LONG CHAINS IN THE RUDIN-FROLÍK ORDER FOR UNCOUNTABLE CARDINALS

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ABSTRACT. We point out that a construction by Butkovičová of a chain of length $\mathfrak{c}^+$ in the Rudin-Frolík order on $\beta\omega$ can easily be adapted to produce, given an uncountable cardinal κ, a chain of length $(2^\kappa)^+$ in the Rudin-Frolík order on $\beta\kappa$.

INTRODUCTION

The Rudin-Frolík order $\leqslant_{\mathrm{RF}}$ of ultrafilters was defined by Frolík in [4] and used to prove the non-homogeneity of $\beta\omega_0 \setminus \omega_0$. The order is tree-like on $\beta\omega_0 \setminus \omega_0$; the predecessors of an element are linearly ordered and every point has $2^\mathfrak{c}$ successors.

Many natural questions about this order have been answered for $\beta\omega_0 \setminus \omega_0$. For example, every point has at most $\mathfrak{c}$ many predecessors, hence a chain can have cardinality at most $\mathfrak{c}^+$ and in [1] Butkovičová constructed a chain of that maximum possible order type.

The definition of $\leqslant_{\mathrm{RF}}$ was give by Frolík for ultrafilters on arbitrary sets and its stand to reason that one asks whether the results for ultrafilters on ω_0 can be generalized to uncountable cardinals.

It is the purpose of this note to point out that Butkovičová's construction can be used almost verbatim to produce in $\beta\kappa$, where κ is uncountable, an $\leqslant_{\mathrm{RF}}$-chain of length $(2^\kappa)^+$.

1. PRELIMINARIES

1.1. **The Rudin-Frolík order.** Let κ be an infinite cardinal. An indexed set $\{u_\alpha : \alpha \in \kappa\}$ is said to be κ-discrete if there is a partition $\{U_\alpha : \alpha \in \kappa\}$ into sets of cardinality κ such that $U_\alpha \in u_\alpha$ for all α.

In case $\kappa = \omega_0$ this notion coincides with relative discreteness of the set in the Čech-Stone compactification $\beta\omega_0$ because in Hausdorff spaces one can separate the elements of a countable relatively discrete set can be separated by pairwise disjoint open sets.

If $X = \{u_\alpha : \alpha \in \kappa\}$ is κ-discrete then its closure in $\beta\kappa$ is homeomorphic to $\beta\kappa$; the map $f : \alpha \to \beta\kappa$ given by $f(\alpha) = u_\alpha$ induces a homeomorphism between $\beta\kappa$ and the closure of $\{u_\alpha : \alpha \in \kappa\}$. If $u \in \beta\kappa$ then, following Frolík, one usually denotes $\beta f(u)$ by $\sum(X, u)$.

The Rudin-Frolík order $\leqslant_{\mathrm{RF}}$ on $\beta\kappa$ is defined by $u \leqslant_{\mathrm{RF}} v$ iff there is a κ-discrete set X such that $v = \sum(X, u)$. The proof in [5] that $\leqslant_{\mathrm{RF}}$ is a partial order of the types of ultrafilters on ω_0 is readily adapted to other cardinals.

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1.2. **Stratified sets of filters.** The main tool in the construction is that of a stratified set of filters.

Let κ be an infinite cardinal. A stratified set of filters on κ is an $\alpha \times \kappa$-matrix $\langle \mathcal{F}_{\beta,\eta} : \langle \beta, \eta \rangle \in \alpha \times \kappa \rangle$ of filters on κ, where α is an ordinal, such that

- there is a choice $F_{\beta,\eta} \in \mathcal{F}_{\beta,\eta}$ of elements such that for every $\beta \in \alpha$ the family $\{F_{\beta,\eta} : \eta \in \kappa\}$ is pairwise disjoint, and
- if $\beta < \delta < \alpha$, $\eta \in \kappa$ and $F \in \mathcal{F}_{\beta,\eta}$ then $\{\zeta \in \kappa : F \in \mathcal{F}_{\delta,\zeta}\}$ has cardinality κ.

Note that the filters in a stratified set are uniform, except possibly for those in the last row, if α is a successor. For if $F \in \mathcal{F}_{\beta,\eta}$ and $\beta + 1 < \alpha$ then F intersects κ many of the pairwise disjoint sets $F_{\beta+1,\zeta}$, so it has cardinality κ.

In our construction it will always be the case that if $\alpha = \beta + 1$ the filters $\mathcal{F}_{\beta,\zeta}$ are all uniform.

1.3. **Independent matrix.** A secondary tool in the construction is that of an independent matrix of sets, that is, a matrix $\langle A_{\xi,\eta} : \langle \xi, \eta \rangle \in 2^\kappa \times \kappa \rangle$ of subsets of κ such that

(1) for each ξ the family $\{A_{\xi,\eta} : \eta \in \kappa\}$ is a partition of κ into sets of cardinality κ, and
(2) for each finite function $p \subset 2^\kappa \times \kappa$ the intersection $\bigcap_{\xi \in \operatorname{dom} p} A_{\xi,p(\xi)}$ has cardinality κ.

For a construction of such a family see [3, Theorem 3].

2. The main result

As in [1] we construct a triangular array of κ-discrete families of ultrafilters. That is, an array

$$\left\langle \mathfrak{U}_{\alpha,\beta} : \beta < \alpha < (2^\kappa)^+ \right\rangle$$

where each $\mathfrak{U}_{\alpha,\beta}$ is an indexed κ-discrete family $\langle u(\alpha,\beta,\eta) : \eta \in \kappa \rangle$ of ultrafilters on κ.

The demands on this array are that for every $\alpha < (2^\kappa)^+$

(1) the row $\langle \mathfrak{U}_{\alpha,\beta} : \beta < \alpha \rangle$ is a stratified family of ultrafilters, and
(2) if $\beta < \delta < \alpha$ and $\eta \in \kappa$ then

$$u(\alpha,\beta,\eta) = \sum \bigl(\mathfrak{U}_{\alpha,\delta}, u(\delta,\beta,\eta)\bigr)$$

This will yield very many chains of length $(2^\kappa)^+$ in the Rudin-Frolík order on $\beta\kappa$. Indeed, for each fixed pair $\langle \beta, \eta \rangle$ condition (2) above shows that $u(\delta,\beta,\eta) <_{\mathrm{RF}} u(\alpha,\beta,\eta)$ whenever $\beta < \delta < \alpha$ so that the sequence

$$\left\langle u(\alpha,\beta,\eta) : \beta < \alpha < (2^\kappa)^+ \right\rangle$$

is $<_{\mathrm{RF}}$-increasing.

The construction. We construct our array recursively, one row at a time. At the same time we construct an array

$$\left\langle \mathfrak{A}_{\alpha,\beta} : \beta < \alpha < (2^\kappa)^+ \right\rangle$$

of partitions, where $\mathfrak{A}_{\alpha,\beta} = \{U(\alpha,\beta,\eta) : \eta \in \kappa\}$ is a row in the independent matrix $\langle A_{\xi,\eta} : \langle \xi, \eta \rangle \in 2^\kappa \times \kappa \rangle$ from subsection 1.3. We will always have $U(\alpha,\beta,\eta) \in u(\alpha,\beta,\eta)$, so $\mathfrak{A}_{\alpha,\beta}$ witnesses the κ-discreteness of $\mathfrak{U}_{\alpha,\beta}$.

To begin the construction we take the first row $\{A_{0,\eta} : \eta \in \kappa\}$ from our matrix, so $U(1,0,\eta) = A_{0,\eta}$, and we choose, for each η, a uniform ultrafilter $u(1,0,\eta)$ such that $U(1,0,\eta) \in u(1,0,\eta)$. Then the set $\mathfrak{U}_{1,0} = \{u(1,0,\eta) : \eta \in \kappa\}$ is κ-discrete, and this one-element row is stratified, vacuously.

Now assume that $\alpha \in (2^\kappa)^+$ is given and that we have an array

$$\langle \mathfrak{U}_{\beta,\gamma} : \gamma < \beta < \alpha \rangle$$

that meets requirements (1) and (2) up to α. So each row $\langle \mathfrak{U}_{\beta,\gamma} : \gamma < \beta \rangle$ is stratified and we have

$$u(\beta, \gamma, \eta) = \sum \bigl(\mathfrak{U}_{\beta,\delta}, u(\delta, \gamma, \eta) \bigr)$$

whenever $\gamma < \delta < \beta$ and $\eta \in \kappa$.

We construct the αth row.

Since $\alpha < (2^\kappa)^+$ we can take an injective map $i : \alpha \to 2^\kappa$ and make an independent matrix $\langle U(\alpha, \beta, \eta) : \langle \beta, \eta \rangle \in \alpha \times \kappa \rangle$ by setting $U(\alpha, \beta, \eta) = A_{i(\xi),\eta}$ for all ξ and η. This then also gives us the partitions $\mathfrak{A}_{\alpha,\beta}$ for $\beta < \alpha$.

Using this matrix, and the ultrafilters constructed thus far, we construct a stratified family

$$\langle \mathcal{F}_{\beta,\eta} : \langle \beta, \eta \rangle \in \alpha \times \kappa \rangle$$

of filters, as follows.

Let $\langle \beta, \eta \rangle \in \alpha \times \kappa$. We let $\mathcal{F}_{\beta,\eta}$ be the filter generated by the union of the following families:

1. the Fréchet filter $\{F \subseteq \kappa : |\kappa \setminus X| < \kappa\}$,
2. for $\gamma < \beta$ the singleton set $\{U(\alpha, \gamma, \zeta)\}$, where ζ is such that $\eta \in U(\beta, \gamma, \zeta)$,
3. the singleton set $\{U(\alpha, \beta, \eta)\}$,
4. for $\gamma \in (\beta, \alpha)$ the family $\bigl\{ \bigcup_{\zeta \in U} U(\alpha, \gamma, \zeta) : U \in u(\gamma, \beta, \eta) \bigr\}$

Using the fact that $\bigl\langle U(\alpha, \gamma, \zeta) : \langle \gamma, \zeta \rangle \in \alpha \times \kappa \bigr\rangle$ is an independent matrix one readily checks that the union of the families given above has the finite intersection property and that $\mathcal{F}_{\beta,\eta}$ is indeed a uniform filter. It remains to show that the resulting family is stratified.

For this we use that the formulas above are used at every stage of the construction and hence that (1)–(4) hold for all triples $\gamma < \beta < \delta$ of ordinals below α.

So let $\langle \beta, \eta \rangle$ and $\delta \in (\beta, \alpha)$ be given. We calculate for every element G of the generating family of $\mathcal{F}_{\beta,\eta}$ the set $X_G = \{\zeta : G \in \mathcal{F}_{\delta,\zeta}\}$ and show that it belongs to $u(\delta, \beta, \eta)$. This will show that $X_G \in u(\delta, \beta, \eta)$ for all G in $\mathcal{F}_{\beta,\eta}$, and hence that all these sets have cardinality κ.

1. If G belongs to the Fréchet filter then $X_G = \kappa$ because all filters are uniform.
2. If $G = U(\alpha, \gamma, \xi)$, with $\gamma < \beta$, then $G \in \mathcal{F}_{\delta,\zeta}$ iff $\zeta \in U(\delta, \gamma, \xi)$ and so $X_G = U(\delta, \gamma, \xi)$. But since $U(\alpha, \gamma, \xi) \in \mathcal{F}_{\beta,\eta}$ we also have $\eta \in U(\beta, \gamma, \xi)$, which means that during the construction of row δ we ensured via (2) that $U(\delta, \gamma, \xi) \in u(\delta, \beta, \eta)$, that is, $X_G \in u(\delta, \beta, \eta)$.
3. If $G = U(\alpha, \beta, \eta)$ then $G \in \mathcal{F}_{\delta,\zeta}$ iff $\zeta \in U(\delta, \beta, \eta)$, so $X_G = U(\delta, \beta, \eta)$, and $X_G \in u(\delta, \beta, \eta)$.
4. If $G = \bigcup_{\xi \in U} U(\alpha, \gamma, \xi)$ with $\beta < \gamma < \delta$ and $U \in u(\gamma, \beta, \eta)$, then $G \in \mathcal{F}_{\delta,\zeta}$ iff $\zeta \in U(\delta, \gamma, \xi)$ for some ξ in U, and so $X_G = \bigcup_{\xi \in U} U(\delta, \gamma, \xi)$ and so $X_G \in \sum \bigl(\mathfrak{U}_{\delta,\gamma}, u(\gamma, \beta, \eta) \bigr) = u(\delta, \beta, \eta)$.
5. If $G = \bigcup_{\xi \in U} U(\alpha, \delta, \xi)$ for some $U \in u(\delta, \beta, \eta)$ then $X_G = U$ because if $\zeta \in \kappa$ then $G \in \mathcal{F}_{\delta,\zeta}$ iff $G \supseteq U(\alpha, \delta, \zeta)$ iff $\zeta \in U$.
6. If $G = \bigcup_{\xi \in U} U(\alpha, \gamma, \xi)$ with $\delta < \gamma$ and $U \in u(\gamma, \beta, \eta)$ then, because $u(\gamma, \beta, \eta) = \sum \bigl(\mathfrak{U}_{\gamma,\delta}, u(\delta, \beta, \eta) \bigr)$, we have $\{\zeta : U \in u(\gamma, \delta, \zeta)\} \in u(\delta, \beta, \eta)$. But the definition of the $\mathcal{F}_{\delta,\zeta}$ then implies that $X_G = \{\zeta : U \in u(\gamma, \delta, \zeta)\}$ and so $X_G \in u(\delta, \beta, \eta)$.

In the next section we will show how to find a stratified set $\langle u(\alpha, \beta, \eta) : \langle \beta, \eta \rangle \in \alpha \times \kappa \rangle$ of ultrafilters such that $\mathcal{F}_{\beta,\eta} \subseteq u(\alpha, \beta, \eta)$ for all $\langle \beta, \eta \rangle$.

We now show that that such a stratified set will satisfy

$$u(\alpha,\beta,\eta) = \sum\bigl(\mathfrak{U}_{\alpha,\delta}, u(\delta,\beta,\eta)\bigr)$$

whenever $\beta < \delta < \alpha$ and $\eta \in \kappa$.

Let $V \in u(\alpha,\beta,\eta)$ and $I = \{\zeta : V \in u(\alpha,\delta,\zeta)\}$. We must show that $I \in u(\delta,\beta,\eta)$.

Let $U \in u(\delta,\beta,\eta)$, then $W = \bigcup_{\xi\in U} U(\alpha,\delta,\xi)$ belongs to $\mathcal{F}_{\beta,\eta}$ and so $V \cap W \in u(\delta,\beta,\eta)$, which then implies that $J = \{\zeta : V \cap W \in u(\alpha,\delta,\zeta)\}$ has cardinality κ. But $J \subseteq I \cap U$, so that $I \cap U$ has cardinality κ. As U was arbitrary this shows that $I \in u(\delta,\beta,\eta)$.

3. From stratified sets of filters to stratified sets of ultrafilters

We are given a stratified set $\langle \mathcal{F}_{\beta,\eta} : \langle\beta,\eta\rangle \in \alpha\times\kappa\rangle$ of filters and we must construct a stratified set $\langle u_{\beta,\eta} : \langle\beta,\eta\rangle \in \alpha\times\kappa\rangle$ of ultrafilters such that $\mathcal{F}_{\beta,\eta} \subseteq u_{\beta,\eta}$ for all $\langle\beta,\eta\rangle$.

The following lemma gives us the successor step in the construction below.

Lemma 3.1. *Let $\langle \mathcal{F}_{\beta,\eta} : \langle\beta,\eta\rangle \in \alpha\times\kappa\rangle$ be a stratified set of filters on κ and let $X \subseteq \kappa$. Then there is a stratified set $\langle \mathcal{G}_{\beta,\eta} : \langle\beta,\eta\rangle \in \alpha\times\kappa\rangle$ of filters such that for all $\langle\beta,\eta\rangle$ we have $\mathcal{F}_{\beta,\eta} \subseteq \mathcal{G}_{\beta,\eta}$, and $X \in \mathcal{G}_{\beta,\eta}$ or $\kappa\setminus X \in \mathcal{G}_{\beta,\eta}$.*

Proof. We start by deciding to which filters we add X and to which we add its complement.

We let $C = \{\langle\beta,\eta\rangle : X \in \mathcal{F}_{\beta,\eta}\}$ and close it off in a certain way: define $C_0 = C$, and, recursively, $C_\xi = \bigcup_{\delta<\xi} C_\delta$ when ξ is a limit, and at successor stages

$$C_{\xi+1} = C_\xi \cup \bigl\{\langle\beta,\eta\rangle : (\exists\varepsilon > \beta)(\exists F \in \mathcal{F}_{\beta,\eta})\bigl(\bigl|\{\langle\varepsilon,\zeta\rangle : F \in \mathcal{F}_{\varepsilon,\zeta}\} \setminus C_\xi\bigr| < \kappa\bigr)\bigr\} \qquad (\dagger)$$

This process stops at κ^+ (or earlier) because $C_{\kappa^++1} = C_{\kappa^+}$. To see this note that if $\langle\beta,\eta\rangle \in C_{\kappa^++1}$, as witnessed by $\varepsilon > \beta$ and $F \in \mathcal{F}_{\beta,\eta}$, then the intersection $\{\langle\varepsilon,\zeta\rangle : F \in \mathcal{F}_{\varepsilon,\zeta}\} \cap C_{\kappa^+}$ is of cardinality κ and hence already a subset of C_ξ for some $\xi < \kappa^+$, which then shows that $\langle\beta,\eta\rangle$ is already in $C_{\xi+1}$. We call C_{κ^+} the closure of C and will denote it by $\overline{C}$.

If $\langle\beta,\eta\rangle \in \overline{C}$ then $G \cap X$ has cardinality κ whenever $G \in \mathcal{F}_{\beta,\eta}$. We prove this by induction on ξ.

If $\langle\beta,\eta\rangle \in C_0$ then $X \in \mathcal{F}_{\beta,\eta}$ and we are done.

Going from ξ to $\xi+1$ let $\langle\beta,\eta\rangle \in C_{\xi+1}\setminus C_\xi$ and take $\varepsilon > \beta$ and $F \in \mathcal{F}_{\beta,\eta}$ such that $\{\langle\varepsilon,\zeta\rangle : F \in \mathcal{F}_{\varepsilon,\zeta}\} \setminus C_\xi$ has cardinality less than κ. Now if $G \in \mathcal{F}_{\beta,\eta}$ then $\{\langle\varepsilon,\zeta\rangle : G \cap F \in \mathcal{F}\varepsilon,\zeta\}$ is a subset of $\{\langle\varepsilon,\zeta\rangle : F \in \mathcal{F}_{\varepsilon,\zeta}\}$ of cardinality κ; therefore its intersection I with C_ξ has cardinality κ. But then, by the inductive assumption, $G \cap F \cap F_{\varepsilon,\zeta} \cap X$ has cardinality κ for all $\langle\varepsilon,\zeta\rangle \in I$, hence certainly $G \cap X$ has cardinality κ.

If $\langle\beta,\eta\rangle \notin \overline{C}$ then, in particular $\langle\beta,\eta\rangle \notin C$, hence $X \notin \mathcal{F}_{\beta,\eta}$ and so every member of $\mathcal{F}_{\beta,\eta}$ intersects $\kappa\setminus X$.

Because the ultrafilters are assumed to be uniform this implies that $F \cap (\kappa\setminus X)$ has cardinality κ for all F in $\mathcal{F}_{\beta,\eta}$.

We let $\mathcal{G}_{\beta,\eta}$ be the filter generated by $\mathcal{F}_{\beta,\eta}\cup\{X\}$ if $\langle\beta,\eta\rangle \in \overline{C}$ and by $\mathcal{F}_{\beta,\eta}\cup\{\kappa\setminus X\}$ otherwise.

We need to show that $\langle \mathcal{G}_{\beta,\eta} : \langle\beta,\eta\rangle \in \alpha\times\kappa\rangle$ is stratified.

If $\langle\beta,\eta\rangle \notin \overline{C}$ then for every $F \in \mathcal{F}_{\beta,\eta}$ and every $\gamma > \beta$ the intersection $\{\langle\gamma,\zeta\rangle : F \in \mathcal{F}_{\gamma,\zeta}\}\setminus\overline{C}$ has cardinality κ, and so $\{\langle\gamma,\zeta\rangle : F\cap(\kappa\setminus X) \in \mathcal{G}_{\gamma,\zeta}\}$ has cardinality κ as well.

Next let $\langle\beta,\eta\rangle \in \overline{C}$. We show that for every $G \in \mathcal{F}_{\beta,\eta}$ and every $\gamma > \beta$ the set $I_{G,\gamma} = \{\langle\gamma,\zeta\rangle \in \overline{C} : G \in \mathcal{F}_{\gamma,\zeta}\}$ has cardinality κ.

Once this is established we see that for $\langle\gamma,\zeta\rangle \in G_{F,\gamma}$ we have $F, X \in \mathcal{G}_{\gamma,\zeta}$ and so hence $F \cap X \in \mathcal{G}_{\gamma,\zeta}$. And since $\mathcal{G}_{\beta,\eta}$ is generated by $\{F \cap X : F \in \mathcal{F}_{\beta,\eta}\}$ this shows that the family is stratified at each element of $\overline{C}$.

We prove the statement induction on ξ that the statement holds for every C_ξ.

If $\langle\beta,\eta\rangle \in C_0$ then $X \in \mathcal{F}_{\beta,\eta}$ and so for every $G \in \mathcal{F}_{\beta,\eta}$ and every $\gamma > \beta$ the set $\{\langle\gamma,\zeta\rangle : G \cap X \in \mathcal{F}_{\gamma,\zeta}\}$ has cardinality κ and is contained in C_0.

Going from ξ to $\xi+1$ let $\langle\beta,\eta\rangle \in C_{\xi+1}$ as witnessed by $\varepsilon > \beta$ and $F \in \mathcal{F}_{\beta,\eta}$. Let $I = \{\langle\varepsilon,\zeta\rangle : F \in \mathcal{F}_{\varepsilon,\zeta}\}$, then $I \setminus C_\xi$ has cardinality less than κ.

Now let $G \in \mathcal{F}_{\beta,\eta}$, then $\{\langle\varepsilon,\zeta\rangle \in I : G \in \mathcal{F}_{\varepsilon,\zeta}\}$ has cardinality, hence so does $I_{G,\varepsilon} \cap C_\xi$.

By the inductive assumption we find that for $\gamma > \varepsilon$ the set $\{\langle\gamma,\zeta\rangle \in C_\xi : G \in \mathcal{F}_{\gamma,\zeta}\}$ has cardinality κ.

By the definition of $C_{\xi+1}$ we find that for γ in the interval (β,ε) the set $\{\langle\gamma,\zeta\rangle : G \cap F \in \mathcal{F}_{\gamma,\zeta}\}$ is a subset of $C_{\xi+1}$, as witnessed by F and ε. □

To create the stratified set of ultrafilters we enumerate the power set of κ as $\langle X_\nu : \nu \in 2^\kappa\rangle$ and recursively build matrices $\langle\mathcal{F}^\nu_{\beta,\eta} : \langle\beta,\eta\rangle \in \alpha\times\kappa\rangle$, where $\langle\mathcal{F}^0_{\beta,\eta} : \langle\beta,\eta\rangle \in \alpha\times\kappa\rangle$ is the given stratified set, each time $\langle\mathcal{F}^{\nu+1}_{\beta,\eta} : \langle\beta,\eta\rangle \in \alpha\times\kappa\rangle$ is obtained by applying Lemma 3.1 to $\langle\mathcal{F}^\nu_{\beta,\eta} : \langle\beta,\eta\rangle \in \alpha\times\kappa\rangle$ and X_ν, and at limit stages $\mathcal{F}^\nu_{\beta,\eta} = \bigcup_{\mu<\nu}\mathcal{F}^\mu_{\beta,\eta}$ for all $\langle\beta,\eta\rangle$.

Then we can let $u_{\beta,\eta} = \mathcal{F}^{2^\kappa}_{\beta,\eta}$ for all $\langle\beta,\eta\rangle$.

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