d-18 The Čech–Stone Compactifications of ℕ and ℝ

It is safe to say that among the ***Čech–Stone compactifications*** of individual spaces, that of the space ℕ of natural numbers is the most widely studied. A good candidate for second place is $\beta\mathbb{R}$. This article highlights some of the most striking properties of both compactifications.

1. Description of $\beta\mathbb{N}$ and $\mathbb{N}^*$

The space $\beta\mathbb{N}$ (aka $\beta\omega$) appeared anonymously in [10] as an example of a ***compact Hausdorff*** space without non-trivial converging sequences. The construction went as follows: for every $x \in (0,1)$ let $0.a_1(x)a_2(x)\ldots a_n(x)\ldots$ be its dyadic expansion (favouring the one that ends in zeros). This gives us a countable set $A = \{a_n(x)\colon n \in \mathbb{N}\}$ of points in the ***Tychonoff cube*** $[0,1]^{(0,1)}$. The closure of A is the required space. To see that this closure is indeed $\beta\mathbb{N}$ one checks that the map $n \mapsto a_n$ induces a homeomorphism of $\beta\mathbb{N}$ onto $\operatorname{cl} A$. Indeed, it suffices to observe that whenever X is a coinfinite subset of ℕ one has $a_n(x) = 1$ iff $n \in X$, where $x = \sum_{n\in X} 2^{-n}$.

The present-day description of $\beta\mathbb{N}$ is as the ***Stone space*** of the ***Boolean algebra*** $\mathcal{P}(\mathbb{N})$. Thus the underlying set of $\beta\mathbb{N}$ is the set of all ***ultrafilters*** on ℕ with the family $\{\overline{X}\colon X \subseteq \mathbb{N}\}$ as a base for the open sets, where $\overline{X}$ denotes the set of all ultrafilters of which X is an element. The space $\beta\mathbb{N}$ is a ***separable*** and ***extremally disconnected*** compact Hausdorff space and its cardinality is the maximum possible, i.e., $2^{\mathfrak{c}}$.

Most of the research on $\beta\mathbb{N}$ concentrates on its ***remainder*** $\beta\mathbb{N} \setminus \mathbb{N}$, which, as usual, is denoted $\mathbb{N}^*$. By extension one writes $X^* = \overline{X} \setminus \mathbb{N}$ for subsets X of ℕ. The family $\mathcal{B} = \{X^*\colon X \subseteq \mathbb{N}\}$ is precisely the family of ***clopen*** sets of $\mathbb{N}^*$. Because $X^* \subseteq Y^*$ iff $X \setminus Y$ is finite the algebra $\mathcal{B}$ is isomorphic to the quotient algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$ of $\mathcal{P}(\mathbb{N})$ by the ideal of finite sets – hence $\mathbb{N}^*$ is the Stone space of $\mathcal{P}(\mathbb{N})/\mathit{fin}$. Much topological information about $\mathbb{N}^*$ comes from knowledge of the combinatorial properties of this algebra. In practice one works in $\mathcal{P}(\mathbb{N})$ with all relations taken modulo finite. We use, e.g., $X \subseteq^* Y$ to denote that $X \setminus Y$ is finite, $X \subset^* Y$ to denote that $X \subseteq^* Y$ but not $Y \subseteq^* X$, and so on. In this context the word 'almost' is mostly used in place of 'modulo finite', thus 'A and B are almost disjoint' means $A \cap B =^* \emptyset$.

2. Basic properties of $\mathbb{N}^*$

Many results about $\mathbb{N}^*$ are found by constructing special families of subsets of ℕ, although the actual work is often done on a suitable countable set different from ℕ.

The proof that $\beta\mathbb{N}$ has cardinality $2^{\mathfrak{c}}$ employs an **independent family**, which we define on the countable set $\{\langle n,s\rangle\colon n \in \mathbb{N}, s \subseteq \mathcal{P}(n)\}$. For every subset x of ℕ put $I_x = \{\langle n,s\rangle\colon x \cap n \in s\}$. Now if F and G are finite disjoint subsets of $\mathcal{P}(\mathbb{N})$ then $\bigcap_{x\in F} I_x \setminus \bigcup_{y\in G} I_y$ is non-empty – this is what independent means. Sending $u \in \beta\mathbb{N}$ to the point $p_u \in {}^{\mathcal{P}(\mathbb{N})}2$, defined by $p_u(x) = 1$ iff $I_x \in u$, gives us a continuous map from $\beta\mathbb{N}$ onto ${}^{\mathfrak{c}}2$. Thus the existence of an independent family of size $\mathfrak{c}$ easily implies the well-known fact that the ***Cantor cube*** ${}^{\mathfrak{c}}2$ is separable; conversely, if D is dense in ${}^{\mathfrak{c}}2$ then setting $I_\alpha = \{d \in D\colon d_\alpha = 0\}$ defines an independent family on D. Any closed subset F of $\mathbb{N}^*$ such that the restriction to F of the map onto ${}^{\mathfrak{c}}2$ is irreducible is (homeomorphic with) the ***absolute*** of ${}^{\mathfrak{c}}2$. If a point u of $\mathbb{N}^*$ belongs to such an F then *every* compactification of the space $\mathbb{N} \cup \{u\}$ contains a copy of $\beta\mathbb{N}$.

Next we prove that $\mathbb{N}^*$ is very non-separable by working on the tree $2^{<\mathbb{N}}$ of finite sequences of zeros and ones. For every $x \in {}^{\mathbb{N}}2$ let $A_x = \{x\restriction n\colon n \in \mathbb{N}\}$. Then $\{A_x\colon x \in {}^{\mathbb{N}}2\}$ is an **almost disjoint family** of cardinality $\mathfrak{c}$ and so $\{A_x^*\colon x \in {}^{\mathbb{N}}2\}$ is a disjoint family of clopen sets in $\mathbb{N}^*$.

We can use this family also to show that $\mathbb{N}^*$ is not extremally disconnected. Let Q denote the points in ${}^{\mathbb{N}}2$ that are constant on a tail (these correspond to the endpoints of the ***Cantor set*** in $[0,1]$) and let $P = {}^{\mathbb{N}}2 \setminus Q$. Then $O_Q = \bigcup_{x\in Q} A_x^*$ and $O_P = \bigcup_{x\in P} A_x^*$ are disjoint open sets in $\mathbb{N}^*$, yet $\operatorname{cl} O_Q \cap \operatorname{cl} O_P \neq \emptyset$, for if A is such that $A_x \subseteq^* A$ for all $x \in P$ then the ***Baire Category Theorem*** may be applied to find $x \in Q$ with $A_x \subseteq^* A$. This example shows that $\mathbb{N}^*$ is not **basically disconnected**: O_Q is an open F_σ-set whose closure is not open.

The algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$ has two countable (in)completeness properties. The first states that when $\langle b_n\rangle_n$ is a decreasing sequence of non-zero elements there is an x with $b_n > x > 0$ for all n; in topological terms: nonempty ***G_δ-sets*** on $\mathbb{N}^*$ have nonempty interiors and $\mathbb{N}^*$ has no isolated points. The second property says that when $\langle b_n\rangle_n$ is as above and, in addition, $\langle a_n\rangle_n$ is an increasing sequence with $a_n < b_n$ for all n there is an x with $a_n < x < b_n$ for all n; in topological terms: disjoint open F_σ-sets in $\mathbb{N}^*$ have disjoint closures, i.e., $\mathbb{N}^*$ is an **F-space**.

We now turn to sets A and B where no interpolating x can be found, that is, we look for A and B such that $\bigvee A' < \bigwedge B'$ whenever $A' \in [A]^{<\omega}$ and $B' \in [B]^{<\omega}$ but for which there is no x with $a \leqslant x \leqslant b$ for all $a \in A$ and $b \in B$. The minimum cardinalities of sets like these are called **cardinal characteristics of the continuum** and these cardinal numbers play an important role in the study of $\beta\mathbb{N}$ and $\mathbb{N}^*$.

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We have already seen such a situation with a countable A and a B of cardinality $\mathfrak{c}$: the families $\{A_x^*\}_{x\in Q}$ and $\{\mathbb{N}^* \setminus A_x^*\}_{x\in P}$. However, this case, of a countably infinite A, is best visualized on the countable set $\mathbb{N}\times\mathbb{N}$. For $n\in\mathbb{N}$ put $a_n = (n\times\mathbb{N})^*$ and for $f\in{}^{\mathbb{N}}\mathbb{N}$ put $b_f = \{\langle m,n\rangle\colon n\geqslant f(m)\}^*$; then $A=\{a_n\colon n\in\mathbb{N}\}$ and $B=\{b_f\colon f\in{}^{\mathbb{N}}\mathbb{N}\}$ are as required: if $a_n < x$ for all n then one readily finds an f such that $b_f < x$ (make sure $\{n\}\times[f(n),\infty)\subset x$). A pair like (A,B) above is called an (unfillable) **gap**; gap, because, as in a Dedekind gap, one has $a<b$ whenever $a\in A$ and $b\in B$, and unfillable because there is no x such that $a\leqslant x\leqslant b$ for all a and b. Because unfillable gaps are the most interesting one drops the adjective and speaks of gaps.

One does not need the full set ${}^{\mathbb{N}}\mathbb{N}$ to create a gap; it suffices to have a subset U such that for all $g\in{}^{\mathbb{N}}\mathbb{N}$ there is $f\in U$ such that $\{n\colon f(n)\geqslant g(n)\}$ is infinite. The minimum cardinality of such a set is denoted $\mathfrak{b}$.

The properties of $\mathfrak{b}$ and its bigger brother $\mathfrak{d}$ are best explained using the relation $<^*$ on ${}^{\mathbb{N}}\mathbb{N}$ where, in keeping with the notation established above, $f <^* g$ means that $\{n\colon f(n)\geqslant g(n)\}$ is finite. The definition of $\mathfrak{b}$ given above identifies it as the minimum cardinality of an unbounded – with respect to $<^*$ – subset of ${}^{\mathbb{N}}\mathbb{N}$; unbounded is not the same as dominating (cofinal), the minimum cardinality of a dominating subset is denoted $\mathfrak{d}$ and it is called the **dominating number**.

If one identifies $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$, as above, then one recognizes $\mathfrak{d}$ as the character of the closed set $F=\operatorname{cl}\bigcup_n A_n^*$ and $\mathfrak{b}$ as the minimum number of clopen sets needed to create an open subset of $\mathbb{N}^*\setminus F$ whose closure meets F. In either characterization of $\mathfrak{b}$ the defining family can be taken to be well-ordered.

In case where A is finite one can simplify matters by letting $A=\{0\}$. If B is an ultrafilter then there is no x with $0<x<b$ for all $b\in B$, hence there are filters whose only lower bound is 0; the minimum cardinality of a base for such a filter is denoted $\mathfrak{p}$. Alternatively one can ask for chains without positive lower bound: the minimum length of such a chain is denoted $\mathfrak{t}$ – it is called the **tower number**. A defining family for the cardinal $\mathfrak{t}$ can be used to create a separable, normal, and sequentially compact spaces that is not compact.

When we let A become uncountable we encounter two important types of objects of cardinality $\aleph_1$: Hausdorff gaps and Lusin families. A **Hausdorff gap** [5] is a pair of sequences $A=\langle a_\alpha\colon \alpha<\omega_1\rangle$ and $B=\langle b_\alpha\colon\alpha<\omega_1\rangle$ of elements such that $a_\alpha<a_\beta<b_\beta<b_\alpha$ whenever $\alpha<\beta$ but for which there is no x with $a_\alpha<x<b_\alpha$ for all α. A **Lusin family** [7] is an almost disjoint family $\mathcal{A}$ of cardinality $\aleph_1$ such that no two uncountable and disjoint subfamilies of $\mathcal{A}$ can be separated, i.e., if $\mathcal{B}$ and $\mathcal{C}$ are disjoint and uncountable then there is no set X such that $B\subseteq^* X$ and $X\cap C=^*\emptyset$ for all $B\in\mathcal{B}$ and $C\in\mathcal{C}$. One can parametrize the F-space property: an **F_κ-space** is one in which disjoint open sets that are the union of fewer than κ many closed sets have disjoint closures. The two families above show that $\mathbb{N}^*$ is not an $F_{\aleph_2}$-space.

By the Axiom of Choice every almost disjoint family can be extended to a **Maximal Almost Disjoint family** (a **MAD family**). If $\mathcal{A}$ is a MAD family then $\mathbb{N}^*\setminus\bigcup\{A^*\colon A\in\mathcal{A}\}$ is ***nowhere dense*** and every nowhere dense set is contained in such a ‘canonical’ nowhere dense set. The minimum number of nowhere dense sets whose union is dense in $\mathbb{N}^*$ is denoted $\mathfrak{h}$ and is called the **weak Novák number**. It is equal to the minimum number of MAD families without a common MAD refinement – in Boolean terms, it is the minimum κ for which $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is *not* **(κ,∞)-distributive**.

Interestingly [1], one can find a sequence $\langle\mathcal{A}_\alpha\colon\alpha<\mathfrak{h}\rangle$ of MAD families, without common refinement and such that (1) $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$ whenever $\alpha<\beta$; (2) if $A\in\mathcal{A}_\alpha$ then $\{B\in\mathcal{A}_{\alpha+1}\colon B\subseteq^* A\}$ has cardinality $\mathfrak{c}$; and (3) the family $\mathcal{T}=\bigcup_\alpha\mathcal{A}_\alpha$ is dense in $\mathcal{P}(\mathbb{N})/\mathit{fin}$ – topologically: $\{A^*\colon A\in\mathcal{T}\}$ is a ***π-base***. This all implies that, with hindsight, there is an increasing sequence of closed nowhere dense sets of length $\mathfrak{h}$ whose union is dense and that $\mathfrak{h}$ is a regular cardinal; also note that $\mathcal{T}$ is a ***tree*** under the ordering $\supset^*$.

One can use such a tree, of minimal height, in inductive constructions, e.g., as in [2] to show that $\mathbb{N}^*$ is very non-extremally disconnected: every point is a **$\mathfrak{c}$-point**, which means that one can find a family of $\mathfrak{c}$ many disjoint open sets each of which has the point in its closure. An important open problem, at the time of writing, is whether this can be proved for *every* nowhere dense set, i.e., if for every nowhere dense subset of $\mathbb{N}^*$ one can find a family of $\mathfrak{c}$ disjoint open sets each of which has this set in its boundary – in short, whether every nowhere dense set is a **$\mathfrak{c}$-set**.

The latter problem is equivalent to a purely combinatorial one on MAD families: for every MAD family $\mathcal{A}$ the family $\mathcal{I}^+(\mathcal{A})$ should have an **almost disjoint refinement**. Here $\mathcal{I}(\mathcal{A})$ is the ideal generated by $\mathcal{A}$ and $\mathcal{I}^+(\mathcal{A})=\mathcal{P}(\mathbb{N})\setminus\mathcal{I}(\mathcal{A})$; an **almost disjoint refinement** of a family $\mathcal{B}$ is an indexed almost disjoint family $\{A_B\colon B\in\mathcal{B}\}$ with $A_B\subseteq^* B$ for all B. Another characterization asks for enough (on even one) MAD families of true cardinality $\mathfrak{c}$, i.e., if $X\in\mathcal{I}^+(\mathcal{A})$ then $X\cap A\neq^*\emptyset$ for $\mathfrak{c}$ many members of $\mathcal{A}$. If the minimum size of a MAD family, denoted $\mathfrak{a}$, is equal to $\mathfrak{c}$ then this is clearly the case, however there are various models with $\mathfrak{a}<\mathfrak{c}$.

3. Homogeneity

Given that the algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is homogeneous it is somewhat of a surprise to learn that the space $\mathbb{N}^*$ is not a **homogeneous space**, i.e., there are two points x and y for which there is no autohomeomorphism h of $\mathbb{N}^*$ with $h(x)=y$.

The first example of this phenomenon is from [9]: the ***Continuum Hypothesis*** (CH) implies that $\mathbb{N}^*$ has **P-points**; a P-point is one for which the family of neighbourhoods is closed under countable intersections. Clearly $\mathbb{N}^*$ has non-P-points (as does every infinite compact space), so CH implies $\mathbb{N}^*$ is not homogeneous. To construct P-points one does not need the full force of CH, the equality $\mathfrak{d}=\mathfrak{c}$ suffices.

In [4] one finds a proof of the non-homogeneity of $\mathbb{N}^*$ that avoids CH; it uses the **Rudin–Frolík order** on $\mathbb{N}^*$, which is defined by: $u \sqsubset v$ iff there is an embedding $f : \beta\mathbb{N} \to \mathbb{N}^*$ such that $f(u) = v$. If $u \sqsubset v$ then there is no autohomeomorphism of $\mathbb{N}^*$ that maps u to v. This proof works to show that no compact F-space is homogeneous: if X is such a space then one can embed $\beta\mathbb{N}$ into it, no autohomeomorphism of X can map (the copy of) u to (the copy of) v.

That this proof avoiding CH was really necessary became clear when the consistency of "$\mathbb{N}^*$ has no P-points" was proved. In [6] the P-point proof was salvaged partially: $\mathbb{N}^*$ has **weak P-point**s, i.e., points that are not accumulation points of any countable subset.

Though not all points of $\mathbb{N}^*$ are P-points one may still try to cover $\mathbb{N}^*$ by nowhere dense P-sets. Under CH this cannot be done but it is consistent that it can be done. The principle NCF implies that $\mathbb{N}^*$ is the union of a chain, of length $\mathfrak{d}$, of nowhere dense P-sets. The principle NCF (**Near Coherence of Filters**) says that any two ultrafilters on $\mathbb{N}$ are **nearly coherent**, i.e., if $u, v \in \mathbb{N}^*$ then there is a finite-to-one $f : \mathbb{N} \to \mathbb{N}$ such that $\beta f(u) = \beta f(v)$.

This in turn implies that the **Rudin–Keisler order** is downward directed; we say $u \prec v$ if there is some $f : \mathbb{N} \to \mathbb{N}$ such that $u = f(v)$. This is a preorder on $\beta\mathbb{N}$ and a partial order on the **type**s of $\beta\mathbb{N}$: if $u \prec v$ and $v \prec u$ then there is a permutation of $\mathbb{N}$ that send u to v. The Rudin–Frolík and Rudin–Keisler orders are related: $u \sqsubset v$ implies $u \prec v$, so $\sqsubset$ is a partial order on the types as well.

Both orders have been studied extensively; we mention some of the more salient results. There are $\sqsubset$-minimal points in $\mathbb{N}^*$: weak P-points are such. Points that are $\prec$-minimal in $\mathbb{N}^*$ are called **selective ultrafilters**; they exist if CH holds but they do not exist in the ***random real*** model. There are $\prec$-incomparable points (even a family of $2^{\mathfrak{c}}$ many $\prec$-incomparable points) but it is not known whether for every point there is another point $\prec$-incomparable to it. The order $\sqsubset$ is tree-like: types below a fixed type are linearly ordered.

4. The continuum hypothesis and $\beta\mathbb{N}$

The behaviour of $\beta\mathbb{N}$ and, especially, $\mathbb{N}^*$ under the assumption of the Continuum Hypothesis (CH) is very well understood. The principal reason is that the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathit{fin}$ is characterized by the properties of being atomless, countably saturated and of cardinality $\mathfrak{c} = \aleph_1$. Topologically, $\mathbb{N}^*$ is, up to homeomorphism, the unique compact zero-dimensional without isolated points, which is an F-space in which nonempty G_δ-sets have non-empty interior and which is of weight $\mathfrak{c}$. This is known as Parovičenko's characterization of $\mathbb{N}^*$ and it implies that whenever X is compact zero-dimensional and of weight $\mathfrak{c}$ (or less) the remainder $(\mathbb{N} \times X)^*$ is homeomorphic to $\mathbb{N}^*$. This provides us with many incarnations of $\mathbb{N}^*$, e.g., as $(\mathbb{N} \times {}^{\mathfrak{c}}2)^*$, which immediately provides us with $2^{\mathfrak{c}}$ many autohomeomorphisms of $\mathbb{N}^*$, or as $(\mathbb{N} \times (\omega + 1))^*$, which gives us a P-set in $\mathbb{N}^*$, viz. $(\mathbb{N} \times \{\omega\})^*$, that is homeomorphic to $\mathbb{N}^*$ itself.

Parovičenko's 'other theorem' states that every compact space of weight $\aleph_1$ (or less) is a continuous image of $\mathbb{N}^*$, whence under CH the space $\mathbb{N}^*$ is a universal compactum of weight $\mathfrak{c}$, in the mapping onto sense.

Virtually everything known about $\mathbb{N}^*$ under CH follows from Parovičenko's theorems. To give the flavour we show that a compact zero-dimensional space can be embedded into $\mathbb{N}^*$ if (and clearly only if) it is an F-space of weight $\mathfrak{c}$ (or less). Indeed, let X be such a space and take a P-set A in $\mathbb{N}^*$ that is homeomorphic to $\mathbb{N}^*$ and a continuous onto map $f : A \to X$. This induces an upper-semi-continuous decomposition of $\mathbb{N}^*$ whose quotient space, the ***adjunction space*** $\mathbb{N}^* \cup_f X$, can be shown to have all the properties that characterize $\mathbb{N}^*$, hence it is $\mathbb{N}^*$ and we have embedded X into $\mathbb{N}^*$, even as a P-set.

In the absence of CH very few of the consequences of Parovičenko's theorems remain true. The characterization theorem is in fact equivalent to CH and for many concrete spaces, like $\mathbb{N}^* \times \mathbb{N}^*$, $\mathbb{R}^*$, $(\mathbb{N} \times (\omega+1))^*$ and the Stone space of the ***measure algebra*** it is *not* a theorem of ZFC that they are continuous images of $\mathbb{N}^*$. It is also consistent with ZFC that all autohomeomorphisms of $\mathbb{N}^*$ are **trivial**, i.e., induced by bijections between cofinite sets. This is a far cry from the $2^{\mathfrak{c}}$ autohomeomorphisms that we got from CH. It is also consistent that $\mathbb{N}^*$ cannot be homeomorphic to a nowhere dense P-subset of itself; this leaves open a major question: is there a non-trivial copy of $\mathbb{N}^*$ in itself, i.e., one not of the form $\operatorname{cl} D \setminus D$ for some countable discrete subset of $\mathbb{N}^*$.

A good place to start exploring $\beta\mathbb{N}$ is van Mill's survey [KV, Chapter 11].

5. Cardinal numbers

The cardinal numbers mentioned above are all uncountable and not larger than $\mathfrak{c}$, so the Continuum Hypothesis (CH) implies that all are equal to $\aleph_1$. More generally, ***Martin's Axiom*** implies all are equal to $\mathfrak{c}$. One can prove certain inequalities between the characteristics, e.g., $\aleph_1 \leqslant \mathfrak{p} \leqslant \mathfrak{t} \leqslant \mathfrak{h} \leqslant \mathfrak{b} \leqslant \mathfrak{d}$. Intriguingly it is as yet unknown whether $\mathfrak{p} = \mathfrak{t}$ is provable, what is known is that $\mathfrak{p} = \aleph_1$ implies $\mathfrak{t} = \aleph_1$.

One proves $\mathfrak{b} \leqslant \mathfrak{a}$ but neither $\mathfrak{a} \leqslant \mathfrak{d}$ nor $\mathfrak{d} \leqslant \mathfrak{a}$ is provable in ZFC.

Two more characteristics have received a fair amount of attention, the **splitting number** $\mathfrak{s}$ is the minimum cardinality of a splitting family, that is, a family $\mathcal{S}$ such that for every infinite X there is $S \in \mathcal{S}$ with $X \cap S$ and $X \setminus S$ infinite; and its dual, the **reaping number** $\mathfrak{r}$, which is the minimum cardinality of a family that cannot be reaped (or split), that is, a family $\mathcal{R}$ such that for every infinite X there is $R \in \mathcal{R}$ such that one of $X \cap R$ and $X \setminus R$ is not infinite.

Topologically $\mathfrak{s}$ is the minimum κ for which ${}^{\kappa}2$ is not ***sequentially compact*** and $\mathfrak{r}$ is equal to the minimum ***π-character*** of points in $\mathbb{N}^*$.

Further inequalities between these characteristics are, e.g., $\mathfrak{h} \leqslant \mathfrak{s} \leqslant \mathfrak{d}$ and $\mathfrak{b} \leqslant \mathfrak{r}$.

These ‘small’ cardinals are the subject of ongoing research; good introductions are [KV, Chapter 3] and [vMR, Chapter 11].

6. Properties of $\beta\mathbb{R}$

Instead of $\beta\mathbb{R}$ one usually considers $\beta\mathbb{H}$, where $\mathbb{H}$ is the positive half line $[0,\infty)$. This is because $x \mapsto -x$ induces an autohomeomorphism of $\beta\mathbb{R}$ that shows that $\beta[0,\infty)$ and $\beta(-\infty,0]$ are the same thing.

In a sense $\beta\mathbb{H}$ looks like $\beta\mathbb{N}$ in that it is a thin locally compact space with a large compact lump at the end; this remainder $\mathbb{H}^*$ has some properties in common with $\mathbb{N}^*$: both are F-spaces in which nonempty G_δ-sets have nonempty interior, both have tree π-bases and neither is an $F_{\aleph_2}$-space. Under CH the spaces $\mathbb{H}^*$ and $\mathbb{N}^*$ have homeomorphic dense sets of P-points. That is where the superficial similarities end because $\beta\mathbb{H}$ and $\mathbb{H}^*$ are connected and $\beta\mathbb{N}$ and $\mathbb{N}^*$ most certainly are not.

A deeper similarity is a version of Parovičenko’s universality theorem: every continuum of weight $\aleph_1$ (or less) is a continuous image of $\mathbb{H}^*$, so that, under CH, the space $\mathbb{H}^*$ is a universal continuum of weight $\mathfrak{c}$. A version of Parovičenko’s characterization theorem for $\mathbb{H}^*$ is yet to be found.

The structure of $\mathbb{H}^*$ as-a-continuum is very interesting. Most references for what follows can be found in [HvM, Chapter 9].

The following construction is crucial for our understanding of the structure of the continuum $\mathbb{H}^*$: take a discrete sequence $\langle [a_n,b_n]\rangle_n$ of non-trivial closed intervals and an ultrafilter u on $\mathbb{N}$. The intersection

$$I = \bigcap_{U\in u} \operatorname{cl}\Bigl(\bigcup_{n\in U} [a_n,b_n]\Bigr)$$

is a continuum, often called a **standard subcontinuum** of $\mathbb{H}^*$. What is striking about this construction is not its simplicity but that these (deceptively) simple continua govern the continuum-theoretic properties of $\mathbb{H}^*$. Every proper subcontinuum (in particular every point) is contained in a standard subcontinuum – it is in fact the intersection of some family of standard subcontinua. From this one shows that $\mathbb{H}^*$ is hereditarily ***unicoherent*** – every finite intersection of standard subcontinua is a standard subcontinuum or a point – and ***indecomposable*** – standard subcontinua are nowhere dense.

From the other direction every subcontinuum is also the union of a suitable family of standard subcontinua. Thus, no subcontinuum of $\mathbb{H}^*$ is hereditarily indecomposable, as standard subcontinua have ***cut points***. Indeed, I_u contains the ultraproduct $\prod_n [a_n,b_n]/u$, as a dense set: the equivalence class of a sequence $\langle x_n\rangle_n$ corresponds to its own u-limit, denoted x_u. The subspace topology of this ultraproduct coincides with its ***order topology*** and every point of the ultraproduct (except a_u and b_u) is a cut point of I_u and so a **weak cut point** (i.e., cut point of some subcontinuum) of $\mathbb{H}^*$. Although the continuum I_u is an **irreducible continuum**, i.e., no smaller continuum contains its ***end point***s a_u and b_u, it is definitely not an ordered continuum. It is an F-space so the closure of an increasing sequence of points in the ultraproduct is homeomorphic to $\beta\mathbb{N}$ (in an ordered continuum it would have to be $\omega+1$). The ‘supremum’ of such a sequence of points in I_u is an irreducibility layer of I_u; this layer is non-trivial (it contains $\mathbb{N}^*$) and indecomposable. Adding all these facts together we can deduce that a maximal chain of indecomposable subcontinua of $\mathbb{H}^*$ has a one-point intersection; such a point is not a weak cut point of $\mathbb{H}^*$. Thus $\mathbb{H}^*$ is shown to be not homogeneous by purely continuum-theoretic means. There is a natural quasi-order on an irreducible continuum: in the present case $x \leqslant y$ means “every continuum that contains a_u and y also contains x”. An **irreducibility layer** is an equivalence class for the equivalence relation “$x \leqslant y$ and $y \leqslant x$”.

The weak cut points constructed above are ***near point***s and, in fact, every near point is a weak cut point of $\mathbb{H}^*$. Under CH one can construct different kinds of weak cut points: ***far*** but not ***remote*** and even remote. Under CH it is also possible to map a remote weak cut point to a near point by an autohomeomorphism of $\mathbb{H}^*$. On the other hand it is consistent that all weak cut points are near points and hence that the far points are topologically invariant in $\mathbb{H}^*$. A similar consistency result for remote points is still wanting.

The ***composant*** of a point x of $\mathbb{H}^*$ meets $\mathbb{N}^*$ in the ultrafilter $\{U\colon x \in \operatorname{cl}\bigcup_{n\in U}[n,n+1]\}$ and two points from $\mathbb{N}^*$ are in the same composant of $\mathbb{H}^*$ iff they are nearly coherent. Therefore, the structure of the set of composants of $\mathbb{H}^*$ is determined by the family of finite-to-one maps of $\mathbb{N}$ to itself. This implies that the number of composants may be $2^{\mathfrak{c}}$ (e.g., if CH holds), 1 (this is equivalent to the NCF principle) or 2 (in other models of set theory); whether other numbers are possible is unknown.

The number of (homeomorphism types of) subcontinua of $\mathbb{H}^*$ is as yet a function of Set Theory: in ZFC one can establish the lower bound of 14. Under CH there are $\aleph_1$ types even though in one respect CH act as an equalizer: CH is equivalent to the statement that all standard subcontinua are mutually homeomorphic. Most of the ZFC continua are found as intervals in an I_u with points and non-trivial layers at their ends and with varying cofinalities.

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