Some realcompact spaces

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Our question

Give examples of realcompact spaces with closed subsets that are C^*-embedded but not C-embedded.

Why?

We didn't know any.

Seriously: why?

Seriously: we didn't know any.

Remember: a space is normal
iff every closed subspace is C^*-embedded
iff every closed subspace is C-embedded

Our question

In *Rings of Continuous Functions* you will find just $\beta\mathbb{R} \setminus \mathbb{N}^*$.

Here $\mathbb{N}$ is a closed subspace that is C^*-embedded, the space itself is pseudocompact, so ...

The example is due to Katětov (1951)

This amounts to cheating: just make sure there are no unbounded continuous functions on the ambient space.

Our question

We wanted *realcompact* examples, because these are complementary to pseudocompact spaces in the sense that:

$$\text{compact} = \text{pseudocompact} + \text{realcompact}$$

There are spaces that are neither pseudocompact nor realcompact, just watch our initial attempts . . .

A difficulty

An indication that finding examples is not completely trivial is

Theorem

Assume X is realcompact (but not compact) and A is a subset of X whose closure is not compact. Then A contains a copy of $\mathbb{N}$ that is closed and C-embedded in X.

Proof.

Take $a \in \overline{A} \cap (\beta X \setminus X)$. By realcompactness there is a continuous $f : \beta X \to \mathbb{R}$ such that $f(a) = 0$ and $f(x) > 0$ whenever $x \in X$.
Take a sequence $\langle x_n : n \in \omega \rangle$ in A such that $2^{-n} > f(x_n) > f(x_{n+1})$ for all n.
Then the set $\{x_n : n \in \omega\}$ is closed, discrete, and C-embedded in X. □

So, realcompact non-compact spaces are rife with closed C-embedded copies of $\mathbb{N}$.

Planks

Our examples will be variations on the Tychonoff and Dieudonné planks.
Both have $(\omega_1 + 1) \times (\omega_0 + 1) \setminus \{\langle \omega_1, \omega_0 \rangle\}$ as their underlying sets.

The Tychonoff plank $\mathbb{T}$ has the subspace topology of the product of the two *ordered* spaces $\omega_1 + 1$ and $\omega_0 + 1$.

The Dieudonné plank $\mathbb{D}$ has the subspace topology of the product of the two *ordered* spaces $\omega_1 + 1$ and $\omega_0 + 1$, but where each ordinal in ω_1 is made isolated.

Both have a closed copy of $\mathbb{N}$, namely the right-hand side: $R = \{\omega_1\} \times \omega_0$.

And in both cases that copy is not even C^*-embedded.

Why is that?

Planks

Take the map $f : R \to [0, 1]$ given by $f(\omega_1, n) = n \bmod 2$.

If g were a continuous extension of f then there would be an ordinal α in ω_1 such that $g(\beta n) = f(\omega_1, n)$ for all n and all $\beta \geqslant \alpha$.

But then g would not be continuous at $\langle \beta, \omega_0 \rangle$ for all $\beta \geqslant \alpha$.

Another plank

A third variation: $\mathbb{J}$ has the same underlying set but now $\omega_1 + 1$ is the one-point compactification of the discrete space ω_1.

Again, no success:

- R is not C^*-embedded (same argument as before),
- $\mathbb{J}$ is not realcompact (no infinite subset of R is C^*-embedded)
- $\mathbb{J}$ is not pseudocompact either ($\bigl\{\langle n, n\rangle : n \in \omega\bigr\}$ is clopen and discrete)

Now what?

A variation

We blow up the top line:
take the product $X = (\omega_1 + 1) \times \beta\omega_0$ and the plank $P = X \setminus (\{\omega_1\} \times \omega_0^*)$.

Some success: R is now C^*-embedded.
Given $f : R \to [0, 1]$ extend it in the obvious way to all of $(\omega_1 + 1) \times \omega$.
And the extend it vertically using that we have $\beta\omega_0$ on each vertical line.

Some failure: no infinite subset of R is C-embedded, so this plank is not realcompact.

It is also not pseudocompact because it maps (perfectly) onto $\mathbb{J}$.

So, at least we have improved upon Katětov's example by removing pseudocompactness.

Another variation

Time for drastic measures: $\Pi = \beta\omega_1 \times \beta\omega_0$ and $\mathbb{V} = \Pi \setminus (\omega_1^* \times \omega_0^*)$.

Now R is much thicker: $R = \omega_1^* \times \omega$.

But it is C^*-embedded: given $f : R \to [0,1]$ extend it over every horizontal line $\beta\omega_1 \times \{n\}$ and then vertically again.

And, ..., R is not C-embedded: the obvious map $f : R \to \mathbb{R}$, given by $f(u,n) = n$ for $u \in \omega_1^*$ cannot be extended.
For an extension g we would have an α such that $g(\beta, n) = n$ for all n and all $\beta \geqslant \alpha$.

And, ..., $\mathbb{V}$ is realcompact: with a bit of work, considering a few cases, you can show that every zero-set ultrafilter with the countable intersection property converges.

But, every closed copy of $\mathbb{N}$ that is C^*-embedded is also C-embedded.
So, ...

Yet Another variation

. . . , time for one more variation.
Actually, we drop ω_1 and ω_0 and work with totally different sets, but the space will still be plank-like.

Horizontally we take $\mathfrak{C} = 2^\omega \cup \{\infty\}$, where each point of 2^ω is isolated and ∞ has co-countable neighbourhoods.

Vertically we start with the binary tree $D = 2^{<\omega}$ with the discrete topology and compactify it by laying the Cantor set 2^ω on top of it, so $cD = 2^{\leqslant\omega}$.
A basic neighbourhood of $x \in 2^\omega$ is of the form

$$U(x, n) = \{s \in cD : x \restriction n \subseteq s\}$$

We let $e : \beta D \to cD$ be the extension of the identity map.

Bear with me, . . .

Yet Another variation

We use e to partition D^* into closed sets: for $x \in 2^\omega$ we let $K_x = \{u \in D^* : e(u) = x\}$

We take the following subspace of the product $\mathfrak{C} \times \beta D$:

$$\mathbb{A} = (\mathfrak{C} \times D) \cup \bigcup_{x \in 2^\omega} \{x\} \times K_x$$

$R = \{\infty\} \times D$ is now a closed copy of $\mathbb{N}$.

R is C^*-embedded: just like before, the vertical lines are now subsets of βD so we can still extend vertically.

Yet Another variation

R is not C-embedded: define $f(\infty, s) = |s|$.
As before, given a potential extension g we have a co-countable subset B of 2^ω such that $g(x, s) = f(\infty, s)$ for all $x \in B$ and all $s \in D$.
And $g \restriction \big(\{x\} \times D\big)$ would not be extendible to the points of $\{x\} \times K_x$ (for $x \in B$).

Yet Another variation

The realcompactness of $\mathbb{A}$ is now the trickiest bit.

The top line $T = \bigcup_{x \in 2^\omega} \{x\} \times K_x$ is a zero-set (use $f(x,s) = 2^{-|s|}$).
The horizontal lines $\mathfrak{C} \times \{s\}$ are clopen, hence zero-sets.

A zero-set ultrafilter z with the countable intersection property picks out one of them.

If it picks $\mathfrak{C} \times \{s\}$ then it is elementary to show that z converges.

If it picks the top line T then every clopen subset C of the Cantor set determines a clopen subset of T, and hence a zero-set: $T_C = \bigcup_{x \in C} \{x\} \times K_x$.
So z picks T_C or its complement, for every C. But there are (only) countably many clopen sets in the Cantor set and all the choices that z makes intersect down to one of the compact sets $\{x\} \times K_x$, which then is in z.
We find that z converges, because K_x is compact.

Yet Another variation

Our right-hand side R should have many infinite C-embedded subsets.

Indeed, we can point out a few:
Every branch in R is C-embedded, as is every infinite antichain.

Light Reading

$\mathbb{A}$lan Dow, $\mathbb{J}$an van Mill, Klaas Pieter Hart, Hans $\mathbb{V}$ermeer
Some Realcompact spaces, arXiv:2211.16545 [math.GN]