What is . . . 'Finite'?

Tá scéilín agam

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Delft, 15 February 2017

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Dutch Science Agenda

The words ‘oneindig’ and ‘oneindigheid’ together occur 25 times as keyword in a question for the Dutch Science Agenda.

The words ‘eindig’ and ‘eindigheid’ together occur 6 times.

They are antonyms, so why the imbalance?

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Van Dale

De allereerste editie (1864):

- eindig: bn. en bijw. een einde hebbende.
- oneindig: bn. en bijw. zonder einde;
  (fig.) buitengemeen groot;
  oneindig groot: door geene maat te bepalen;
  oneindig klein: nul.

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Chambers

13th Edition (2014):

finite *adj* having an end or limit; subject to limitations or conditions, opp to *infinite*. [l. *finitus*, pap of *finire* to limit]

infinite *adj* without end or limit; greater than any quantity that can be assigned [*maths*]; extending to infinity; vast; in vast numbers; inexhaustible; infinitated (*logic*)

infinitate *vt* to make infinite; to turn into a negative term (*logic*).

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Van Dale

De wiskunde is wel gedefinieerd als de wetenschap van het oneindige, die dit met eindige middelen tracht te beheersen

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Interesting . . .

'finite' is opposite to 'infinite', yet . . .

'infinite' generates many more words than 'finite'

again an imbalance

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Simple relation

Mathematics keeps it simple:

‘infinite’ is not ‘finite’

hence

‘finite’ is not ‘infinite’

once you define one you define the other.

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Finite

Without further ado

A set, X, is finite if there are a natural number n and a bijection $f : n \to X$.

In Set Theory we define natural numbers in such a way that $n = \{i \in \mathbb{N} : i < n\}$.
So $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, etc

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Infinite

So, ..., a set, X, is infinite if there is no natural number n with a bijection $f : n \to X$.

Basically, when you have an infinite set, you are empty-handed.

Or are you ...

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Characterizations of finiteness

Do we need the external natural numbers to define finiteness?

Alfred Tarski: no.

A set, X, is finite iff every subfamily of $\mathcal{P}(X)$ has a *maximal* element
(with respect to $\subseteq$).

(maximal: nothing bigger)

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Characterizations of finiteness

Proof.

Only if: by induction.
If: let $\mathcal{F} = \{F \in \mathcal{P}(X) : F \text{ is finite}\}$. $\square$

This is used a lot in Finite(!) Combinatorics.

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Infinite again

So, with an infinite set X you get a subfamily $\mathcal{F}$ of $\mathcal{P}(X)$ without a maximal element.

Well, that's something, but is it useful?

Dictionary to the rescue ...

From Chambers

infinite set *n* (*maths*) a set that can be put into one-one correspondence with part of itself

Actually: *proper* part (of course)

This is actually Dedekind's definition of 'infinite'

Thus, Dedekind-*finite* would mean: every injection from the set to itself is surjective.

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Dedekind-infinite is better

Theorem

TFAE

1. X *is Dedekind-infinite*
2. *there is an injective map* $f : \mathbb{N} \to X$.
3. X *has as many elements as* $X \cup \{p\}$ *for (some)* p *not in* X

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Dedekind-infinite is better

Proof.

1) $\to$ 2): take injective-not-surjective $g : X \to X$ and $x \in X \setminus g[X]$; define $f : \mathbb{N} \to X$ by $f(n) = g^n(x)$.

2) $\to$ 3): take injective $f : \mathbb{N} \to X$ and define $g : X \cup \{p\} \to X$ by $g(p) = f(0)$, $g(f(n)) = f(n+1)$, and $g(x) = x$ otherwise

3) $\to$ 1) Take bijective $h : X \cup \{p\} \to X$ and let $g = h \restriction X$ □

Yep, Dedekind-infinite rocks!

Relationship

‘finite’ implies ‘Dedekind-finite’

and so, contrapositively: ‘Dedekind-infinite’ implies ‘infinite’

How about the converse?

Unfortunately . . .

The implications do not reverse, at least not without some form of the Axiom of Choice.

There's a Bachelor project available if you want to know why this is.

For algebraists

Dedekind had an other idea:

a set, X, is finite iff there is a map $f : X \to X$ such that $\emptyset$ and X are the only f-invariant sets: if $f[A] \subseteq A$ then $A = \emptyset$ or $A = X$.

Such a map for $n \in \mathbb{N}$ is a permutation represented by an n-cycle.

This was a problem on my last Set Theory exam; have a go at it yourself.

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