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CO-ABSOLUTES OF $U(\omega_1)$

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Abstract. Our main results are: 1) after adding any number of Cohen reals to a model of GCH the space $U(\omega_1)$ is co-absolute with a power of ω_2 and 2) there is a model in which $U(\omega_1)$ is not co-absolute with any product of discrete spaces.

Introduction

This note deals with the density and co-absolutes of the space $U(\omega_1)$ of uniform ultrafilters on ω_1. Recall that in [1976] Baumgartner showed that in every **c.c.c.** extension of a model of $2^{\omega_1} = \omega_2$ the cellularity of $U(\omega_1)$ is still ω_2. In [1982] Miller shows that if κ many Cohen reals, where $\kappa \le 2^{\omega_2}$, are added (to a model of $2^{\omega_1} = \omega_2$) then even the density of $U(\omega_1)$ is still ω_2 (Miller points out that the case $\kappa = \omega_3$ is due to Kunen). We remove Miller's restriction on κ by showing that if any number of Cohen reals are added to a model of $2^{\omega_1} = \omega_2$ the space $U(\omega_1)$ is co-absolute with a product of discrete spaces—just as in the Balcar-Vopěnka Theorem. This allows us to conclude that the density of $U(\omega_1)$ is ω_2, because none of the factors can be larger than ω_2 and the number of factors can be at most 2^{ω_1}. However, we also construct a model in which $U(\omega_1)$ is *not* co-absolute with a product of discrete spaces. In both models the π-weight of $U(\omega_1)$ is 2^{ω_1} and hence by a result of Shelah and Juhász there is a dense subspace of $U(\omega_1)$ whose density is 2^{ω_1}. We show that, if 2^{ω_1} is regular, one can find a point in $U(\omega_1)$ whose type has density 2^{ω_1}. In the final section we discuss some older results and raise a few questions about the density of $U(\omega_1)$.

1. Co-absolutes of $U(\omega_1)$

The Stone space of the regular open algebra of a space is known as the absolute of a space. Two spaces are said to be co-absolute if their respective algebras of regular open sets are isomorphic—i.e., their absolutes are homeomorphic. If $2^{\omega_1} = \omega_2$ then the absolute of $U(\omega_1)$ is very simple to describe by the following result of Balcar and Vopěnka in [1972].

Theorem 1.1. *If $2^{\omega_1} = \omega_2$ then the space $U(\omega_1)$ is co-absolute with the metric space ω_2^ω, moreover there is an $\omega \times \omega_2$-matrix*

$$\langle A_{n,\alpha} : n < \omega, \alpha < \omega_2 \rangle$$

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of subsets of ω_1 such that for every n and every φ in ${}^n\omega_2$ the set $A_\varphi = \bigcap_{k<n} A_{k,\varphi(k)}$ is uncountable and for every uncountable X there is a φ in $\bigcup_n {}^n\omega$ such that $A_\varphi \setminus X$ is countable.

An equivalent formulation of co-absoluteness is that the spaces must have π-bases which form isomorphic structures when ordered by (reverse) set inclusion (hence the 'moreover' in the above theorem). A π-*base* for a topological space is a collection of nonempty open subsets of that space such that every nonempty open set contains an element from that collection (and the π-*weight* is the minimum cardinality of a π-base). The π-base for $U(\omega_1)$ mentioned above consists of clopen subsets of $U(\omega_1)$ (co-absoluteness is not always witnessed by clopen sets). Nonempty clopen subsets of $U(\omega_1)$ can be coded by uncountable subsets of ω_1: if $O \subseteq U(\omega_1)$ is clopen then there is an uncountable $A \subseteq \omega_1$ such that $O = U(\omega_1) \cap \overline{A}$ (closure of A in $\beta\omega_1$). Furthermore if $O = U(\omega_1) \cap \overline{A}$ and $P = U(\omega_1) \cap \overline{B}$ then $O \subseteq P$ if and only if $A \setminus B$ is countable. We shall use these facts freely and call a family $\mathcal{A}$ of uncountable subsets of ω_1 a π-base for $U(\omega_1)$ if for every $X \in [\omega_1]^{\omega_1}$ there is $A \in \mathcal{A}$ such that $A \setminus X$ is countable.

It is apparent that co-absolute compact spaces have the same density; indeed a compact space has density κ if and only if its regular open algebra can be written as the union of κ many centered subsets. To compute the density of $U(\omega_1)$ it is natural to try and find a generalization of the Balcar-Vopěnka result—in fact this is a more important problem. We show that it is consistent with $2^{\omega_1} > \omega_2$ that $U(\omega_1)$ is again co-absolute with a power of ω_2. Such a power has at most 2^{ω_1} factors and therefore its density is ω_2. It will actually be easier to show that $U(\omega_1)$ is co-absolute with the product space $\omega_2^\omega \times 2^\kappa$. It is interesting to note that many products of discrete spaces are co-absolute with one another.

Theorem 1.2. *For any two infinite cardinals κ and λ the following spaces are co-absolute:*

$$\lambda^\omega \times 2^\kappa, \quad \lambda^\omega \times \omega^\kappa \quad \text{and} \quad \lambda^\kappa.$$

Proof. It is well-known that for infinite κ the spaces 2^κ and ω^κ are co-absolute, hence so are $\lambda^\omega \times 2^\kappa$ and $\lambda^\omega \times \omega^\kappa$. We deal with the latter two spaces of our triple. Identify λ^κ with $\lambda^\omega \times \lambda^\kappa$. Forcing with its regular open algebra is equivalent to forcing with the product of $\mathrm{Fn}(\omega, \lambda)$ and $\mathrm{Fn}(\kappa, \lambda)$. (As usual, $\mathrm{Fn}(\alpha, \beta)$ denotes the set of functions into β which have finite domain contained in α.) However, after forcing with $\mathrm{Fn}(\omega, \lambda)$ the ordinal λ is countable and hence $\mathrm{Fn}(\kappa, \lambda)$ is then obviously isomorphic with $\mathrm{Fn}(\kappa, \omega)$. This in turn implies that the products

$$\mathrm{Fn}(\omega, \lambda) \times \mathrm{Fn}(\kappa, \lambda)$$

and

$$\mathrm{Fn}(\omega, \lambda) \times \mathrm{Fn}(\kappa, \omega)$$

are forcing equivalent, i.e., their completions are isomorphic. The completion of the last poset is exactly the regular open algebra of $\lambda^\omega \times \omega^\kappa$. □

For the reader who does not like a forcing proof of a seemingly elementary topological fact we offer the following isomorphism between a base for $X = \lambda^\omega \times \omega^\kappa$ and a π-base for $\lambda^\kappa = \lambda^\omega \times \lambda^\kappa$ (its construction is a translation of the forcing proof): Every element p of $\mathrm{Fn}(\omega, \lambda)$ whose domain is a natural number determines in a canonical way a finite one-to-one function s_p: Let $s_p(0) = p(0)$ and if $i > 0$ let

$s_p(i) = p(l)$, where l is the first element of the domain of p such that $p(l) \notin \{s_p(j) : j < i\}$.

Let us now define a π-base for λ^κ that is isomorphic with the canonical base for $\lambda^\omega \times \omega^\kappa$. As above we write $\lambda^\kappa = \lambda^\omega \times \lambda^\kappa$. For $n < \omega$ and $\alpha < \lambda$ we let $O_{n,\alpha}$ be the basic open set $\{x \in \lambda^\omega \times \lambda^\kappa : x_n = \alpha\}$. For $\beta < \kappa$ and $n < \omega$ we let

$$W_{\beta,n} = \bigcup \{U_{p,\beta,\alpha} : n \in \mathrm{dom}(s_p) \text{ and } s_p(n) = \alpha\},$$

where

$$U_{p,\beta,\alpha} = \{x \in \lambda^\omega \times \lambda^\kappa : x \restriction \omega \text{ extends } p \text{ and } x_\beta = \alpha\}.$$

It is not hard to verify that

$$\{O_{n,\alpha} : n < \omega, \alpha < \lambda\} \cup \{W_{\beta,n} : \beta < \kappa, n < \omega\}$$

is the desired π-base for λ^κ.

The π-base from the Balcar-Vopěnka theorem will account for the factor ω_2^ω in our co-absoluteness result.

Theorem 1.3. *Assume $2^{\omega_1} = \omega_2$ and that $\kappa \geq \omega_3$ satisfies $\kappa^{\omega_1} = \kappa$. Then after adding κ many Cohen reals the space $U(\omega_1)$ is co-absolute with $\omega_2^\omega \times 2^\kappa$.*

Proof. Using the Balcar-Vopěnka theorem we may fix, in V, a π-base $\mathcal{A}$ for $U(\omega_1)$ that is isomorphic with $\mathrm{Fn}(\omega, \omega_2)$. Let G be generic on $\mathrm{Fn}(\kappa, 2)$. We want to find, in $V[G]$, an independent family $\mathcal{E} = \{E^i_\alpha : \alpha < \kappa, i < 2\}$ such that

1. $\mathcal{A}$ and the Boolean subalgebra $\mathbb{B}_\mathcal{E}$ of $\mathcal{P}(\omega_1)$ generated by $\mathcal{E}$ are independent, i.e., the cardinality of $A \cap B$ is ω_1 whenever $A \in \mathcal{A}$ and $B \in \mathbb{B}_\mathcal{E}$ and
2. the Boolean algebra $\mathbb{B}$ generated by $\mathcal{A} \cup \mathcal{E}$ is a π-base for $U(\omega_1)$.

This Boolean algebra will then be isomorphic to the algebra generated by the basic clopen sets of $\omega_2^\omega \times 2^\kappa$.

For the construction we fix, in V, a sequence $\langle f_\alpha : \alpha < \kappa \rangle$ of one-to-one functions from ω_1 to κ such that

(i) the functions are almost disjoint, i.e., if $\alpha \neq \beta$ then there is γ_0 such that $f_\alpha(\gamma) \neq f_\beta(\gamma)$ for $\gamma > \gamma_0$, and
(ii) for every partial one-to-one function g from ω_1 to κ whose domain has size ω_1 there is an α such that $\{\gamma \in \mathrm{dom}(g) : f_\alpha(\gamma) = g(\gamma)\}$ is uncountable.

This is possible by the assumption $\kappa = \kappa^{\omega_1}$.

From the f_α's and the generic set G we construct the sets E^i_α as follows. First we let $f_G = \bigcup G$ and then we put

$$E^i_\alpha = \left\{\gamma \in \omega_1 : f_G\big(f_\alpha(\gamma)\big) = i\right\}, \qquad \alpha < \kappa, i < 2.$$

We verify condition 1. Let $A \in \mathcal{A}$ and $\phi \in \mathrm{Fn}(\kappa, 2)$. We write

$$E_\phi = \bigcap \big\{E^{\phi(\alpha)}_\alpha : \alpha \in \mathrm{dom}(\phi)\big\}.$$

We must show that $A \cap E_\phi$ is uncountable. For this we should show that for every γ the set

$$D_\gamma = \left\{p : \exists \xi > \gamma \left[p \Vdash \xi \in A \cap \dot{E}_\phi\right]\right\}$$

is dense in $\mathrm{Fn}(\kappa, 2)$. To this end we fix ξ_0 such that for all $\xi > \xi_0$ the values $f_\alpha(\xi)$, where $\alpha \in \mathrm{dom}(\phi)$, are distinct (this can be done because of (i)). Now let $\gamma \in \omega_1$ and $p \in \mathrm{Fn}(\kappa, 2)$. Since $\mathrm{dom}(p)$ is finite we may find a $\xi > \xi_0$ such that $\xi \in A$, $\xi > \gamma$ and $f_\alpha(\xi) \notin \mathrm{dom}(p)$ for all $\alpha \in \mathrm{dom}(\phi)$ (the functions f_α are one-to-one).

Now we may extend p to a condition q such that $q\bigl(f_\alpha(\xi)\bigr) = \phi(\alpha)$ for all α. But then $q \in D_\gamma$.

We verify condition 2. Let X be an uncountable subset of ω_1, in $V[G]$. We should find an $A \in \mathcal{A}$ and $\phi \in \mathrm{Fn}(\kappa, 2)$ such that $(A \cap E_\phi) \setminus X$ is countable. In other words we must show that the set

$$D = \Bigl\{p : \exists A \in \mathcal{A}\, \exists \phi \in \mathrm{Fn}(\kappa, 2)\ \bigl[p \Vdash (A \cap \dot{E}_\phi) \setminus \dot{X} \text{ is countable}\bigr]\Bigr\}$$

is dense. Let $p \in \mathrm{Fn}(\kappa, 2)$ and consider the set

$$I = \Bigl\{\gamma : \exists q \le p\ \bigl[q \Vdash \gamma \in \dot{X}\bigr]\Bigr\}.$$

The set I is uncountable since p forces $\dot{X}$ to be uncountable. For every $\gamma \in I$ we pick $q_\gamma \le p$ such that $q_\gamma \Vdash \gamma \in \dot{X}$ and we write $\mathrm{dom}(q_\gamma) = \bigl\{\beta(\gamma, 1), \dots, \beta(\gamma, k_\gamma)\bigr\}$ in increasing order. After some standard reductions we may assume that

- all q_γ are of the same size k,
- their domains form a Δ-system with root x, and
- the functions q_γ are all isomorphic, i.e., the set x is in the same position in $\mathrm{dom}(q_\gamma)$ for every γ and there is one function $\psi : k \to 2$ such that $q_\gamma\bigl(\beta(\gamma, i)\bigr) = \psi(i)$ for every γ and every i.

Observe that all $q_\gamma \restriction x$ are the same function—call it q—and note that $q \le p$. For $i \le k$ we define $g_i : I \to \kappa$ by $g_i(\gamma) = \beta(\gamma, i)$. Using property (ii) k times we can find an uncountable subset I_k of I and ordinals $\alpha_1, \dots, \alpha_k$ such that $g_i \restriction I_k \subseteq f_{\alpha_i}$ for every i. Now take an $A \in \mathcal{A}$ such that $A \subseteq I_k$ and define $\phi \in \mathrm{Fn}(\kappa, 2)$ by $\phi(\alpha_i) = \psi(i)$ and observe that $\phi(\alpha_i) = q_\gamma\bigl(\beta(\gamma, i)\bigr)$ for all γ and i.

We now show that A and ϕ witness the fact that $q \in D$. In fact we can show

$$q \Vdash A \cap \dot{E}_\phi \subseteq \dot{X}.$$

For suppose we have $\gamma \in A$ and $q' \le q$ such that $q' \Vdash \gamma \in \dot{E}_\phi$. This means that if $q' \in G$ then by the definition of E_ϕ we have

$$f_G\bigl(f_{\alpha_i}(\gamma)\bigr) = \phi(\alpha_i),$$

for every $i \le k$. That is, $f_{\alpha_i}(\gamma) \in \mathrm{dom}(q')$ and $q'\bigl(f_{\alpha_i}(\gamma)\bigr) = \phi(\alpha_i)$ for all i. Hence it follows that $q' \le q_\gamma$ and $q' \Vdash \gamma \in \dot{X}$. □

Corollary 1.4. *Assume the* GCH *and let $\kappa \ge \omega_3$ be regular. After adding κ many Cohen reals one obtains a model in which the cellularity and density of $U(\omega_1)$ are both ω_2 but the π-weight of $U(\omega_1)$ is κ.*

Proof. In $V[G]$ one has $2^\omega = 2^{\omega_1} = \kappa$ hence the density of 2^κ is ω. Since the density of ω_2^ω is ω_2 we deduce that the density of $U(\omega_1)$ is ω_2. On the other hand the π-weight of 2^κ is κ and the π-weight of ω_2^ω is ω_2 so the π-weight of $U(\omega_1)$ is κ as well. □

Remark 1.5. The assumption that $\kappa^{\omega_1} = \kappa$ in Theorem 1.3 is mainly for technical convenience. One can still show for arbitrary $\kappa \ge \omega_2$ that $U(\omega_1)$ is co-absolute with a power (not necessarily the κth power) of ω_2. The main difficulty is that we cannot prove that there is a family of functions, f_α, as above—indeed it seems likely that, in general, such a family would not exist. In such a case we are unable to find a suitable π-base consisting of clopen sets. Rather than defining the E^i_α to be the valuations of a single function we would have to take a union of the clopen sets associated with interpretations of families of partial functions (e.g., see

Theorem 2.4). An alternative proof that the density is still ω_2 is to simply add more Cohen reals and check that adding them does not lower the density.

2. Non co-absolutes of $U(\omega_1)$

Inspired by the Balcar-Vopěnka theorem and our Theorem 1.3 both of which say that under certain circumstances the space $U(\omega_1)$ is co-absolute with a power of ω_2 one might be tempted to conjecture that $U(\omega_1)$ is co-absolute with a power of some discrete space. We shall construct a model where this conjecture does not hold.

We begin by describing the poset that we shall iterate. In [1984] Shelah showed that a Cohen real produces a Suslin tree. Therefore we may fix a $\mathrm{Fn}(\omega,2)$-name $\dot{\trianglelefteq}$ of a partial order on ω_1 such that

$$\Vdash_{\mathrm{Fn}(\omega,2)} \dot{\mathbb{S}} = \langle \omega_1, \dot{\trianglelefteq} \rangle \text{ is a Suslin tree.}$$

Our poset is $\mathbb{Q} = \mathrm{Fn}(\omega,2) * \dot{\mathbb{S}}$. We let $\dot{B}$ be a $\mathbb{Q}$-name for the branch of $\dot{\mathbb{S}}$ added by $\mathbb{Q}$; note that $\dot{B}$ will be an uncountable subset of ω_1. Our aim is to show that if G is a generic filter on Q then in $V[G]$ the set B has the following property: For every uncountable subset X of ω_1 from V there is an uncountable subset Y of X, also from V, such that $Y \cap B = \emptyset$.

We begin with a lemma on forcing with Suslin trees.

Lemma 2.1. *Let $\mathbb{S} = \langle \omega_1, \trianglelefteq \rangle$ be a Suslin tree and $\dot{B}$ a name for its generic branch. Then for any uniform ultrafilter u on ω_1*

$$\Vdash_{\mathbb{S}} \exists U \in u\, [U \cap \dot{B} = \emptyset].$$

Proof. Let $p \in \mathbb{S}$ be arbitrary and pick incomparable q_1 and q_2 above p. We set $\mathbb{S}_i = \{r : q_i \trianglelefteq r\}$ for $i = 1, 2$. Note that $\mathbb{S}_1 \cap \mathbb{S}_2 = \emptyset$ and that $q_i \Vdash \dot{B} \subset \mathbb{S}_i$ for $i = 1, 2$. If u is a uniform ultrafilter then one of $\mathbb{S}_1$ or $\mathbb{S}_2$, say $\mathbb{S}_1$ is not in u. Its complement U is in u and q_1 forces $\dot{B}$ to be disjoint from U. $\square$

Corollary 2.2. *If $\mathbb{S}$ and $\dot{B}$ are as above, and X is an uncountable subset of ω_1 then in $V[B]$ there is an uncountable ground model subset of ω_1 that is contained in $X \setminus B$.*

We now show that $\mathbb{Q}$ has the property mentioned above.

Lemma 2.3. *Suppose G is generic on $\mathbb{Q}$ and that X is an uncountable ground model subset of ω_1. Then there is an uncountable ground model subset Y of ω_1 that is contained in $X \setminus U$.*

Proof. Factor G as $G_1 * G_2$, with G_1 generic on $\mathrm{Fn}(\omega,2)$. By the corollary we can find an uncountable subset Y_1 of X, *from* $V[G_1]$, and a p in G_2 such that $p \Vdash_{\mathbb{S}} Y_1 \subseteq X \setminus \dot{B}$.

Now apply the well-known fact about $\mathrm{Fn}(\omega,2)$ that every uncountable subset of ω_1 in $V[G_1]$ contains an uncountable subset from V. (If $p \Vdash$ "$\dot{A}$ is uncountable" then for some q below p the set $\{\alpha : q \Vdash \alpha \in \dot{A}\}$ is uncountable.) $\square$

We summarize: $\mathbb{Q}$ is a **c.c.c.** poset of size ω_1 that adds an uncountable subset B of ω_1 with the following property: If $X \subseteq \omega_1$ is uncountable and from the ground model then there is another uncountable set from the ground model that is contained in $X \setminus B$.

Theorem 2.4. *There is a model in which $U(\omega_1)$ is not co-absolute with a product of discrete spaces.*

Proof. Assume the GCH in V, and set up a finite-support iteration of length ω_3 where every factor is $\mathbb{Q}$. Thus, $\mathbb{P}_0 = \{0\}$, $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}$ for every α and $\mathbb{P}_\alpha$ is the direct limit of $\langle \mathbb{P}_\beta : \beta < \alpha \rangle$ for limit α.

Let $\mathbb{P} = \mathbb{P}_{\omega_3}$, let G be a generic filter on $\mathbb{P}$ and assume that, in $V[G]$, $U(\omega_1)$ is co-absolute with a product of discrete spaces $\prod_i D_i$. We shall show that we may assume that this product is in fact the power $\omega_2^{\omega_3}$. Since $\mathbb{P}$ is **c.c.c.** it follows from Baumgartner's result that $c\bigl(U(\omega_1)\bigr) = \omega_2$; Corollary 2.2 implies $\pi w\bigl(U(\omega_1)\bigr) = \omega_3$. Every D_i has cardinality at most ω_2 because of the cellularity of $U(\omega_1)$. This then implies that the number of factors must be ω_3. Furthermore we use the fact that the cellularity of every non-empty subset of $U(\omega_1)$ is ω_2 to conclude that infinitely many D_i have cardinality ω_2. We now write the product as $\omega_2^\omega \times \prod_i D_i$. This product is co-absolute with $\omega_2^{\omega_3}$ as can be seen by modifying the proof of Theorem 1.2. Therefore, as noted above, we may fix an $\omega_3 \times \omega_2$-matrix $\langle O_{\alpha,\beta} : \alpha < \omega_3, \beta < \omega_2 \rangle$ of open subsets of $U(\omega_1)$ such that:

1. for every finite function $\varphi \in \mathrm{Fn}(\omega_3, \omega_2)$ the set $O_\varphi = \bigcap_{\alpha \in \mathrm{dom}(\varphi)} O_{\alpha,\varphi(\alpha)}$ is nonempty, and
2. if O is a nonempty open subset of $U(\omega_1)$ then there is a $\varphi \in \mathrm{Fn}(\omega_3, \omega_2)$ such that $O_\varphi \subseteq O$.

For every pair $\langle \alpha, \beta \rangle$ we take an almost disjoint family $\{A_{\alpha,\beta,\gamma} : \gamma < \omega_2\}$ of uncountable subsets of ω_1 such that $\bigcup_\gamma A^u_{\alpha,\beta,\gamma}$ is a dense open subset of $O_{\alpha,\beta}$. In fact we may and do replace each $O_{\alpha,\beta}$ by this corresponding dense open subset without losing properties 1 and 2. (For an uncountable set X we abbreviate $U(\omega_1) \cap \overline{X}$ by X^u.)

It follows that we may take, in V, a 3-dimensional array

$$\mathcal{A} = \langle \dot{A}_{\alpha,\beta,\gamma} : \alpha < \omega_3 \text{ and } \beta, \gamma < \omega_2 \rangle$$

of $\mathbb{P}$-names of uncountable subsets of ω_1 that behave as above.

We take κ large enough and an elementary submodel M of $H(\kappa)$ of cardinality ω_2 and such that $M^{\omega_1} \subseteq M$. We demand that $\langle \mathbb{P}_\alpha : \alpha < \omega_3 \rangle$ and $\mathcal{A}$ are in M.

Consider $\delta = M \cap \omega_3$ and the set U added by $\mathbb{Q}$ at step $\delta + 1$. Let $G_\delta = G \cap \mathbb{P}_\delta$.

Using the fact that M is an elementary submodel of $H(\kappa)$ that is closed under ω_1-sequences it is not hard to show that

$$\mathcal{A} \restriction \delta = \langle \dot{A}_{\alpha,\beta,\gamma} : \alpha < \delta \text{ and } \beta, \gamma < \omega_2 \rangle$$

is an array of $\mathbb{P}_\delta$-names and that, in $V[G_\delta]$, the collection

$$\bigl\{ O_\varphi : \varphi \in \mathrm{Fn}(\delta, \omega_2) \bigr\}$$

is a π-base for $U(\omega_1)$. Let us indicate how this last statement can be expressed in terms of the array $\mathcal{A}$ only: If X is uncountable then there is a finite function φ such that $O_\varphi \subseteq X^u$. Now we note that O_φ is the union of all sets of the form

$$A_{\varphi,f} = \bigcap_{\alpha \in \mathrm{dom}(\varphi)} A_{\alpha,\varphi(\alpha),f(\alpha)},$$

where f runs through $\omega_2^{\mathrm{dom}(\varphi)}$. In this way the inclusion "$O_\varphi \subseteq X^u$" becomes a convenient shorthand for: "for every f the difference $A_{\varphi,f} \setminus X$ is countable".

In $V[G]$ we take $\varphi \in \mathrm{Fn}(\omega_3, \omega_2)$ such that $O_\varphi \subseteq U^u$. We reach a contradiction by considering $\varphi' = \varphi \restriction \delta$. We may choose a function $f : \mathrm{dom}(\varphi') \to \omega_2$ such that $A_{\varphi',f}$ is uncountable. The set $A_{\varphi',f}$ is in $V[G_\delta]$ and hence there is an uncountable set Y from $V[G_\delta]$ such that $Y \subseteq A_{\varphi',f} \setminus U$. Choose an extension $\psi \in \mathrm{Fn}(\delta, \omega_2)$ of φ' such that $O_\psi \subseteq Y$. But now we have a finite function $\sigma = \varphi \cup \psi$ whose associated set O_σ is empty: On the one hand $O_\sigma \subseteq O_\varphi \subseteq U^u$ and on the other hand $O_\sigma \subseteq O_\psi \subseteq Y^u$; but $U^u \cap Y^u = \emptyset$.

This contradiction shows that in $V[G]$ the space $U(\omega_1)$ is not co-absolute with a power of a discrete space. □

3. A point in $U(\omega_1)$ whose type has maximum density

For a point p of $U(\omega_1)$ we let $\tau(p) = \{\pi(p) : \pi \in S_{\omega_1}\}$—the type of p (S_{ω_1} denotes the permutation group of ω_1). The purpose of this section is to show that some types can have the maximum possible density, namely 2^{ω_1}.

In fact let κ be a regular cardinal greater than or equal to 2^{ω_1}. We prove the following theorem.

Theorem 3.1. *After adding κ many Cohen reals there is a point p in $U(\omega_1)$ whose type has density $\pi w(U(\omega_1)) = 2^{\omega_1}$ and, even stronger, every subset of $\tau(p)$ of size less than 2^{ω_1} is relatively discrete.*

Proof. In the extension we have $2^{\omega_1} = \kappa = \pi w(U(\omega_1))$, the last equality follows from Theorem 1.3. It remains to find the point p.

Let $\{\pi_\alpha : \alpha < 2^{\omega_1}\}$ enumerate the group S_{ω_1}, and let $\{X_\alpha : \alpha < 2^{\omega_1}\}$ enumerate $[\omega_1]^{\omega_1}$. We construct an increasing sequence $\langle \mathcal{F}_\alpha : \alpha < 2^{\omega_1}\rangle$ of uniform filters on ω_1 such that for every α

1. either X_α or its complement belongs to $\mathcal{F}_\alpha$,
2. the character of $\mathcal{F}_\alpha$ is at most $|\alpha| + \omega_1$, and
3. there is an uncountable subset P_α of ω_1 such that $P_\alpha \in \mathcal{F}_\alpha$ and for $\beta, \gamma < \alpha$: if $\pi_\beta(\mathcal{F}_\alpha) \neq \pi_\gamma(\mathcal{F}_\alpha)$ then $\omega_1 \setminus \pi_\beta[P_\alpha] \in \pi_\gamma(\mathcal{F}_\alpha)$.

So $p = \bigcup_\alpha \mathcal{F}_\alpha$ is a uniform ultrafilter and condition 3 implies that for every α the set $\{\pi_\beta(p) : \beta < \alpha\}$ is relatively discrete: $\{\pi_\beta(p) : \beta < \alpha\} \cap \pi_\beta[P_\alpha^*] = \{\pi_\beta(p)\}$. This and the fact that 2^{ω_1} is regular implies that every small subset of $\tau(p)$ is nowhere dense.

To start the induction we let $\mathcal{F}_0$ be the co-countable filter on ω_1. If we assume for convenience that $X_0 = \omega_1$ then 1 and 2 are satisfied; condition 3 is vacuous for $\alpha = 0$.

At stage α we let $\mathcal{G} = \bigcup_{\beta<\alpha} \mathcal{F}_\beta$. By assumption the character of $\mathcal{G}$ is at most $|\alpha| + \omega_1$.

In order to satisfy condition 3 we first refine the filter $\mathcal{G}$ as follows: Fix some well-ordering in type $|\alpha|$ of the set of pairs $\{\gamma, \beta\}$ with $\gamma, \beta < \alpha$. Assume some refinement $\mathcal{G}'$ of $\mathcal{G}$ is found and consider the next pair $\{\gamma, \beta\}$. If $\mathcal{G}'$ has an element on which π_γ and π_β agree do nothing to $\mathcal{G}'$. Otherwise the set $D = \{\eta : \pi_\gamma(\eta) \neq \pi_\beta(\eta)\}$ has uncountable intersection with every element of $\mathcal{G}'$.

The familiar three-set lemma implies that we can split D into three sets D_1, D_2 and D_3 such that $\pi_\gamma[D_i] \cap \pi_\beta[D_i] = \emptyset$ for each i. One of these sets may be added to $\mathcal{G}'$ so that the resulting filter is uniform. We replace $\mathcal{G}$ by the filter obtained in this way and we note that the character of $\mathcal{G}$ is still at most $|\alpha| + \omega_1$.

The filter $\mathcal{G}$ now has the property that for every pair $\{\gamma, \beta\}$ below α there is an element G of $\mathcal{G}$ such that π_γ and π_β agree on G or else $\pi_\gamma[G] \cap \pi_\beta[G] = \emptyset$.

Since κ is regular we may choose $\delta_\alpha < \kappa$ such that the above discussion takes place in $V[f \restriction \delta_\alpha]$, where f is the generic point of 2^κ.

Consider the subset P_α of ω_1 coded by $f \restriction [\delta_\alpha, \delta_\alpha + \omega_1)$. We want to add the set

$$P_{\gamma,\beta} = \omega_1 \setminus \pi_\gamma^{\leftarrow}\bigl[\pi_\beta[P_\alpha]\bigr]$$

to $\mathcal{G}$, whenever $\pi_\gamma[G] \cap \pi_\beta[G] = \emptyset$ for some $G \in \mathcal{G}$.

Let K be a finite subset of α and $H \in \mathcal{G}$; we must show that

$$H \cap P_\alpha \cap \bigcap \bigl\{P_{\gamma,\beta} : \{\gamma, \beta\} \in [K]^2\bigr\} \tag{$*$}$$

is uncountable. That is, we must find uncountably many η in H for which $\eta \in P_\alpha$ and $\pi_\gamma(\eta) \notin \pi_\beta[P_\alpha]$, whenever $\gamma \neq \beta$ in K.

Upon shrinking H somewhat we may assume that for every $\gamma, \beta \in K$ either π_γ and π_β agree on H or $\pi_\gamma[H] \cap \pi_\beta[H] = \emptyset$ — we may as well assume that the second alternative holds for any two distinct γ and β in K.

We replace, for convenience, the interval $[\delta_\alpha, \delta_\alpha + \omega_1)$ by ω_1. Let $p \in \mathrm{Fn}(\omega_1, 2)$ and let

$$L = \bigcup_{\gamma,\beta \in K} \pi_\gamma^{\leftarrow}\Bigl[\pi_\beta\bigl[\mathrm{dom}(p)\bigr]\Bigr].$$

This is a finite set, so there is $\eta \in H \setminus L$. Note that $\eta \notin \mathrm{dom}(p)$ and that if $\gamma \neq \beta$ then $\eta \neq \pi_\beta^{\leftarrow}\bigl(\pi_\gamma(\eta)\bigr)$ and $\pi_\beta^{\leftarrow}\bigl(\pi_\gamma(\eta)\bigr) \notin \mathrm{dom}(p)$. Hence we may extend p to a condition q such that $q(\eta) = 1$ and $q\Bigl(\pi_\beta^{\leftarrow}\bigl(\pi_\gamma(\eta)\bigr)\Bigr) = 0$; i.e., q forces that η is in the intersection in $(*)$. An obvious density argument now shows that $(*)$ is indeed uncountable.

Finally in this step decide whether X_α or its complement can be added so as to obtain the uniform filter $\mathcal{F}_\alpha$. ☐

Remark 3.2. This gives us, in the Cohen model, explicit examples of dense subsets of $U(\omega_1)$ whose density is $\pi w\bigl(U(\omega_1)\bigr)$ (compare this with the Shelah-Juhász result that for compact spaces the π-weight equals the supremum of densities of dense subsets; Juhász and Shelah [1989]).

4. Some questions and additional remarks

We assume that V satisfies GCH.

Baumgartner's theorem [1976] that in every extension of V by a **c.c.c.** poset the cellularity of $U(\omega_1)$ is ω_2 suggests the question whether the same is true for the density of $U(\omega_1)$.

An interesting fact due to Miller [1982] is that in the extension of V obtained by adding $\aleph_\omega$ Cohen reals the π-weight of $U(\omega_1)$ is at most $\aleph_\omega$ and hence smaller than 2^{ω_1}. Is there a model in which $\pi w\bigl(U(\omega_1)\bigr) < 2^{\omega_1} < \aleph_\omega$?

Clearly the density of a type is at most $\pi w\bigl(U(\omega_1)\bigr)$, hence there need not be one of density 2^{ω_1}. Is there always a type of density $\pi w\bigl(U(\omega_1)\bigr)$? Can there be types of different densities?

Let us also repeat the question implicitly raised by Miller in [1982]: are the density and cellularity of $U(\omega_1)$ equal? A negative answer to the first question would, of course, also be a negative answer to this question.

Remark 4.1. In connection with this last question (and the first one) we note the following consequence of Lemma 2.1: If one forces over V with a Suslin tree then in $V[B]$ the subset of $U(\omega_1)$ of those ultrafilters that extend a ground model ultrafilter is not dense in $U(\omega_1)$, because the clopen set B^u is disjoint from it. This would mean that after forcing with enough Suslin trees one could push the density of $U(\omega_1)$ above ω_2. The problem with this strategy is that, because of the GCH, there will not be enough Suslin trees to keep the iteration going longer than ω_2 steps. This also explains why we used the Cohen reals: they provided us with new Suslin trees.

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