When Linguists do Set Theory

A Cautionary Tale

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Background

Back in October 2019 there was a short discussion on twitter about the nature of books.

A bit condensed: a book is a finite sequence of symbols from some alphabet (including spaces, punctuation, etc).
The discussion was about a paper that argued that all books past, present, and future already exist, as members of product sets of such alphabets.

This is not what I want to talk about ...

Proper classes?

In the paper, by Paul M. Postal, that argued about this status of books, there was a curious sentence:

Then, appealing to the reasoning of Langendoen and Postal (1984), one can show further that the universe of books is truly vast, amounting to what is called a *proper class* in some varieties of set theory.

Well, I could not let that one lie.

The Vastness of Natural Languages

"Langendoen and Postal (1984)" refers to the book *The Vastness of Natural Languages*, which argues that the collection of sentences in a Natural Language is not a set but larger in magnitude than any set.

The argument that I want to look at today is also available in a paper called *Sets and Sentences* by the same authors (advantage over the book: *shorter*, and easily available via Langendoen's website).

Size restrictions

The book devotes a whole chapter to arguing (quite vociferously I might add) that there should be no size restrictions on sentences in Natural Languages.

The paper argues this too, but less extensively.

Rules for languages

The authors give descriptions of rules that show how to combine constituents into Co-ordinate compound constituents using connectives.

The nature of these is not important for the general argument. Think of sets of sentences being combined into sentences using connectives, commas, (semi-)colons, and what not.

And of course there are no rules that limit the cardinality of the sets of constituents that can be used.

Existence

Central to the argument that a Natural Language is a proper class is an existence theorem.

Claim

Every set of constituents has a Co-ordinate compound constituent.

The authors give a ‘straightforward’ argument for this.

The existence proof, 1

In steps (the Q below is an unspecified category of sentences).

- Take a set U of constituents and let k be its cardinality (finite or infinite).

The existence proof, 2

- "Clearly, from the purely formal point of view, there is a co-ordinate compound W belonging to the category Q." Sounds impressive but it proves nothing; no arguments, no indication where that W should/could come from.

But, . . . , to be fair, every language should have at least one sentence, so we'll let this one slide.

As an aside: this sentence is representative of the pontificating style that the authors adopted.

The existence proof, 3

- “Since there are no size restrictions on co-ordinate compounds, W can have any number, finite (more than one) or transfinite, of immediate constituents;”
  Bad mathematical style: from “there is an individual” to “there are individuals of all sorts”.

Real mathematical error: the authors use the absence of size-restricting axioms to ‘deduce’ that there are arbitrarily large individuals.
It seems that by leaving out axioms you can prove more ...

The existence proof, 4

- "W can then, in particular, have exactly k such constituents."
  So the seemingly arbitrary W from the existence statement has become quite specific.

The existence proof, 5

- "The subconjuncts of W form a set V of cardinality exactly k."
  True, because every constituent contains/has exactly one subconjunct.

The existence proof, 6

Brace yourselves. Remember our arbitrary set U of constituents?

- "To show that W is a co-ordinate projection of U, it then in effect suffices that there exist a one-to-one mapping from U to V."
  Right . . .
  At the outset W and U were completely unrelated. And a bijection does not make sets equal, last time I checked.
- "But this is trivial, since the two sets have the same number of elements." Well, yes, that is the definition . . .

Closure Principle

Remember the indefinite article from the Claim?
Well . . .

The Closure Principle for Co-ordinate Compounding

If U is a set of constituents each belonging to the collection, S_w, of (well-formed) constituents of category Q of any NL, then S_w contains the Co-ordinate compound constituent of U.

That 'a' has become a 'the'.

Closure Principle

Below S is the category of sentences of the (nameless) language under discussion.

Closure under Co-ordinate Compounding of Sentences

Let L be the collection of all members of the category S of an NL and let $\mathrm{CP}(U)$ be the co-ordinate projection of the set of sentences U. Then:

$$(\forall U)(U \subset L \longrightarrow \mathrm{CP}(U) \in L)$$

This is taken as a truth about all Natural Languages.

Closure Principle

The Closure Principle implies that Natural Languages form proper classes ("megacollections")

The NL Vastness Theorem

NLs are not sets (are megacollections).

The proof is just
If L is a set then it has a cardinality, but it contains

$$Z = \{\mathrm{CP}(y) : y \subseteq L \text{ and } |y| \geqslant 2\}$$

and the cardinality of Z is larger than that of L.
Contradiction.

However

The proof of the existence claim uses, without justification, that there are compounds (sentences) of arbitrarily large cardinality.

We can at best treat that as an assumption, but that assumption can only be true if the sentences form a proper class, so . . .

the proof of the main result can be summarized as:
if the sentences in the natural language form a proper class
then the sentences in the natural language form a proper class

There is more

The book and paper contain more fun (or better: cringeworthy) material.

But if I timed this right my time is up.

You can read more in . . .

Light reading

Website: https://fa.ewi.tudelft.nl/~hart

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A critique of 'The vastness of Natural Languages',
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D. Terence Langendoen and Paul M. Postal,
Sets and Sentences, in *The Philosophy of Linguistics*, ed. by Jerrold J. Katz, 225–248. (1985) Oxford: Oxford University Press

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