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ČECH-STONE REMAINDERS OF SPACES THAT LOOK LIKE $[0,\infty)$

ALAN DOW AND KLAAS PIETER HART

Abstract. We show that many spaces that look like the half line $\mathbb{H}=[0,\infty)$ have, under CH, a Čech-Stone-remainder that is homeomorphic to $\mathbb{H}^*$. We also show that CH is equivalent to the statement that all standard subcontinua of $\mathbb{H}^*$ are homeomorphic.

The proofs use Model-theoretic tools methods like reduced products and elementary equivalence.

Introduction

The purpose of this note is to answer (partially) some natural questions about the Čech-Stone remainder of the real line or rather the remainder of the space $\mathbb{H}=[0,\infty)$ as the remainder of $\mathbb{R}$ is just a sum of two copies of $\mathbb{H}^*$.

Our first result says that, under CH, the space $\mathbb{H}^*$ is, to a certain extent, unique: if X is a space that looks a bit like $\mathbb{H}$ then X^* and $\mathbb{H}^*$ are homeomorphic. To 'look a bit like $\mathbb{H}$' the space X must be a connected ordered space with a first element, without last element, of countable cofinality and of weight at most $\mathfrak{c}$. The weight restriction is necessary, because if the weight of X is larger than $\mathfrak{c}$ then so is the weight of X^* and therefore X^* cannot be homeomorphic with $\mathbb{H}^*$.

As a consequence various familiar connected ordered spaces have a Čech-Stone remainder that is homeomorphic to $\mathbb{H}^*$. So the remainders of the lexicographic ordered square (minus the vertical line on the right) and of any Suslin line are homeomorphic to $\mathbb{H}^*$.

The second result is concerned with the so-called standard subcontinua of $\mathbb{H}^*$: take a discrete sequence $\langle I_n\rangle_n$ of closed intervals in $\mathbb{H}$ and put, for any $u\in\omega^*$, $I_u=\bigcap U\in u\,\mathrm{cl}\left(\bigcup_{n\in U} I_n\right)$. Then I_u is a standard subcontinuum of $\mathbb{H}^*$.

We show that CH is equivalent to the statement that all standard subcontinua are homeomorphic. This solves Problem 264 from Hart and van Mill [1990].

Our final result shows that certain subcontinua of the standard subcontinua are homeomorphic to $\mathbb{H}^*$; the precise statement is in Section 3, here suffice it to say that these continua are natural candidates for being homeomorphic to $\mathbb{H}^*$.

As may be expected we shall not directly construct homeomorphisms between the spaces in question—it's too hard to take care of 2^{ω_1} points in ω_1 steps—but we show that the spaces have isomorphic bases for the closed sets (isomorphic as

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lattices). That this works follows from the results of Wallman from [1938], to be described in Section 1 below.

A few words on how we show that the bases are isomorphic as lattices: We implicitly and explicitly use a powerful result from Model Theory which says that under quite general circumstances various structures are isomorphic. In each case the bases are identified as reduced products of families of easily described lattices. The factors of these products are pairwise elementary equivalent and hence so are the products themselves. Furthermore these products satisfy an certain saturation property. The combination of elementary equivalence and this saturation property implies that the lattices are isomorphic. A more detailed explanation can be found in Section 3.

The paper is organized as follows. Section 1 contains some preliminary remarks. In Section 2 we prove the first result, the proof is self-contained (i.e., requires no model theory). In Section 3 we prove the results about the standard subcontinua, here we appeal to standard fact from Model Theory to keep the proofs pleasantly short. The final Section 4 deals with a special case of Theorem 2.1 that can be proved under weaker assumptions.

1. Preliminaries

1.1. **Sums of compact spaces.** We shall be dealing with sums of compact spaces a lot, so it's worthwhile to fix some notation. So let $X = \bigoplus_{n\in\omega} X_n$ be a topological sum of compact spaces; we always take $\bigcup_{n\in\omega}\{n\}\times X_n$ as the underlying set of the space. The map $q : X \to \omega$ defined by $q(n, x) = n$ extends to $\beta q : \beta X \to \beta\omega$. We shall always denote the fiber of $u \in \omega^*$ under the map βq by X_u.

1.2. **The half line.** Our main objects of interest are the half line $\mathbb{H} = [0, \infty)$ and its Čech-Stone remainder $\mathbb{H}^*$. The space $\mathbb{H}^*$ is a quotient of another space—that is somewhat easier to handle—by a very simple map.

Indeed, consider the space $\mathbb{M} = \omega \times \mathbb{I}$—the sum of ω many copies of the unit interval $\mathbb{I}$. The map $\pi_{\mathbb{H}} : \mathbb{M} \to \mathbb{H}$ defined by $\pi_{\mathbb{H}}(n, x) = n + x$ maps $\mathbb{M}$ onto $\mathbb{H}$ and the map $\pi^*_{\mathbb{H}} = \beta\pi_{\mathbb{H}} \restriction \mathbb{M}^*$ maps $\mathbb{M}^*$ onto $\mathbb{H}^*$.

A key point in our proof is to see what kind of identifications are made by $\pi^*_{\mathbb{H}}$, so we take a better look at the components of $\mathbb{M}^*$. Because ω^* is zero-dimensional and because $\mathbb{I}_u$ is connected for every u we know exactly what the components of $\mathbb{M}^*$ are: the sets $\mathbb{I}_u$.

Furthermore, each $\mathbb{I}_u$ has a natural top and bottom: we call the point $0_u = u\text{-}\lim\langle n, 0\rangle$ the bottom point and $1_u = u\text{-}\lim\langle n, 1\rangle$ the top point. The continuum $\mathbb{I}_u$ has many cut points: for every sequence $\langle x_n\rangle_n$ in $(0, 1)$ the point $x_u = u\text{-}\lim\langle n, x_n\rangle$ is a cut point of $\mathbb{I}_u$ and this set of cut points is dense. It follows that $\mathbb{I}_u$ is irreducible between 0_u and 1_u, which means that there is no proper subcontinuum of $\mathbb{I}_u$ that contains 0_u and 1_u.

We can put a preorder on $\mathbb{I}_u$: say $x \leqslant_u y$ iff every subcontinuum of $\mathbb{I}_u$ that contains 0_u and y also contains x. The layer of the point x is the set $\{y : y \leqslant_u x \text{ and } x \leqslant_u y\}$. This order is continuous in the sense that $\{y : y \leqslant_u x\}$ is the closure of $\{y : y <_u x\}$. We shall use this order in the proof of Theorem 3.2.

We turn back to the map $\pi^*_{\mathbb{H}}$; using standard properties of the Čech-Stone compactification one can easily prove the next lemma (if $u \in \omega^*$ then $u + 1$ is the ultrafilter generated by $\{U + 1 : U \in u\}$).

Lemma 1.1. *For every $u \in \omega^*$ the map $\pi_{\mathbb{H}}^*$ identifies the points 1_u and 0_{u+1} and these are the only identifications made.*

The continua $\mathbb{I}_u$ govern most of the structure of $\mathbb{H}^*$; they are known as the standard subcontinua of $\mathbb{H}^*$. More information on $\mathbb{H}^*$ can be found in the survey Hart [1992].

1.3. **Wallman spaces.** As mentioned above we construct the homeomorphisms indirectly via isomorphisms between certain lattices of closed sets of the spaces in question.

This is justified by the results of Wallman from [1938]; Wallman generalized the familiar Stone duality for Boolean algebras and zero-dimensional spaces to a duality for lattices and compact spaces. We briefly describe this 'Wallman duality'.

If L is a lattice then a filter on L is a subset F such that $0 \notin F$, if $x_1, x_2 \in F$ then $x_1 \wedge x_2 \in F$ and if $x_1 \in F$ and $x_1 \leqslant x_2$ then $x_2 \in F$. An ultrafilter on L is just a maximal filter. The set X_L of ultrafilters on L is topologized by taking the family of all sets of the form $x^+ = \{F : x \in F\}$ with $x \in L$ as a base for the closed sets of X_L. The space X_L is always compact, it is Hausdorff iff L satisfies a certain technical condition.

If $\mathcal{B}$ is a base for the closed sets of a compact Hausdorff space then $\mathcal{B}$ satisfies this condition. Thus, $X = X_{\mathcal{B}}$ whenever $\mathcal{B}$ is a (lattice) base for the closed sets of $\mathcal{B}$. It is now easy to see that two compact Hausdorff spaces with isomorphic (lattice) bases for the closed sets are homeomorphic.

2. Remainders of spaces that look like $\mathbb{H}$

This section is devoted to a proof of the result mentioned in the introduction, namely

Theorem 2.1 (CH). *Let X be a connected ordered space with a first element, with no last element, of countable cofinality and of weight $\mathfrak{c}$. Then X^* and $\mathbb{H}^*$ are homeomorphic.*

We shall construct the homeomorphism indirectly, via spaces that are mapped onto $\mathbb{H}^*$ and X^* respectively.

Remember from 1.2 that $\mathbb{H}^*$ is the quotient of $\mathbb{M}^*$ obtained by identifying 1_u and 0_{u+1} for every $u \in \omega^*$ and that the map is called $\pi_{\mathbb{H}}$.

We can construct a similar situation for X^*: take a strictly increasing and cofinal sequence $\langle a_n \rangle_n$ in X with $a_0 = \min X$. For every n let $\mathbb{J}_n = [a_n, a_{n+1}]$ and consider the sum $Y = \bigoplus_n \mathbb{J}_n$. The map $\pi : Y \to X$ defined by $\pi(n,x) = x$ identifies $\langle n, a_{n+1} \rangle$ and $\langle n+1, a_{n+1} \rangle$ for every n.

As in the case for $\mathbb{H}^*$ and $\mathbb{M}^*$ the only identifications made by $\pi^* = \beta\pi \restriction Y^*$ are of $u\text{-}\lim_n \langle n, a_{n+1} \rangle$ and $u\text{-}\lim_n \langle n+1, a_{n+1} \rangle = u+1\text{-}\lim_n \langle n, a_n \rangle$ for every $u \in \omega^*$. In other words, for every $u \in \omega^*$ the top point of $\mathbb{J}_u$ is identified with the bottom point of $\mathbb{J}_{u+1}$. We denote the top point of $\mathbb{J}_u$ by t_u and the bottom point by b_u.

This gives rise to the following lemma.

Lemma 2.2. *If $h : \mathbb{M}^* \to Y^*$ is a homeomorphism that maps $\mathbb{I}_u$ to $\mathbb{J}_u$ and moreover maps 1_u to t_u for every u then h induces a homeomorphism from $\mathbb{H}^*$ onto X^*.*

Proof. The maps $\pi \circ h$ and $\pi_{\mathbb{H}}$ have exactly the same fibers. Both are closed, being continuous between compact spaces, hence quotient mappings. Hence $\mathbb{H}^*$ (the quotient of $\mathbb{M}^*$ by $\pi_{\mathbb{H}}$) and X^* (the quotient of $\mathbb{M}^*$ by $\pi \circ h$) are homeomorphic. □

Our efforts then will be directed towards constructing a homeomorphism between $\mathbb{M}^*$ and Y^* that satisfies the assumptions of Lemma 2.2.

Rather than constructing a homeomorphism we shall construct two bases $\mathcal{B}$ and $\mathcal{C}$ for the closed sets of $\mathbb{M}^*$ and Y^* respectively and an isomorphism between them that will induce the desired homeomorphism.

To construct $\mathcal{B}$ we consider the lattice generated by the closed intervals in $\mathbb{I}$. It is a base for the closed sets of $\mathbb{I}$.

We let $\mathbb{L}_n$ be the corresponding lattice for $\mathbb{I}_n$. The product lattice $\mathbb{L} = \prod_n \mathbb{L}_n$ corresponds in a natural way to a base for the closed sets of $\mathbb{M}$. The reduced product $\mathbb{L}^* = \prod_n \mathbb{L}_n/\mathrm{fin}$—obtained by identifying x and y whenever $\{n : x(n) \neq y(n)\}$ is finite—will then correspond in a natural way to a base for the closed sets of $\mathbb{M}^*$. This will be the base $\mathcal{B}$.

In a similar way we find $\mathcal{C}$: let $\mathbb{K}_n$ be the lattice generated by the closed intervals of $\mathbb{J}_n$ and consider $\mathbb{K} = \prod_n \mathbb{K}_n$ and the reduced product $\mathbb{K}^* = \prod_n \mathbb{K}_n/\mathrm{fin}$. The lattice corresponds to a base $\mathcal{C}$ for the closed sets of Y^*.

Finding an isomorphism between $\mathbb{L}^*$ and $\mathbb{K}^*$ is the same thing as finding a bijection φ between $\mathbb{L}$ and $\mathbb{K}$ such that for all $x, y \in \mathbb{L}$ we have $x \leqslant^* y$ iff $\varphi(x) \leqslant^* \varphi(y)$, where $x \leqslant^* y$ means that $\{n : x_n \leqslant y_n\}$ is cofinite.

To ensure that the induced homeomorphism maps $\mathbb{I}_u$ to X_u for every u, it suffices to ensure that whenever $y = \varphi(x)$ the sets $\{n : x_n = \varnothing\}$ and $\{n : y_n = \varnothing\}$ as well as the sets $\{n : x_n = \mathbb{I}_n\}$ and $\{n : y_n = X_n\}$ differ by a finite set only.

Furthermore, to get $h(0_u) = b_u$ and $h(1_u) = t_u$ for every u we simply map the closed set $b_{\mathbb{M}} = \{\langle n, 0\rangle : n \in \omega\}$ to $b_X = \{\langle n, a_n\rangle : n \in \omega\}$ and the set $t_{\mathbb{M}} = \{\langle n, 1\rangle : n \in \omega\}$ to $t_X = \{\langle n, a_{n+1}\rangle : n \in \omega\}$. We leave it to the reader to check that this will indeed suffice.

We shall construct a bijection φ from $\mathbb{L}$ to $\mathbb{K}$ that satisfies the following conditions:

(α) $\varphi(b_{\mathbb{M}}) = b_X$ and $\varphi(t_{\mathbb{M}}) = t_X$, and

(β) for every x and y in $\mathbb{K}$ there is an $N \in \omega$ such that for every $n \geqslant N$ the sets of endpoints of $\varphi(x)(n)$ and $\varphi(y)(n)$ have the same configuration as the sets of endpoints of $x(n)$ and $y(n)$. By this we mean the following.
   1. The closed sets $x(n)$ and $\varphi(x)(n)$ have the same number of intervals and the families of intervals are similar in that if the ith interval of $x(n)$ consists of one point then so does the ith interval of $\varphi(x)(n)$ and vice versa. The same is demanded of $y(n)$ and $\varphi(y)(n)$.
   2. If $\{a_i : i < k\}$, $\{b_j : j < l\}$, $\{c_i : i < k\}$ and $\{d_j : j < l\}$ are the sets of endpoints of $x(n)$, $y(n)$, $\varphi(x)(n)$ and $\varphi(y)(n)$ respectively (all sets in increasing order) then for all $i < k$ and $j < l$ we have $a_i <,=,> b_j$ iff $c_i <,=,> d_j$.

Condition (α) is one of the demands made at the outset; in combination with (β) it ensures that for example the sets $\{n : x(n) = \mathbb{I}_n\}$ and $\{n : \varphi(x)(n) = X_n\}$ differ by a finite set.

Condition (β) also readily implies that $x \leqslant^* y$ iff $\varphi(x) \leqslant^* \varphi(y)$ for all $x, y \in \mathbb{K}$.

By CH we can construct φ in an induction of length ω_1; but rather than setting up the whole bookkeeping apparatus we show how to perform a typical inductive step. So assume we have a bijection $\varphi : A \to B$ that satisfies (α) and (β), where

A and B are countable subsets of $\mathbb{K}$ and $\mathbb{L}$ respectively with $t_{\mathbb{M}}, b_{\mathbb{M}} \in A$ and $t_X, b_X \in B$.

Let $\langle x_i \rangle_i$ be an enumeration of A and let $y_i = \varphi(x_i)$ for all i. We show how to find $\varphi(x)$ for an arbitrary $x \in \mathbb{K} \setminus A$ (the task of finding $\varphi^{-1}(y)$ for $y \in \mathbb{L} \setminus B$ is essentially the same).

First find an increasing sequence $\langle n_k \rangle_k$ of natural numbers such that whenever $i, j < k$ and $n \geqslant n_k$ the endpoints of $x_i(n)$ and $x_j(n)$ and those of $y_i(n)$ and $y_j(n)$ are in the same configuration. Using the fact that the intervals $\mathbb{I}_n$ and X_n are densely ordered it is now an easy matter to find $y \in \mathbb{L}$ such that the endpoints of $x(n)$ and $x_i(n)$ and those of $y(n)$ and $y_i(n)$ have the same configuration whenever $n_k \leqslant n < n_k + 1$ and $i < k$. We put $\varphi(x) = y$ of course.

This completes the proof of Theorem 2.1.

Remark 2.3. Lemma 2.2 brings up an interesting question. If $h : \mathbb{M}^* \to Y^*$ were just any homeomorphism then it would have to map components of $\mathbb{M}^*$ to components of Y^* and thus would induce a map φ from ω^* to ω^* by $h[\mathbb{I}_u] = \mathbb{J}_{\varphi(u)}$. It is readily seen that φ is an autohomeomorphism of ω^*: Note that the set $C = \{u : h(0_u) = t_{\varphi(u)}\}$ is clopen so that we may change h by first turning the $\mathbb{I}_u$ with $u \in C$ upside-down. But then φ merely mirrors the action of h on the set $\{0_u : u \in \omega^*\}$ and hence it is an autohomeomorphism.

The problem is now to find an autohomeomorphism of $\mathbb{M}^*$ that permutes the $\mathbb{I}_u$ in the same way as φ^{-1} permutes the points of ω^* for then we could simply say: if $\mathbb{M}^*$ and Y^* are homeomorphic then $\mathbb{H}^*$ and X^* are homeomorphic. We formulate this as an explicit question.

Question 2.4. Is there for every autohomeomorphism φ of ω^* an autohomeomorphism h of $\mathbb{M}^*$ such that $h[\mathbb{I}_u] = \mathbb{I}_{\varphi(u)}$ for all $u \in \omega^*$?

3. More homeomorphic continua

The argument given in Section 2 is actually a careful proof of a special case of a general Model-Theoretic result. We shall give a brief sketch of this result and then show how it may be used to show that a few more continua of interest are homeomorphic.

The result says "elementary equivalent and countably saturated models of size ω_1 are isomorphic".

Two models for a theory are said to be elementary equivalent if they satisfy the same sentences, where a sentence is a formula without free variables. This may be rephrased in a more algebraic way; two models A and B are elementary equivalent iff the following holds: if $\{x_1, \dots, x_n\} \subseteq A$ and $\{y_1, \dots, y_n\} \subseteq B$ are such that for every formula φ with n free variables $\varphi(x_1, \dots, x_n)$ holds in A iff $\varphi(y_1, \dots, y_n)$ holds in B then for every formula ψ for which there is an $x \in A$ such that $\psi(x_1, \dots, x_n, x)$ holds there is also a $y \in B$ such that $\psi(y_1, \dots, y_n, y)$ holds (and vice versa of course).

By way of example consider dense linear orders with first and last points. Any two such sets are elementary equivalent: if $F = \{x_1, \dots, x_n\}$ and $G = \{y_1, \dots, y_n\}$ are as in the previous paragraph then we simply know that $x_i \leqslant x_j$ iff $y_i \leqslant y_j$ and x_i is the first (last) element iff y_i is. The conclusion will then be: for every x that is in a certain position with respect to F then there is a y in the same position with respect to G.

A countably saturated model is one in which, loosely speaking, every countable system of equations has a solution iff every finite subsystem of it has a solution.

A countably saturated dense linear order is generally known as an η_1-set: if A and B are countable and $a < b$ for every $a \in A$ and $b \in B$ then there is an x such that $a < x < b$ for all a and b.

The well-known theorem of HAUSDORFF from [1914] that under CH any two η_1-sets of cardinality $\mathfrak{c}$ are isomorphic can now be seen as a special case of the general isomorphism theorem.

The 'typical inductive step' from Section 2 may be modifies that the reduced product modulo the finite sets is countably saturated: we were looking for an element of $\mathbb{E}$ that satisfied the same equations as x and we used the fact that we could always satisfy any finite number of these equations.

We refer to the book CHANG and KEISLER [1977] for the necessary background on Model Theory.

We shall now use this Model-Theoretic approach to show that many more continua are homeomorphic, under CH. As noted in the introduction, the first result solves Problem 264 from HART and VAN MILL [1990].

Theorem 3.1. *The Continuum Hypothesis is equivalent to the statement that all standard subcontinua of $\mathbb{H}^*$ are homeomorphic.*

Proof. One direction was done by DOW in [1984]: under ¬CH there are u and v for which $\mathbb{I}_u$ and $\mathbb{I}_v$ are *not* homeomorphic.

For the other direction we note that we can obtain a base for the closed sets of $\mathbb{I}_u$ simply by taking the ultraproduct $\mathbb{L}/u$. This product is actually an ultrapower because the $\mathbb{L}_n$ are all the same.

The proof is finished by noting that $\mathbb{L}/u$ and $\mathbb{L}/v$ are elementary equivalent (both are elementary equivalent to $\mathbb{L}_0$) and countably saturated (CHANG and KEISLER [1977, Theorem 6.1.1.]); by the general isomorphism theorem the ultrapowers are isomorphic. □

The next result shows that, again under CH, all layers of countable cofinality are homeomorphic. Indeed the following, stronger, theorem is true.

Theorem 3.2 (CH). *Let $\langle a_n \rangle_n$ be an increasing sequence of cut points in some $\mathbb{I}_u$ and let L be the 'supremum' layer for this sequence. Then L is homeomorphic to $\mathbb{H}^*$.*

Proof. To begin we note that, because $\mathbb{I}_u$ is an F-space, the closed interval $[a_0, L]$ is the Čech-Stone compactification of the interval $[a_0, L)$.

We now follow the proof of Theorem 2.1.

Form the intervals $\mathbb{J}_n = [a_n, a_{n+1}]$ and the topological sum $Y = \bigoplus_n \mathbb{J}_n$. The map $\pi : Y \to [a_0, L)$ that identifies $\langle n, a_{n+1} \rangle$ and $\langle n+1, a_{n+1} \rangle$ for every n induces a map from Y^* onto L (the restriction of $\beta\pi$). This map is of the same nature as $\pi_{\mathbb{H}}^*$: it identifies the top point of $\mathbb{J}_u$ and the bottom point of $\mathbb{J}_{u+1}$ for every $u \in \omega^*$.

Our aim is to find a homeomorphism $h : \mathbb{M}^* \to Y^*$ that satisfies the assumptions of Lemma 2.2. We shall do this, again, via an isomorphism between bases for the closed sets of $\mathbb{M}^*$ and Y^* respectively.

We shall use the lattice $\mathbb{E}$ as a base for $\mathbb{M}^*$ and we make a base for Y^* as follows: For each n the interval $\mathbb{J}_n$ is homeomorphic with $\mathbb{I}_u$ and hence it has a base $\mathbb{K}_n$ for the closed sets that is elementary equivalent to $\mathbb{L}_n$. The reduced product $\mathbb{K}^* = \prod_n \mathbb{K}_n$ is then a base for the closed sets of Y^*.

We may now copy the inductive construction of a bijection from $\mathbb{L}$ to $\mathbb{K}$ from the proof of Theorem 2.1. The only difference is that we can no longer rely on the linear order of $\mathbb{J}_n$ when we are constructing the images coordinatewise. Instead we enumerate the countably many formulas from lattice theory with parameters from A and use elementary equivalence to produce for every x a y such that, as n gets bigger, there are more and more formulas that $x(n)$ and $y(n)$ both satisfy or both do not satisfy (for the y's we replace the parameters from A with their images under φ of course). □

Remark 3.3. The lattice $\mathbb{K}_n$ is *not* the lattice generated by the intervals of $\mathbb{J}_n$; indeed, the Wallman space of the latter lattice is a linearly ordered continuum and in fact the continuum that one gets by collapsing the layers of $\mathbb{J}_n$ to points.

4. A special case

Consider the long line L of length $\omega_1 \times \omega$; that is, we take the ordinal $\omega_1 \times \omega$ and stick an open unit interval between α and $\alpha + 1$ for every $\alpha < \omega_1 \times \omega$.

Apparently Eric van Douwen raised the question whether L^* and $\mathbb{H}^*$ could be homeomorphic. Theorem 2.1 implies that the answer is yes, under CH.

A slight modification of the methods in Section 2 will show that the answer is even yes if $\mathfrak{d} = \omega_1$.

Theorem 4.1. *If $\mathfrak{d} = \omega_1$ then L^* and $\mathbb{H}^*$ are homeomorphic.*

Proof. We shall find, of course, a homeomorphism $h : \mathbb{M}^* \to Y^*$ of the familiar kind, where $Y = \omega \times [0, \omega_1]$ and $[0, \omega_1]$ denotes the long segment of length ω_1.

Now, because $\mathfrak{d} = \omega_1$, we may take a sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ of points in $\mathbb{I}^\omega$ with the following properties:

1. For all α and all n we have $0 < x_\alpha(n) < 1$.
2. If $\beta < \alpha$ then $x_\beta <^* x_\alpha$.
3. If x is such that $x(n) < 1$ for all n then there is an α such that $x <^* x_\alpha$.

It is then an easy matter to define a sequence $\langle h_\alpha : \alpha < \omega_1 \rangle$ of homeomorphisms, where

$$h_\alpha : \bigcup_n \{n\} \times [0, x_\alpha(n)] \to \omega \times [0, \alpha],$$

such that

- $h_\alpha\big[\{n\} \times [0, x_\alpha(n)]\big] = \{n\} \times [0, \alpha]$ for all α and n, and
- if $\beta < \alpha$ then h_α extends h_β except on a finite number of vertical lines.

It is then straightforward to check that this sequence induces the desired homeomorphism from $\mathbb{M}^*$ onto Y^*. □

Question 4.2. Is $\mathfrak{d} = \omega_1$ equivalent to the statement that L^* and $\mathbb{H}^*$ are homeomorphic?

We note that $\mathfrak{d} = \omega_1$ iff $\mathbb{M}^*$ and Y^* are homeomorphic; this is so because $\mathfrak{d} = \omega_1$ iff the character of the set of top points of $\mathbb{M}^*$ is ω_1. We have just seen that this implies that $\mathbb{M}^*$ and Y^* are homeomorphic; on the other hand if $\mathbb{M}^*$ and Y^* are homeomorphic then clearly the set of top points of $\mathbb{M}^*$ has character ω_1.

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Department of Mathematics, York University, 4700 Keele Street, North York, Ontario, Canada M3J 1P3
E-mail address: `dowa@nexus.yorku.ca`

Faculty of Technical Mathematics and Informatics, TU Delft, Postbus 5031, 2600 GA Delft, the Netherlands
E-mail address: `wiawkph@dutrun2.tudelft.nl`