2022-06-14

TWO PROBLEMS ONE MODEL.

1. CUT POINTS IN STANDARD SUBCONTINUA OF $\beta\mathbb{R}\setminus\mathbb{R}$.

IN $\mathbb{R}$ TAKE INTERVALS $[a_n, b_n]$ FOR $n\in\omega$
SUCH THAT $a_n < b_n < a_{n+1}$ ALWAYS.

LET $u \in \omega^*$ BE A FREE ULTRAFILTER

$$[a_u, b_u] = \bigcap_{U\in u} \overline{\bigcup_{n\in U} [a_n, b_n]}$$

IS A STANDARD SUBCONTINUUM OF $\beta\mathbb{R}\setminus\mathbb{R}$.

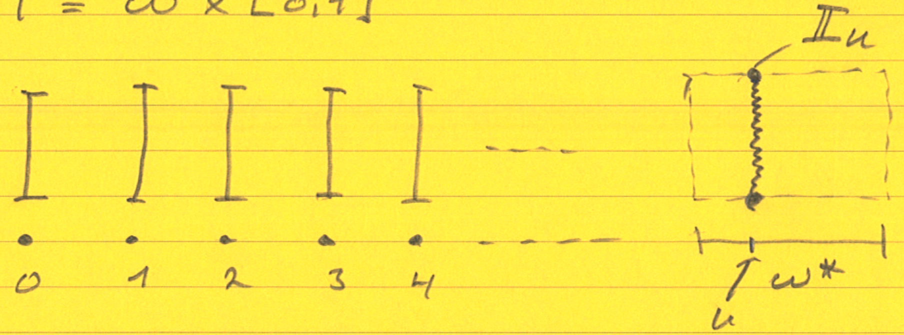
IT HAS CUT-POINTS: LET $x_n \in (a_n, b_n)$ $(n\in\omega)$

EASY TO CHECK $[a_u, b_u] = [a_u, x_u] \cup [x_u, b_u]$
$\{x_u\} = [a_u, x_u] \cap [x_u, b_u]$.
SO $[a_u, b_u]\setminus\{x_u\}$ IS NOT CONNECTED.

QUESTION: ANY OTHERS?

ANSWER: THAT DEPENDS ...

FOR CONVENIENCE WORK IN $\beta\mathbb{M}$, WHERE
$\mathbb{M} = \omega \times [0,1]$



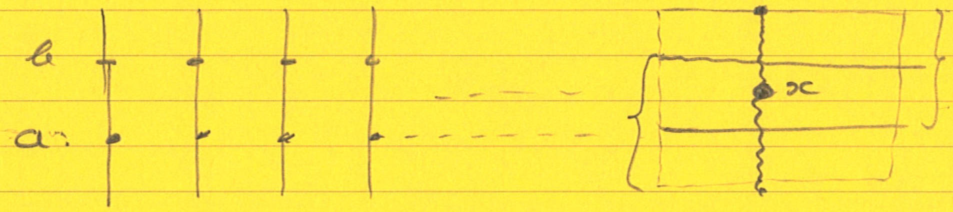
- $\beta\mathbb{M}\setminus\mathbb{M}$ AND HENCE $\mathbb{I}_u$ IS AN F-SPACE
SO IF $\langle z_n : n\in\omega\rangle$ IS AN INCREASING
SEQUENCE OF CUTPOINTS THEN
$\overline{\{z_n : n\in\omega\}} = \beta\omega$
THE 'SUPREMUM' OF $\{z_n : n\in\omega\}$ IS AN
INDECOMPOSABLE CONTINUUM
SO THERE ARE MANY NON-CUTPOINTS IN $\mathbb{I}_u$

• How to recognize cut points?

Let $x \in \mathbb{I}_u$

Define $A_x = \{\langle a_n\rangle_n : x \in \overline{\bigcup_{n\in U}\{n\}\times[a_n,1]}\}$

$B_x = \{\langle b_n\rangle_n : x \in \overline{\bigcup_{n\in U}\{n\}\times[0,b_n]}\}$



Note if x is of the form x_u for some $\langle x_n\rangle_n$

then $\langle x_n\rangle_m \in A_x \cap B_x$

otherwise $A_x \cap B_x = \emptyset$.

So for other cut points x we will have $A_x \cap B_x = \emptyset$.

Well: if $x \in \mathbb{I}_u$ and $A_x \cap B_x = \emptyset$ then TFAE

(1) x is a cut point

(2) if $f : \omega \to \omega$ then there are $a \in A_x$ and $b \in B_x$ such that $\{m : b_m - a_m < 2^{-f(m)}\} \in u$

(3) $\{ \mathrm{Ex}(\bigcup_{n\in U}\{n\}\times(a_n, b_n)) : U \in u, a \in A_x, b \in B_x \}$ is a local base at x

Slightly more set-theoretical:

(2a) if $f : \omega \to \mathbb{N}$ there is a $g : \omega \to \omega$ such that $g(n) < f(n)$ for all n and $\bigcup_n \{n\}\times\left[\frac{g(n)}{f(n)}, \frac{g(n)+1}{f(n)}\right]$ belongs to x (or $\langle \frac{g(n)}{f(n)}\rangle_n \in A_x$ and $\langle \frac{g(n)+1}{f(n)}\rangle_n \in B_x$)

• One more property for non-trivial cut points

If x is a cut point of $\mathbb{I}_u$ and <u>not</u> of the form x_u then x is a far point: if $D \subseteq \mathbb{M}$ is closed and discrete then $x \notin \overline{D}$

WHY: $D \cap \{n\} \times [0,1]$ IS FINITE

LET $f : \omega \to \omega$ BE SUCH THAT IF $\langle n, d\rangle$ AND $\langle n, e\rangle \in D$ THEN $|d - e| > 2^{-f(n)}$

BY (2a): THERE IS A $g < 2^f$ SUCH THAT $x \in \underbrace{\bigcup_n \{n\} \times [g(n) \cdot 2^{-f(n)}, (g(n)+1) 2^{-f(n)}]}_{F}$

BUT $|F \cap D \cap \{n\} \times [0,1]| \leq 1$ FOR ALL n.

SO IT DETERMINES A POINT OF THE FORM z_u (OR NO POINT AT ALL)

BUT $x \neq z_u$ SO $x \notin \overline{D}$

BY INDUCTION: IF D IS SCATTERED AND OF FINITE HEIGHT THEN $x \notin \overline{D}$

CH: THERE IS A CLOSED SET E IN $\mathbb{M}$ SUCH THAT $\overline{E} \cap \mathbb{I}_u$ CONTAINS A NON-TRIVIAL CUT POINT, FOR EVERY u

SO THE GOAL WILL BE: TO SHOW IN THE LAVER MODEL: IF x IS A FAR POINT THEN x DOES NOT SATISFY CONDITION (2a).

THE BOREL CONJECTURE

$X \subseteq \mathbb{R}$ HAS STRONG MEASURE ZERO IF FOR EVERY SEQUENCE $\langle \varepsilon_n : n \in \omega\rangle$ OF POSITIVE REAL NUMBERS THERE IS A SEQUENCE OF INTERVALS $\langle I_n : n \in \omega\rangle$ WITH LENGTH$(I_n) \leq \varepsilon_n$ FOR ALL n AND $X \subseteq \bigcup_{n \in \omega} I_n$.

BOREL: ALL STRONG MEASURE ZERO SETS ARE COUNTABLE.

A LUSIN SET IS UNCOUNTABLE AND HAS STRONG MEASURE ZERO

MAKING IT A BIT MORE SET-THEORETICAL

WORK IN THE CANTOR SET $^{\omega}2$

BASIC OPEN SET $[s] = \{x : s \subseteq x\}$
$s \in {}^{<\omega}2$

IF $s \in {}^{n}2$ THEN $[s]$ GETS MEASURE 2^{-n}.

$X \subseteq {}^{\omega}2$ HAS STRONG MEASURE ZERO IFF
FOR ALL $f \in {}^{\omega}\omega$ THERE IS $g : \omega \to {}^{<\omega}2$
SUCH THAT - $g(n) \in {}^{f(n)}2$ (ALL n)
- $X \subseteq \bigcup\{[g(n)] : n \in \omega\}$

SO WE HAVE TWO QUITE SIMILAR THINGS TO PROVE IN THE LAVER MODEL

1. IF x IS A FAR POINT
THEN THERE IS A FUNCTION $f : \omega \to \mathbb{N}$
SUCH THAT FOR NO $g : \omega \to \omega$
WITH $g(n) < f(n)$ DO WE HAVE
$x \in \overline{\bigcup_{n\in\omega} \{n\} \times [g(n)/f(n), (g(n)+1)/f(n)]}$

2. IF $X \subseteq {}^{\omega}2$ IS UNCOUNTABLE
THEN THERE IS A FUNCTION $f : \omega \to \omega$
SUCH THAT FOR NO $g : \omega \to {}^{<\omega}2$
WITH $g(n) \in {}^{f(n)}2$ DO WE HAVE
$X \subseteq \bigcup_{n\in\omega} [g(n)]$

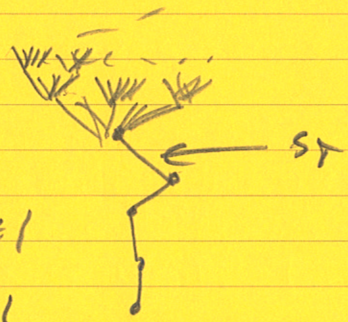
HOW? THE LAVER MODEL IS OBTAINED BY A LENGTH ω_2 ITERATION OF LAVER FORCING $\mathbb{L}$ WITH COUNTABLE SUPPORTS OVER A MODEL OF CH.

SO ALWAYS $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{L}}$ AND SUPPORTS ARE COUNTABLE.
LET $\dot{x}$ AND $\dot{X}$ BE NAMES FOR OUR FAR POINT AND OUR UNCOUNTABLE SET, WLOG $\Vdash |\dot{X}| = \aleph_1$
IF G IS GENERIC ON $\mathbb{P}_{\omega_2}$ THEN THERE IS AN $\alpha < \omega_2$ SUCH THAT
- $X \in V[G{\restriction}\alpha]$
- $\mathrm{VAL}(\dot{x}{\restriction}\alpha, G{\restriction}\alpha)$ IS A FAR POINT OF $\mathbb{M}$ IN $V[G{\restriction}\alpha]$.

NOW WE DEFINE $\mathbb{L}$:

$\mathbb{L}$ CONSISTS OF THE SUBTREES OF ${}^{<\omega}\omega$ OF THE FOLLOWING FORM

SO $T \in \mathbb{L}$ IFF
- THERE IS $s_T \in T$ SUCH THAT $T \subseteq \{t : t \subseteq s_T \vee s_T \subseteq t\}$
- IF $t \in T$ AND $s_T \subseteq t$ THEN $\{n : t^\frown n \in T\}$ IS INFINITE



IF G IS GENERIC ON $\mathbb{L}$ THEN $\bigcap\{[T] : T \in G\}$ CONSISTS OF ONE FUNCTION $f_G = \bigcup\{s_T : T \in G\}$ AND f_G IS VERY FAST GROWING WITH RESPECT TO THE GROUND MODEL.

BACK TO $V[G\restriction\alpha]$:
- THIS IS STILL A MODEL OF CH
- THE EXTENSION $V[G]$ OF $V[G\restriction\alpha]$ IS SIMPLY THE EXTENSION BY AN ω_2 LONG ITERATION OF $\mathbb{L}$ AS DEFINED IN $V[G\restriction\alpha]$
- SO, ..., WHAT IS THE f THAT WE ARE LOOKING FOR?
  THE FIRST LAVER FUNCTION ADDED BY THE REST OF THE ITERATION.

SO WE SIMPLIFY MATTERS AND PROVE
IF
- V SATISFIES CH
- $X \subseteq {}^{\omega}2$ IS UNCOUNTABLE
- $x \in \mathbb{M}^*$ IS A FAR POINT

THEN THE FIRST LAVER REAL f IN THE ITERATION IS AS REQUIRED BY 1 AND 2
THAT IS IN $V[G]$
- IF $g : \omega \to {}^{<\omega}2$ IS SUCH THAT $g(n) \in {}^{f(n)}2$ FOR ALL n THEN $X \nsubseteq \bigcup_{new}[g(n)]$
- IF $g : \omega \to \omega$ IS SUCH THAT $g(n) < f(n)$ FOR ALL n THEN THERE IS AN F IN x SUCH THAT $F \cap \bigcup_{new} \{n\} \times \left[\frac{g(n)}{f(n)}, \frac{g(n)+1}{f(n)}\right] = \emptyset$.