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FIXED-POINT SETS OF AUTOHOMEOMORPHISMS OF COMPACT F-SPACES

K. P. HART AND J. VERMEER

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Abstract. We investigate fixed-point sets of autohomeomorphisms of compact F-spaces. If the space in question is finite dimensional (in the sense of covering dimension), then the fixed-point set is a P-set; on the other hand there is an infinite-dimensional compact F-space with an involution whose fixed-point set is not a P-set.

In addition we show that under CH a closed subset of ω^* is a P-set iff it is the fixed-point set of an autohomeomorphism.

Introduction

In this note we investigate the fixed-point sets of autohomeomorphisms of compact F-spaces. In Vermeer [6, 7] the second author studied fixed-point sets of continuous self-maps of extremally and basically disconnected spaces. It was proved that whenever X is a compact κ-basically disconnected space (i.e., the Stone space of a κ-complete Boolean algebra) and $\phi : X \to X$ is injective and continuous, the fixed-point set of ϕ is a P_κ-set of X. In particular for a basically disconnected (i.e., ω_1-basically disconnected) space the fixed-point set of a self-embedding is always a P-set.

The methods used to obtain the above-mentioned result do not readily generalize to the natural extension of the class of basically disconnected spaces: the class of F-spaces. The point is that these methods relied heavily on the fact that a countable increasing union of clopen sets in a basically disconnected space has a clopen closure and this last property hardly ever holds nontrivially in general F-spaces.

Here we use results about fixed-point free extensions of fixed-point free maps to obtain the result that the fixed-point set of an autohomeomorphism of a finite-dimensional compact F-space is a P-set of that space. This seems to be new, even for the space ω^*.

If we assume the Continuum Hypothesis, then we can even show that a closed subset of ω^* is a P-set iff it is the fixed-point set of an autohomeomorphism (even an involution) of ω^*. This gives an external characterization of the P-sets in ω^* and is a partial answer to Problem 218 of Hart and van Mill [4]. We finish the paper with an example of an infinite-dimensional compact F-space and an involution on it whose fixed-point set is not a P-set.

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1. Preliminaries

By convention all spaces under consideration are completely regular. We call—as usual—a space X an F-space if every cozero set in it is C^*-embedded, i.e., if M is a cozero set of X and $f : M \to \mathbb{R}$ is a bounded continuous function, then f can be extended to a bounded continuous function from X to $\mathbb{R}$. For compact spaces this takes the following convenient form: A compact space X is an F-space iff for every F_σ-subset F of X the equality $\operatorname{cl} F = \beta F$ holds. A rich supply of compact F-spaces can be gotten from the well-known fact that $\beta X \setminus X$ is an F-space whenever X is σ-compact and locally compact.

We also need the characterization of ω^* given by Parovičenko in [5]. This characterization is valid under the assumption of the Continuum Hypothesis (CH).

Theorem 1.1 (CH)**.** *A compact space X is homeomorphic to ω^* if and only if it is a compact, zero-dimensional F-space of weight $\mathfrak{c}$ without isolated points in which nonempty G_δ-sets have nonempty interiors.*

This theorem is particularly useful when one works with P-sets in ω^*; we recall that a subset of a space is a P-set if every G_δ-set containing it is a neighbourhood of it or, equivalently, a set A is a P-set if for every F_σ-set F disjoint from it one has $A \cap \operatorname{cl} F = \varnothing$.

For example, in the proof of Lemma 1.3 below we use the fact that $\omega^* \setminus \operatorname{Int} A$ is homeomorphic to ω^* whenever A is a P-set of ω^*. A second application occurs in the proof of Theorem 2.2.

¿From van Douwen and van Mill [2] we quote the following theorem, the homeomorphism extension theorem for nowhere dense P-sets.

Theorem 1.2 (CH)**.** *Let A and B be nowhere dense P-sets of ω^* and $h : A \to B$ a homeomorphism. Then there is an autohomeomorphism $\tilde{h}$ of ω^* that extends h.*

We shall need the following mild extension of this theorem.

Lemma 1.3 (CH)**.** *Let A and B be proper P-subsets of ω^*, and let $h : A \to B$ be a homeomorphism that maps the interior of A onto the interior of B. Then there is an autohomeomorphism $\tilde{h}$ of ω^* that extends h.*

Proof. Consider $\omega^* \setminus \operatorname{Int} A$ and $\omega^* \setminus \operatorname{Int} B$. As noted above both spaces are homeomorphic to ω^* because A and B are P-sets.

The homeomorphism extension theorem for nowhere dense P-sets now gives us an extension $h' : \omega^* \setminus \operatorname{Int} A \to \omega^* \setminus \operatorname{Int} B$ of the restriction $h \restriction \operatorname{Fr} A$. To finish we let $\tilde{h} = h \cup h'$. □

The final result that we need is from van Douwen [1]. We use the term 'finite-dimensional' in the sense of the covering dimension dim.

Theorem 1.4. *Let X be a finite-dimensional paracompact space and $f : X \to X$ a closed self-map for which there is a natural number k such that $\left|f^{-1}(x)\right| \le k$ for all $x \in X$. Then f has a fixed point if and only if βf has a fixed point.*

2. Finite-dimensional spaces

We get our first result by a judicious application of van Douwen's theorem.

Theorem 2.1. *Let X be a compact finite-dimensional F-space and $\phi : X \to X$ a continuous and injective map. The fixed-point set F of ϕ is a P-set of X.*

Proof. Let K be an F_σ-subset of X that is disjoint from F. We must show that $\operatorname{cl} K$ is disjoint from F. To this end we take the set $L = \bigcup_{k\in\mathbb{Z}} \phi^k[K]$. Observe that L is also an F_σ-set that is disjoint from F; that L is an F_σ-set is clear. To see that L contains no fixed points of ϕ combine the facts that K contains none and that ϕ is injective. It is also clear that $\phi[L] \subseteq L$. Finally we observe that $\phi \restriction L$ is closed: use the fact that $\phi^{-1}[L] = L$.

Now, because X is an F-space, we have $\operatorname{cl} L = \beta L$. Then van Douwen's theorem implies that $\operatorname{cl} L$ contains no fixed points of ϕ either. It follows that $\operatorname{cl} L \cap F = \varnothing$, so certainly $\operatorname{cl} K \cap F = \varnothing$. □

For the space ω^* we can reverse the implication, provided we assume CH.

Theorem 2.2 (CH). *A closed subset A of ω^* is a P-set iff it is the fixed-point set of some autohomeomorphism of ω^*.*

Proof. Let A be a P-set of ω^*. We shall find an autohomeomorphism ϕ of ω^* of which A is the fixed-point set; indeed, ϕ will be an involution, i.e., ϕ^2 is the identity.

Consider $\omega^* \times 2$ and identify, for every $x \in A$, the points $\langle x, 0\rangle$ and $\langle x, 1\rangle$ (we glue the two copies of ω^* together along the copies of A). Because A is a P-set, the resulting quotient space Q is homeomorphic to ω^*: it satisfies the conditions from Parovičenko's theorem.

Define an autohomeomorphism ψ of Q by sending $\langle x, i\rangle$ to $\langle x, 1-i\rangle$ for every x. Clearly ψ^2 is the identity and the copy A_Q of A in Q is the fixed-point set of ψ.

It remains to turn ψ into an autohomeomorphism of ω^* whose fixed-point set is A itself.

The identity $\mathrm{Id} : A_Q \to A$ is a homeomorphism that maps the interior of A_Q onto the interior of A and so by Lemma 1.3 it may be extended to a homeomorphism $h : Q \to \omega^*$.

In the end we take $\phi = h \circ \psi \circ h^{-1}$ of course. □

3. Infinite-dimensional spaces

In this section we give an example of compact infinite-dimensional F-space X and an autohomeomorphism ϕ of X whose fixed-point set is not a P-set. Again ϕ can be taken to be an involution.

Our starting point is the following example, considered by van Douwen in [1]. Let $\mathbb{S} = \bigoplus_n S^n$, where S^n is the standard n-sphere. Next let $\phi : \mathbb{S} \to \mathbb{S}$ be the sum of the antipodal mappings. Now ϕ has no fixed points, yet $\beta\phi$ does have fixed points; this can be seen as follows: if $\beta\phi$ would have no fixed points, then there would be a finite closed cover $\{F_1, \ldots, F_n\}$ of $\beta\mathbb{S}$ such that $\beta\phi[F_i] \cap F_i = \varnothing$ for all i. However, the Lusternik-Schnirelman-Borsuk Theorem (Dugundji and Granas [3, Theorem 4.4]) implies that $\phi[F_i] \cap F_i \cap S^n \neq \varnothing$ for some i.

To begin we take for every n the closed n-ball B^n. Remove the origin and call the result X_n. The antipodal map e_n on X_n has no fixed points and, as $\dim X_n = n$, neither do βe_n and $e_n^* = \beta e_n \restriction X_n^*$ (apply Theorem 1.4). Also note that X_n^* is an F-space.

Write $X = \bigoplus_n \beta X_n$ and $e = \bigoplus_n \beta e_n$. The map e has no fixed points but βe has many of them: for any sequence $\langle S_n\rangle_n$ of spheres centered at the origins of the B^n we get fixed points of βe in the closure of $\bigoplus_n S_n$.

Now take any neighbourhood of $(\bigoplus_n X_n^*)^*$ in βX; it contains a tail of a sequence of spheres as in the preceding paragraph and hence a fixed point of βe. But then $(\bigoplus_n X_n^*)^*$ contains fixed points of βe as well.

Our example is the closure of $\bigoplus_n X_n^*$ in βX, and the map ϕ is the restriction of βe. It is clearly an F-space, and the (nonempty) fixed-point set of ϕ is contained in the nowhere dense G_δ-set $(\bigoplus_n X_n^*)^*$.

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Faculty of Technical Mathematics and Informatics, TU Delft, Postbus 5031, 2600 GA Delft, the Netherlands

E-mail address: `wiawkph@dutrun2.tudelft.nl`

E-mail address: `J.Vermeer@twi.tudelft.nl`