# CROWDED RATIONAL ULTRAFILTERS

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Abstract. We prove that if every family in $({}^\omega\omega, \leq^*)$ of size less than $\mathfrak{c}$ is bounded then there exists a point $p$ in $\mathbb{Q}^*$ such that $p$ generates an ultrafilter in the set-theoretic sense on $\mathbb{Q}$ and such that $p$ has a base consisting of sets that are homeomorphic to $\mathbb{Q}$. This is a partial answer to Question 30 (Problem 229) in [1].

## 1. Gruff ultrafilters

Let $X$ be a metrizable space without isolated points. We shall call a point $p$ of the Čech-Stone remainder $X^*$ *gruff* if it generates an ultrafilter on the set $X$; conversely, an ultrafilter on the set $X$ will be called gruff if it has a base consisting of closed sets of the space $X$. Thus we are able to speak unambiguously about gruff ultrafilters on $X$.

It is easily seen that every point in $X^*$ that contains a discrete set is gruff. On the other hand, there is no gruff remote point, as every gruff ultrafilter contains a nowhere dense set. E. van Douwen in [2] studied the question whether there can exist a gruff ultrafilter which does not contain a scattered set; such an ultrafilter is said to be *crowded*. One of the reasons for this is that such ultrafilters provide examples of particularly nice points of $X^*$ that are totally non-remote: if $p$ is a crowded gruff ultrafilter and if $A \in p$ then there is $B \in p$ such that $B$ is nowhere dense in $A$.

It is not difficult to see that there are no crowded gruff ultrafilters on the real line $\mathbb{R}$: Every closed non-scattered set is of cardinality $\mathfrak{c}$ and so a crowded gruff ultrafilter would be uniform and would therefore be generated by more than $\mathfrak{c}$ sets. However, $\mathbb{R}$ has only $\mathfrak{c}$ closed sets, so no family of closed sets can generate a uniform ultrafilter.

The situation is somewhat different if we consider the space $\mathbb{Q}$ of rational numbers. E. van Douwen proved in [2] that under $\mathsf{CMA}$ (Martin's Axiom for countable posets) there are crowded gruff ultrafilters on $\mathbb{Q}$. We shall show that the existence of gruff ultrafilters on $\mathbb{Q}$ follows from $\mathfrak{b} = \mathfrak{c}$, where $\mathfrak{b}$ is the minimal cardinality of an unbounded subset in $({}^\omega\omega, \leq^*)$. This is of interest because it shows that there are gruff ultrafilters in Laver's model for the Borel Conjecture; $\mathsf{CMA}$ is certainly false in that model.

**Theorem 1.** *If $\mathfrak{b} = \mathfrak{c}$ then there exists a crowded gruff ultrafilter on $\mathbb{Q}$.*

We shall need two lemmas proved by E. van Douwen in [2], albeit in a slightly different form. Let us call a nonempty set without isolated points *crowded*.

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**Lemma 2.** *Every crowded and unbounded subset of $\mathbb{Q}$ has a closed, crowded and unbounded subset.*

*Proof.* Let $F$ be a crowded and unbounded subset of $\mathbb{Q}$. Let $\mathcal{U}$ be a countable clopen base for $\mathbb{Q}$ which is closed under finite unions and consists of bounded sets. Consider the countable poset $\mathbb{P}$ defined by

$$\langle p, U\rangle \in \mathbb{P} \quad \text{iff} \quad p \in [F]^{<\omega},\ U \in \mathcal{U} \text{ and } p \cap U = \emptyset$$

ordered by

$$\langle p, U\rangle \le \langle q, V\rangle \quad \text{iff} \quad p \supseteq q \text{ and } U \supseteq V.$$

Consider

$$\mathcal{D} = \{C_x : x \in \mathbb{Q}\} \cup \{D_n : n \in \omega\} \cup \{E_{x,n} : x \in \mathbb{Q}, n \in \omega\},$$

where $C_x = \bigl\{\langle p, U\rangle \in \mathbb{P} : x \in p \cup U\bigr\}$, $D_n = \bigl\{\langle p, U\rangle \in \mathbb{P} : (\exists x \in p)\, |x| > n\bigr\}$ and $E_{x,n} = \bigl\{\langle p, U\rangle \in \mathbb{P} : x \in p \Rightarrow (\exists y \in p)\, 0 < |x - y| < 2^{-n}\bigr\}$. The family $\mathcal{D}$ is a countable family of dense subsets of the poset $\mathbb{P}$; hence, by the Rasiowa-Sikorski Theorem, there is a filter $G$ on $\mathbb{P}$ that meets them all. Define

$$K = \bigcup\{p : (\exists U \in \mathcal{U})\, \langle p, U\rangle \in G\}$$

and

$$W = \bigcup\{U : \langle \emptyset, U\rangle \in G\}.$$

Clearly $K \subseteq F$ and $K \cap W = \emptyset$. For every $x \in \mathbb{Q}$ we have $G \cap C_x \neq \emptyset$ so $K \cup W = \mathbb{Q}$. It follows that $K$ is closed. It is also easily seen that $K$ is crowded and unbounded. ☐

**Lemma 3.** *Let $\mathcal{F}$ be a free filterbase consisting of closed and crowded sets which extends the filter of co-bounded clopen sets. Define, for $R \subseteq \mathbb{Q}$ and $F \subseteq \mathbb{Q}$,*

$$K_R(F) = \bigcup\{L \subseteq F : L \text{ is crowded and } L \subseteq \overline{L \cap R}\}.$$

*Let $A \subseteq \mathbb{Q}$. Then either for $R = A$ or for $R = \mathbb{Q} \setminus A$ the collection*

$$\mathcal{F}^+ = \mathcal{F} \cup \bigl\{K_R(F) : F \in \mathcal{F}\bigr\}$$

*is a free filterbase consisting of closed, crowded and unbounded sets.*

*Proof.* First we show that for every $F \in \mathcal{F}$ the set $K_R(F)$ is either empty or closed and crowded. Assume $K_R(F)$ is non-empty. Then it is crowded, being a union of crowded sets. It also satisfies $K_R(F) \subseteq \overline{K_R(F) \cap R}$ and hence we have

$$\overline{K_R(F)} \subseteq \overline{K_R(F) \cap R} \subseteq \overline{\overline{K_R(F)} \cap R},$$

so $K_R(F)$ is closed.

Observe that $K_R(F) \subseteq K_R(G)$ if $F \subseteq G$. Now it is easy to see that for every $F \in \mathcal{F}$ there is $R \in \{A, \mathbb{Q} \setminus A\}$ such that $K_R(F)$ is unbounded. For suppose both $K_A(F)$ and $K_{\mathbb{Q}\setminus A}(F)$ are bounded. Let $H \in \mathcal{F}$ be such that $H \subseteq F \setminus \bigl(K_A(F) \cup K_{\mathbb{Q}\setminus A}(F)\bigr)$. Then both $K_A(H)$ and $K_{\mathbb{Q}\setminus A}(H)$ are empty, which is impossible.

Now we show that for either $R = A$ or $R = \mathbb{Q} \setminus A$ the set $K_R(F)$ is unbounded for every $F \in \mathcal{F}$. If it were not true then there are $F, G \in \mathcal{F}$ with $K_A(F)$ and $K_{\mathbb{Q}\setminus A}(G)$ both bounded. Let $H \in \mathcal{F}$ be such that $H \subseteq F \cap G$. Clearly, $K_A(H) \subseteq K_A(F)$ and $K_{\mathbb{Q}\setminus A}(H) \subseteq K_{\mathbb{Q}\setminus A}(G)$, hence $K_A(H)$ and $K_{\mathbb{Q}\setminus A}(H)$ are both bounded, which is a contradiction.

Let $R \in \{A, \mathbb{Q} \setminus A\}$ be such that $K_R(F)$ is closed, crowded and unbounded for every $F \in \mathcal{F}$ and let $\mathcal{F}^+ = \mathcal{F} \cup \{K_R(F) : F \in \mathcal{F}\}$. To show that $\mathcal{F}^+$ is a filterbase it suffices to show that $\{K_R(F) : F \in \mathcal{F}\}$ is a filterbase because $K_R(F) \subseteq F$ for all $F$. But if $\mathcal{F}_0 \in [\mathcal{F}]^{<\omega}$ then there is $G \in \mathcal{F}$ such that $G \subseteq \bigcap \mathcal{F}_0$; then also $K_R(G) \subseteq \bigcap\{K_R(F) : F \in \mathcal{F}_0\}$. $\square$

*Proof of Theorem 1.* Let $\{A_\xi : \xi \in \mathfrak{c}\}$ enumerate $\mathcal{P}(\mathbb{Q})$. By transfinite recursion on $\xi \in \mathfrak{c}$ we shall construct families $\mathcal{F}_\xi \subseteq \mathcal{P}(\mathbb{Q})$ such that for every $\xi, \eta \in \mathfrak{c}$

(i) if $\xi < \eta$ then $\mathcal{F}_\xi \subseteq \mathcal{F}_\eta$,
(ii) $\mathcal{F}_\xi$ is a free filterbase on $\mathbb{Q}$ consisting of closed, crowded and unbounded subsets of $\mathbb{Q}$;
(iii) $\mathcal{F}_\xi$ is of cardinality less than $\mathfrak{c}$, and
(iv) there is $F \in \mathcal{F}_{\xi+1}$ such that $F \subseteq A_\xi$ or $F \cap A_\xi = \emptyset$.

It is easily seen that $\mathcal{F} = \bigcup_{\xi\in\mathfrak{c}} \mathcal{F}_\xi$ is a base of a crowded gruff ultrafilter.

We proceed to the construction. Let

$$\mathcal{F}_0 = \{[n, \infty) : n \in \omega\}.$$

This guarantees that every filter extending $\mathcal{F}_0$ is free and consists of unbounded sets. If $\xi < \mathfrak{c}$ is a limit ordinal we let $\mathcal{F}_\xi = \bigcup_{\eta\in\xi} \mathcal{F}_\eta$; note that $|\mathcal{F}_\xi| < \mathfrak{c}$ because $\mathfrak{c} = \mathfrak{b}$ is regular.

Suppose $\mathcal{F}_\xi$ is a free filterbase consisting of closed, crowded and unbounded subsets of $\mathbb{Q}$ and of cardinality less than $\mathfrak{c}$. We have to decide $A_\xi$. By Lemma 3 there is $R \in \{A_\xi, \mathbb{Q}\setminus A_\xi\}$ such that $\mathcal{F}^+_\xi = \mathcal{F}_\xi \cup \{K_R(F) : F \in \mathcal{F}_\xi\}$ is a free filterbase consisting of closed, crowded and unbounded sets. Enumerate the complement of $R$:

$$\mathbb{Q} \setminus R = \{x_n : n \in \omega\}.$$

For every $F \in \mathcal{F}^+_\xi$ let $\tilde{F}$ be a closed, crowded and unbounded subset of $F \cap R$; such a set exists by Lemma 2 because $K_R(F) = \overline{K_R(F) \cap R} \subseteq \overline{F \cap R}$ and so $F \cap R$ contains a crowded unbounded set. Define $f_F \in {}^\omega\omega$ by

$$f_F(n) = \min\{m \in \omega : (x_n - 2^{-m}, x_n + 2^{-m}) \cap \tilde{F} = \emptyset\}.$$

The set

$$C(f_F) = \mathbb{Q} \setminus \bigcup_{n\in\omega} \bigl(x_n - 2^{-f_F(n)}, x_n + 2^{-f_F(n)}\bigr)$$

is a closed superset of $\tilde{F}$, hence unbounded and not scattered.

Consider the family $\mathcal{E} = \{f_F : F \in \mathcal{F}^+_\xi\}$. Because $\mathfrak{b} = \mathfrak{c}$ and $|\mathcal{E}| < \mathfrak{c}$ the family $\mathcal{E}$ is bounded. Let $g \in {}^\omega\omega$ be such that $g \mathrel{{}^*{\geq}} f_F$ for every $F \in \mathcal{F}^+_\xi$ and let

$$C(g) = \mathbb{Q} \setminus \bigcup_{n\in\omega} \bigl(x_n - 2^{-g(n)}, x_n + 2^{-g(n)}\bigr).$$

We shall show that for every $F \in \mathcal{F}^+_\xi$ the set $C(g) \cap F$ contains a closed, crowded and unbounded set.

Let $F \in \mathcal{F}^+_\xi$. The set $C(f_F) \setminus C(g)$ is bounded; hence there is a clopen bounded set $D$ containing $C(f_F) \setminus C(g)$. Clearly $\tilde{F} \setminus D$ is closed, crowded and unbounded. We also have $\tilde{F} \subseteq C(f_F)$ and $\tilde{F} \subseteq F$, hence $\tilde{F} \setminus D \subseteq C(f_F) \setminus D \subseteq C(g)$ and so $\tilde{F} \setminus D$ is a closed, crowded and unbounded subset of $F \cap C(g)$.

For every $F \in \mathcal{F}_\xi^+$ let $F' \subseteq C(g) \cap F$ be closed and crowded such that the set $C(g) \cap F \setminus F'$ is scattered. The existence of such a set follows from the Cantor-Bendixson Theorem. The family

$$\mathcal{F}_{\xi+1} = \mathcal{F}_\xi^+ \cup \{F' : F \in \mathcal{F}_\xi^+\}$$

is as required. □

## 2. $n$-gruff ultrafilters

Let $n$ be a positive natural number. A point $p$ in $\mathbb{Q}^*$ is said to be *$n$-gruff* if it is the intersection of $n$ ultrafilters on $\mathbb{Q}$.

The existence of crowded $n$-gruff ultrafilters on $\mathbb{Q}$ follows from $\mathsf{CMA}$, as shown by E. van Douwen in [2]. By slightly modifying the proof of Theorem 1 it is not difficult to show that the same can be proved under $\mathfrak{b} = \mathfrak{c}$:

**Theorem 4.** *If $\mathfrak{b} = \mathfrak{c}$ then there exists a crowded $n$-gruff ultrafilter on $\mathbb{Q}$.*

The proof of Theorem 4 is almost identical to that of Theorem 1 so we will indicate only the main differences.

Let $\mathcal{B}$ be a family of subsets of $\mathbb{Q}$. A set $F \subseteq \mathbb{Q}$ is said to be *$\mathcal{B}$-good* if $F \subseteq \overline{F \cap B}$ for every $B \in \mathcal{B}$.

Fix a collection $\mathcal{H}$ of $n$ disjoint dense subsets of $\mathbb{Q}$ such that $\bigcup \mathcal{H} = \mathbb{Q}$. Observe that every $H \in \mathcal{H}$ must be crowded and unbounded.

**Lemma 5.** *Every crowded, unbounded and $\mathcal{H}$-good subset of $\mathbb{Q}$ has a closed, crowded, unbounded and $\mathcal{H}$-good subset.*

*Proof.* The proof is almost the same as the proof of Lemma 2. The only difference is the choosing of the dense subsets $D_n$ and $E_{x,n}$:

$$D_n = \bigl\{\langle p, U\rangle \in \mathbb{P} : (\forall H \in \mathcal{H})\, (\exists x \in p \cap H)\, |x| > n\bigr\}$$

and

$$E_{x,n} = \bigl\{\langle p, U\rangle \in \mathbb{P} : x \in p \Rightarrow (\forall H \in \mathcal{H})\, (\exists y \in p \cap H)\, 0 < |x - y| < 2^{-n}\bigr\}.$$

□

**Lemma 6.** *Let $\mathcal{F}$ be a free filterbase consisting of closed, crowded and $\mathcal{H}$-good sets and which extends the filter of co-bounded clopen sets. Define, for $F \subseteq \mathbb{Q}$, $H_0 \subseteq \mathcal{H}$ and $R \subseteq H_0$,*

$$K_R(F) = \bigcup\{L \subseteq F : L \text{ is crowded and } \mathcal{H}_R - \text{good}\},$$

*where $\mathcal{H}_R = \bigl(\mathcal{H} \setminus \{H_0\}\bigr) \cup \{R\}$. Let $A \subseteq H_0$. Then either for $R = A$ or for $R = H_0 \setminus A$ the collection*

$$\mathcal{F}^+ = \mathcal{F} \cup \bigl\{K_R(F) : F \in \mathcal{F}\bigr\}$$

*is a free filterbase consisting of closed, crowded, unbounded and $\mathcal{H}$-good sets.*

*Proof.* Follow the proof of Lemma 3. It is easily seen that we can also guarantee $\mathcal{H}$-goodness. □

*Proof of Theorem 4.* Fix an enumeration of $\bigcup_{H\in\mathcal{H}}\mathcal{P}(H)$:

$$\bigcup_{H\in\mathcal{H}}\mathcal{P}(H)=\{A_\xi\subseteq\mathbb{Q}:\xi\in\mathfrak{c}\}.$$

By transfinite recursion on $\xi\in\mathfrak{c}$ we construct families $\mathcal{F}_\xi\subseteq\mathcal{P}(\mathbb{Q})$ such that for every $\xi,\eta\in\mathfrak{c}$ they satisfy the conditions (i), (ii), (iii) in the proof of Theorem 1 together with

(iv)* there is $F\in\mathcal{F}_{\xi+1}$ such that $F\cap H\subseteq A_\xi$ or $F\cap A_\xi=\emptyset$, where $H\in\mathcal{H}$ is such that $A_\xi\subseteq H$, and

(v) each $F\in\mathcal{F}_\xi$ is $\mathcal{H}$-good.

The construction is now exactly the same as in the proof of Theorem 1 except that Lemmas 5 and 6 guarantee $\mathcal{H}$-goodness of the elements of the filterbases $\mathcal{F}_\xi$. Also note that (iv)* ensures that the restriction of $\mathcal{F}$ to $H$ generates an ultrafilter on $H$ for each $H\in\mathcal{H}$, and that $\mathcal{F}$ is the intersection of those ultrafilters because $\mathcal{H}$ is a finite partition of $\mathbb{Q}$. □

## References

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