R}$)
TIETZE-URYSOHN: f_K IS CONTINUOUS AT ALL POINTS NOT IN $\sigma[D]$.

- THE TOPOLOGY OF THE GRAPH, τ_K, IS SEPARABLE METRIZABLE
AND $\operatorname{ind}(K, \tau_K) = 1$

- ALSO $\operatorname{Ind}(X, \tau_K) = 1$
[COUNTABLE CLOSED SUM THEOREM]

DO THE MAIN STEP FOR EVERY $\mathcal{U}$
THAT SATISFIES $\emptyset \neq \mathcal{U}$ AND $\overline{\mathcal{U}} \neq X$,
UNLESS THERE IS $\mathcal{G} \in \mathfrak{G}$ WITH $\mathcal{G} \subseteq \mathrm{Fr}\,\mathcal{U}$.

FINAL TOPOLOGY: $\tau = \bigvee \{ \tau_{K(\mathcal{U})} : \mathcal{U} \ldots \}$

- 'CLEARLY' $\mathrm{Ind}(X, \tau) \geq 2$.
- $\dim(X, \tau) \leq 1$ BY COSMICITY

  EVERY FINITE OPEN COVER
  SITS IN $\bigvee \{ \tau_{K(\mathcal{U})} : \text{-- CTBLY MANY } \mathcal{U} \ldots \}$

THE NOT-REALLY-VERY-GORY DETAILS
CAN BE FOUND AT

FA.ITS.TUDELFT.NL/~HART

DETAILS? WHAT DETAILS?

THE K's MUST NOT INTERFERE
- WITH EACH OTHER
- WITH THE Q's.

DELISTATHIS & WATSON

CH + INVOLVED RECURSIONS

DOW & H.

$MA_{\sigma\text{-CENTERED}}$ + PERSPICUOUS RECURSIONS

CHARALAMBOUS

USES AD FAMILIES OF ANNULI TO GUIDE THE CONSTRUCTION.