A concrete co-existential map that is not confluent

Non impeditus ab ulla scientia

K. P. Hart

Faculty EEMCS
TU Delft

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Outline

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Two maps

Caveat: all spaces are compact Hausdorff

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The map $\beta\pi_\kappa$ helps divide $\beta(\kappa \times X)$ into manageable pieces:

- $\{\alpha\} \times X$ for $\alpha \in \kappa$ (that's just X), and
- $X_u = \beta\pi_\kappa^\leftarrow(u)$ for $u \in \kappa^*$ (an enriched version of X).

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Ultracopowers and Codiagonal maps

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Also, $2^{\kappa\times X}$ is naturally isomorphic to $(2^X)^\kappa$:
$\langle A_\alpha : \alpha < \kappa\rangle$ corresponds to $\bigcup_\alpha \{\alpha\} \times A_\alpha$.

Ultracopowers and Codiagonal maps

Furthermore: $\operatorname{cl}_\beta A \cap X_u = \operatorname{cl}_\beta B \cap X_u$ iff $\{\alpha : A_\alpha = B_\alpha\} \in u$.

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The map π_u corresponds naturally to the diagonal embedding $e_u : 2^X \to (2^X)_u$ defined by $A \mapsto \langle A : \alpha < \kappa \rangle_u$,

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The in-/prefix 'co' is there because π_u and e_u work in opposite directions.

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If 2^X sits nicely inside $(2^X)_u$ then one would expect $\pi_u : X_u \to X$ to be a nice continuous surjection.

Of course niceness is in the mind of the considerer.

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A continuous surjection $f : Y \to X$ is *co-existential* if there are an ultrafilter u on some κ and

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A continuous surjection $f : Y \to X$ is *co-existential* if there are an ultrafilter u on some κ and a continuous surjection $g : X_u \to Y$

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A continuous surjection $f : Y \to X$ is *co-existential* if there are an ultrafilter u on some κ and a continuous surjection $g : X_u \to Y$ such that $f \circ g = \pi_u$.

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The corresponding embedding $(A \mapsto f^{\leftarrow}[A])$ of 2^X into 2^Y is an *existential* embedding.

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Outline

How nice are co-existential and codiagonal maps?

There are various degrees of niceness, but for maps between continua *confluence* and *weak confluence* are quite nice.

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How nice are co-existential and codiagonal maps?

There are various degrees of niceness, but for maps between continua *confluence* and *weak confluence* are quite nice.

A map $f : Y \to X$ is

- **confluent** if for every continuum C in X *every* component of $f^{\leftarrow}[C]$ maps onto C
- **weakly confluent** if for every continuum C in X *some* component of $f^{\leftarrow}[C]$ maps onto C

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Weakly confluent: yes

Every codiagonal map is weakly confluent.

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If $C \subseteq X$ is a continuum then $C_u = X_u \cap \mathrm{cl}_\beta(\kappa \times C)$ is also connected and $\pi_u[C_u] = C$

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Hence every co-existential map is weakly confluent.

If $\pi_u = f \circ g$ then the component of $g[C_u]$ in $f^\leftarrow[C]$ maps onto C.

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Confluent: no

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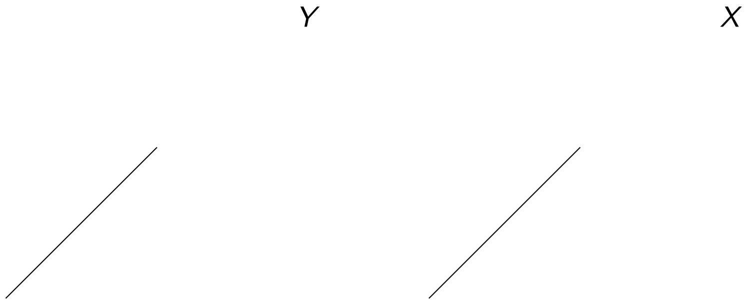
But one would be wrong.

An example



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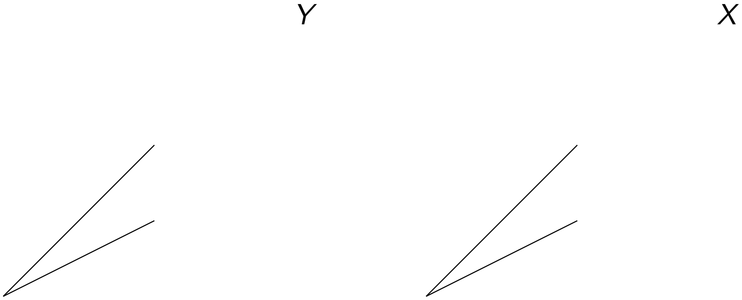
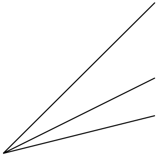
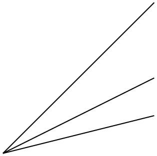


An example

Y X





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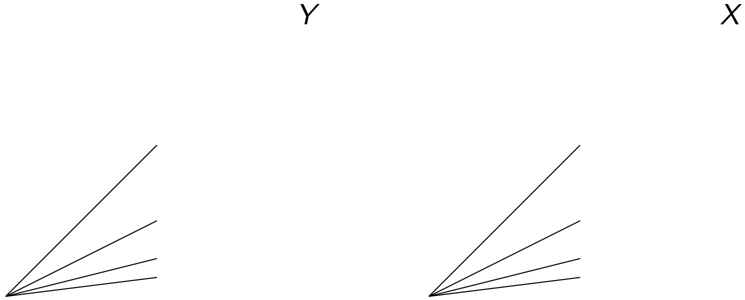

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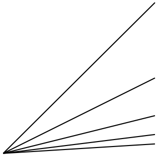

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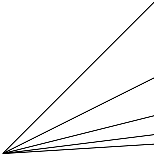

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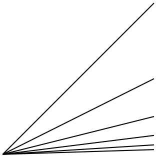

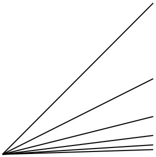

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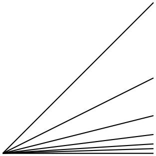
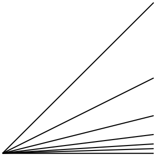
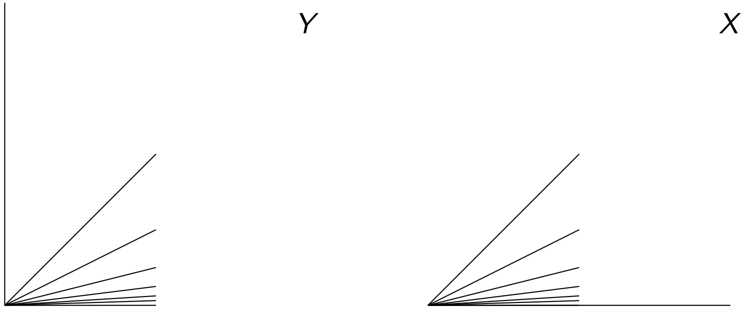




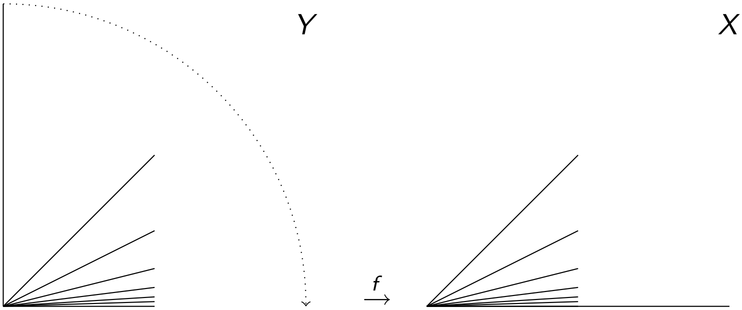
The infinite broom B

An example



The infinite broom B plus an extra hair

An example



The infinite broom B plus an extra hair and the map f.

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An example

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where $H_n = \bigl\{\langle t, t/2^n\rangle : 0 \leqslant t \leqslant 1\bigr\}$ and
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Codomain: $X = B \cup ([0,2] \times \{0\})$.

Domain: $Y = B \cup (\{0\} \times [0,2])$.

Map f: identity on B and $f(0,y) = \langle y, 0\rangle$.

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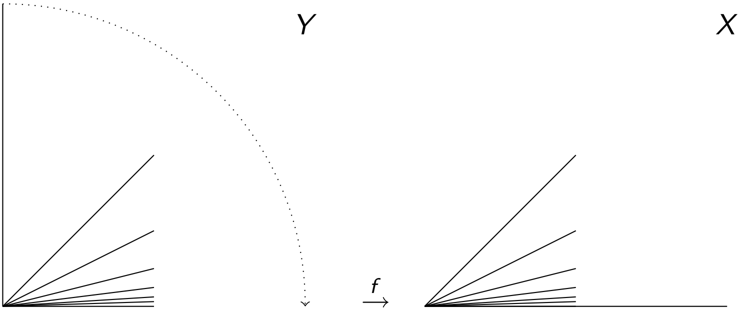
Not confluent



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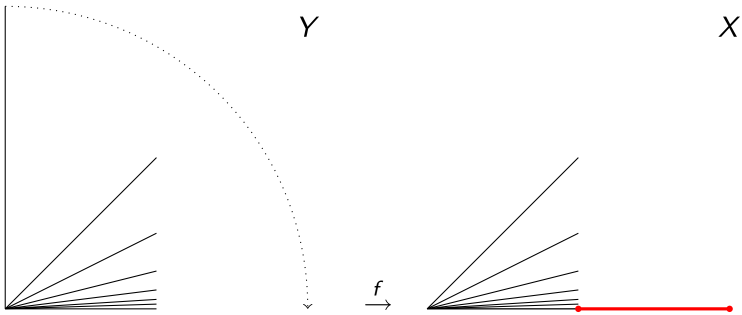

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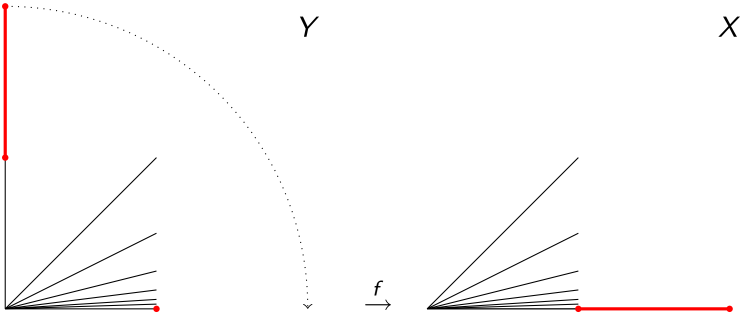

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$$f^{\leftarrow}[C] = \bigl(\{0\} \times [1,2]\bigr) \cup \bigl\{\langle 1,0\rangle\bigr\}$$

The component $\bigl\{\langle 1,0\rangle\bigr\}$ does not map onto C.

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Co-existential

We can define a map $g : \omega \times X \to Y$ such that $f \circ g_u = \pi_u$ for all $u \in \omega^*$; where $g_u = \beta g \restriction X_u$.

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- g is continuous on $F = \bigl\{\langle n, x, y\rangle : \langle x, y\rangle \in \bigcup_{i \leqslant n} H_i\bigr\}$

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g is continuous:

- g is continuous on $F = \{\langle n, x, y\rangle : \langle x, y\rangle \in \bigcup_{i \leqslant n} H_i\}$
- and on $G = \{\langle n, x, y\rangle : \langle x, y\rangle \in \bigcup_{i > n} H_i \cup (\{0\} \times [0, 2])\}$;

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- on $F \cap G = \omega \times \bigl\{\langle 0,0\rangle\bigr\}$ it takes on the value $\langle 0,0\rangle$

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- on $F \cap G = \omega \times \{\langle 0, 0\rangle\}$ it takes on the value $\langle 0, 0\rangle$

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- on $F \cap G = \omega \times \bigl\{\langle 0, 0\rangle\bigr\}$ it takes on the value $\langle 0, 0\rangle$

For each $u \in \omega^*$ we get $g_u[F_u] = B$ and $g_u[G_u] = \{0\} \times [0, 2]$; so g_u is always onto.

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To see that $f \circ g_u = \pi_u$ note that for every $\langle x, y\rangle \in X$ the set $\bigl\{n : f\bigl(g(n,x,y)\bigr) = \langle x, y\rangle\bigr\}$ is cofinite.

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It is $[n,\omega)$ if $\langle x,y\rangle \in H_n$ and it is ω if $y = 0$.

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The codiagonal maps

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$\pi_u^{\leftarrow}[C]$ splits into $g_u^{\leftarrow}\bigl[\{0\} \times [1,2]\bigr]$ and $g_u^{\leftarrow}(1,0)$.

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The codiagonal maps

The codiagonal maps are not confluent either.

$\pi_u^{\leftarrow}[C]$ splits into $g_u^{\leftarrow}\bigl[\{0\} \times [1,2]\bigr]$ and $g_u^{\leftarrow}(1,0)$.

The latter set even maps onto ω^* so it has plenty of components that do not map onto C under π_u.

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Outline

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one wonders whether these maps are as nice topologically as their duals are algebraically.

This merits further investigation.

Outline

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Light reading

Website: `fa.its.tudelft.nl/~hart`

P. Bankston,
Not every co-existential map is confluent, Houston Journal of Mathematics, to appear.

K. P. Hart.
A concrete co-existential map that is not confluent, Topology Proceedings, **34** (2009), 303–306.

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