MANY SUBALGEBRAS OF $\mathcal{P}(\omega)/\mathit{fin}$

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Abstract. In answer to a question on Mathoverflow we show that the Boolean algebra $\mathcal{P}(\omega)/\mathit{fin}$ contains a family $\{\mathcal{B}_X : X \subseteq \mathfrak{c}\}$ of subalgebras with the property that $X \subseteq Y$ implies $\mathcal{B}_Y$ is a subalgebra of $\mathcal{B}_X$ and if $X \nsubseteq Y$ then $\mathcal{B}_Y$ is not embeddable into $\mathcal{B}_X$. The proof proceeds by Stone duality and the construction of a suitable family of separable zero-dimensional compact spaces.

Introduction

The purpose of this note is to give a more leisurely presentation, complete with definitions and references, of an answer to a question on `MathOverflow` [3]:

> Is there a strictly decreasing chain of subalgebras of the Boolean algebra $\mathcal{P}(\omega)/\mathit{fin}$?

The answer to the question as stated is an obvious "yes", but the poser of the question asked for a sequence $\langle B_n : n \in \omega\rangle$ of subalgebras such that $B_{n+1} \subseteq B_n$ *and* B_n is *not* embeddable into B_{n+1}, for all n.

We shall show that the family of subalgebras of $\mathcal{P}(\omega)/\mathit{fin}$ is rich enough to contain such a sequence; in fact, there is a family $\{B_X : X \subseteq \mathfrak{c}\}$ of subalgebras with the property that for all subsets X and Y of $\mathfrak{c}$ we have: if $X \subseteq Y$ then $B_Y \subseteq B_X$ *and* if $X \nsubseteq Y$ then B_Y is *not* embeddable into B_X. This more than answers the question and shows that one can even have a decreasing chain of length $\mathfrak{c}$ or a chain or order type that of the real line.

The construction of the family proceeds via Stone duality: rather than constructing subalgebras of $\mathcal{P}(\omega)/\mathit{fin}$ we construct a family $\{K_X : X \subseteq \mathfrak{c}\}$ of separable compact zero-dimensional spaces with the dual property that there is a set $\{h_{X,Y} : X \subseteq Y \subseteq \mathfrak{c}\}$ of continuous maps, where $h_{X,Y} : K_X \to K_Y$ is a continuous surjection *and* if $X \nsubseteq Y$ then K_X is *not* a continuous image of B_Y. In addition all triangles in the set of maps will commute.

Then $K_\emptyset$ is a continuous image of ω^*, the Stone space of $\mathcal{P}(\omega)/\mathit{fin}$, and hence so are all other spaces K_X. The maps $h_{\emptyset,X}$ embed the algebras of clopen sets of the K_X into $\mathcal{P}(\omega)/\mathit{fin}$, the commutativity of the triangles in the family of continuous surjections yields the desired inclusions, and the nonexistence of further continuous surjections dualizes to the nonexistence of further embeddings.

1. Preliminaries

1.1. **Stone duality.** Stone's duality for Boolean algebras and compact zero-dimensional spaces associates with every compact zero-dimensional space X its Boolean algebra $\mathcal{C}_X$ of closed-and-open subsets and conversely with every Boolean algebra B a compact zero-dimensional space $\mathrm{St}(B)$, its *Stone space*. The associations are each others inverses and they dualize various notions; the most important for us

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is that an embedding $B \to C$ of Boolean algebras becomes a continuous surjection $\mathrm{St}(C) \to \mathrm{St}(B)$, and vice versa.

The book [5, Chapter 3] contains further information on Stone's duality for Boolean algebras and compact zero-dimensional spaces.

1.2. **Bernstein sets.** In our construction we shall uses Bernstein sets in $[0,1]$. We say A is a *Bernstein set* in $[0,1]$ if A and its complement both intersect every uncountable closed set in $[0,1]$. These are also called totally imperfect sets ([1]) because if a set is closed in $[0,1]$ and contained in A then it must be countable.

For other topological material we refer to [2]. The fact used here, that separable compact spaces are continuous images of the remainder ω^* can be proved using Theorem 3.5.13 and Exercise 3.5.H of that book.

2. The spaces

The spaces are variations on Alexandroff's double-arrow space $\mathbb{A}$, called the two arrows space in [2, Exercise 3.10.C].

The underlying set is $D = \bigl([0,1] \times \{0,1\}\bigr) \setminus \bigl\{\langle 0,0\rangle, \langle 0,1\rangle\bigr\}$, ordered lexicographically and endowed with the order topology. (We drop the points $\langle 0,0\rangle$ and $\langle 0,1\rangle$ because they would be (the only) isolated points of $\mathbb{A}$.)

Pictorially we have taken the unit interval $[0,1]$ and split each point x of the open interval $(0,1)$ into two copies, $\langle x,0\rangle$ and $\langle x,1\rangle$. The space $\mathbb{A}$ is compact and separable, hence a continuous image of ω^*.

The variations will be obtained by specifying a subset X of $(0,1)$ and taking $\mathbb{A}_X = \{\langle x,i\rangle \in D : x \in X \to i = 0\}$; that is, by splitting the points of $(0,1) \setminus X$ only. Thus we can write $\mathbb{A} = \mathbb{A}_\emptyset$, and $[0,1] = \mathbb{A}_{(0,1)}$ for example. In all our examples the complement of X will be dense in $(0,1)$ and this will ensure that $\mathbb{A}_X$ is zero-dimensional.

If $X \subseteq Y$ then there is a natural continuous surjection $s : \mathbb{A}_X \to \mathbb{A}_Y$, given by

- $s(x,i) = \langle x,i\rangle$ if $x \notin Y$;
- $s(x,i) = \langle x,0\rangle$ if $x \in Y \setminus X$; and
- $s(x,0) = \langle x,0\rangle$ if $x \in X$.

Our goal will be to create a family $\{S_X : X \subseteq \mathfrak{c}\}$ of subsets of $(0,1)$ such that with $K_X = \mathbb{A}_{S_X}$ for all X we get our family $\{K_X : X \subseteq \mathfrak{c}\}$.

We shall construct a family $\{A_\alpha : \alpha \in \mathfrak{c}\}$ of subsets of $(0,1)$ (all disjoint from $\mathbb{Q}$) and put $S_X = \mathbb{Q} \cup \bigcup_{\alpha\in X} A_\alpha$ for $X \subseteq \mathfrak{c}$.

Clearly then $X \subseteq Y$ implies $S_X \subseteq S_Y$ and hence that K_X maps onto K_Y by $h_{X,Y} : \mathbb{A}_{S_X} \to \mathbb{A}_{S_Y}$ as described above. It is readily seen that $h_{X,Z} = h_{Y,Z} \circ h_{X,Y}$ all triangles in this family commute, as described in the introduction.

It remains to construct the sets A_α in such a way that whenever $X \nsubseteq Y$ each of the A_α with $\alpha \in X \setminus Y$ will prohibit the existence of a continuous surjection from K_Y onto K_X.

To see how this may be accomplished note that since $A_\alpha \subseteq S_X$ the points of A_α are not split in $\mathbb{A}_{S_X}$. In that case the subspace topology that A_α inherits from $\mathbb{A}_{S_X}$ is the same as the subspace topology that it inherits from $[0,1]$.

If $s : K_X \to K_Y$ is continuous then the composition $t \circ s$, where $t : K_Y \to [0,1]$ is the map that send $\langle x,i\rangle$ to x, is continuous as well and its restriction g to A_α is also continuous. We shall arrange matters in such a way that the only maps that can appear in this way will force the range of the map s to be countable.

3. The sets A_α

We can obtain our sets A_α by a direct application of Theorem 2.0 in [4] but to keep this note reasonably self-contained we shall repeat the construction for the special case that we need. The method goes back to [7] and is occasionally referred to as "Sierpiński's technique of killing homeomorphisms" [6], but it can be used to eliminate other maps as well.

In our case we consider the set $\mathcal{F}$ of all maps f that satisfy: $\operatorname{dom} f$ is a co-countable subset of $[0,1]$ and $f : \operatorname{dom} f \to [0,1]$ is continuous. For every $f \in \mathcal{F}$ we let $S(f) = \{x \in \operatorname{dom} f : f(x) \neq x\}$ and $E(f) = \operatorname{dom} f \setminus S(f)$. We choose a subset $C(f)$ of $\operatorname{dom} f$ such that the restriction $f : C(f) \to f\bigl[S(f)\bigr]$ is a bijection.

Before we continue we make some remarks that will be useful later. For each $f \in \mathcal{F}$ the domain is completely metrizable as it is a G_δ-subset of $[0,1]$. As $S(f)$ is open in $\operatorname{dom} f$, and $E(f)$ is closed, both sets are completely metrizable as well. Furthermore, the image $f\bigl[S(f)\bigr]$ is an analytic subset of $[0,1]$. By familiar results from Descriptive Set Theory it follows that each of these sets either is countable or contains a topological copy of the Cantor set. One can modify the construction outlined in Problem 4.5.5 in [2] to prove these results.

This means that in each case we can check whether the set is countable by looking at its intersection with some Bernstein set.

The following proposition yields the family $\{A_\alpha : \alpha \in \mathfrak{c}\}$.

Proposition 3.1. *There is a pairwise disjoint family $\{V\} \cup \{A_\alpha : \alpha \in \mathfrak{c}\}$ of Bernstein sets in $(0,1)$ with the following properties. All are disjoint from $\mathbb{Q}$, and for every $f \in \mathcal{F}$: if $f\bigl[S(f)\bigr]$, and hence $C(f)$, has cardinality $\mathfrak{c}$ then for all α the intersections $C(f) \cap A_\alpha$ and $f\bigl[C(f) \cap A_\alpha\bigr] \cap V$ both have cardinality $\mathfrak{c}$.*

Proof. Since $[0,1]$ has cardinality $\mathfrak{c}$ it also has $\mathfrak{c}$ many co-countable subsets. Since each subset of $[0,1]$ is separable every co-countable set has $\mathfrak{c}$ many continuous functions to $[0,1]$. Hence we may enumerate the subfamily of $\mathcal{F}$ consisting of those f for which $f\bigl[S(f)\bigr]$ is uncountable as $\langle f_\beta : \beta \in \mathfrak{c}\rangle$, in such a way that every f occurs $\mathfrak{c}$ many times in the sequence. We take a similar enumeration $\langle F_\beta : \beta \in \mathfrak{c}\rangle$ of the family of uncountable closed subsets of $[0,1]$ (each set is listed $\mathfrak{c}$ times).

To facilitate the construction we replicate both enumerations $\mathfrak{c}$ times and turn them into $\mathfrak{c} \times \mathfrak{c}$-matrices: $\{f_{\alpha,\beta} : \langle \alpha, \beta\rangle \in \mathfrak{c}^2\}$ and $\{F_{\alpha,\beta} : \langle \alpha, \beta\rangle \in \mathfrak{c}^2\}$, where $f_{\alpha,\beta} = f_\beta$ and $F_{\alpha,\beta} = F_\beta$ for all α and β. We also take a well-order $\prec$ of $\mathfrak{c}^2$ in order type $\mathfrak{c}$,

By recursion on the well-order $\prec$ we will choose points $a_{\alpha,\beta}$, $b_{\alpha,\beta}$, $u_{\alpha,\beta}$, and $v_{\alpha,\beta}$, as follows.

When the points have been found for $\langle \gamma, \delta\rangle \prec \langle \alpha, \beta\rangle$ collect them and the rational numbers in a set: $P = \mathbb{Q} \cup \bigcup_{\langle \gamma,\delta\rangle \prec \langle \alpha,\beta\rangle} \{a_{\gamma,\delta}, b_{\gamma,\delta}, u_{\gamma,\delta}, v_{\gamma,\delta}\}$. Note that the cardinality of P is strictly smaller than $\mathfrak{c}$. Therefore we can find $a_{\alpha,\beta} \in C(f_{\alpha,\beta}) \setminus P$ such that $u_{\alpha,\beta} = f_{\alpha,\beta}(a_{\alpha,\beta}) \notin P$; and note that $u_{\alpha,\beta} \neq a_{\alpha,\beta}$. Next take points $b_{\alpha,\beta}$ and $v_{\alpha,\beta}$ in $F_{\alpha,\beta} \setminus (P \cup \{a_{\alpha,\beta}, v_{\alpha,\beta}\}$ such that $b_{\alpha,\beta} \neq v_{\alpha,\beta}$.

Now, by construction all points chosen in this way are distinct. For every $\alpha \in \mathfrak{c}$ we let

$$A_\alpha = \{a_{\alpha,\beta} : \beta \in \mathfrak{c}\} \cup \{b_{\alpha,\beta} : \beta \in \mathfrak{c}\}$$

we also let

$$V = \{u_{\alpha,\beta} : \langle \alpha, \beta\rangle \in \mathfrak{c}^2\} \cup \{v_{\alpha,\beta} : \langle \alpha, \beta\rangle \in \mathfrak{c}^2\}$$

Because all points chosen are distinct the family $\{V\} \cup \{A_\alpha : \alpha \in \mathfrak{c}\}$ is pairwise disjoint.

The sets are Bernstein sets because $A_\alpha \cap F \supseteq \{b_{\alpha,\beta} : F = F_{\alpha,\beta}\}$ and $V \cap F \supseteq \{v_{\alpha,\beta} : F = F_{\alpha,\beta}\}$, both intersections have cardinality $\mathfrak{c}$.

Likewise, if $f\bigl[S(f)\bigr]$ has cardinality $\mathfrak{c}$ then $A_\alpha \cap C(f) \supseteq \{a_{\alpha,\beta} : f = f_{\alpha,\beta}\}$ and $V \cap f[A_\alpha \cap C(f)] \supseteq \{u_{\alpha,\beta} : f = f_{\alpha,\beta}\}$; again both sets have cardinality $\mathfrak{c}$. □

It now remains to show that the resulting family $\{K_X : X \subseteq \mathfrak{c}\}$ of compact zero-dimensional spaces has the desired properties. We already know that K_X maps onto K_Y if $X \subseteq Y$. We prove the other implication in the next section. There it will become clear what the function of the set V is.

4. Non-existence of continuous surjections

The following lemma implies that if X and Y are subsets of $\mathfrak{c}$ such that $X \nsubseteq Y$ then there is no continuous surjection from K_X onto K_Y.

Lemma 4.1. *Let X and Y be subsets of $(0,1)$ such that $\mathbb{Q} \subseteq X$ and such that there is an α for which $A_\alpha \subseteq X$ and $Y \cap (A_\alpha \cup V) = \emptyset$. Then every continuous map $s : \mathbb{A}_X \to \mathbb{A}_Y$ has a countable range.*

Proof. Let us write A for A_α and let $t : \mathbb{A}_Y \to [0,1]$ be the natural surjection. Also, we identify x and $\langle x, 0\rangle$ when $x \in X$.

As observed before the topology on A in $\mathbb{A}_X$ is the same as its subspace topology in $[0,1]$. Let g be the restriction of $(t \circ s)$ to A.

By one half of Lavrentieff's theorem (Theorem 4.3.20 in [2]) we can find a G_δ-set G that contains A and a continuous map $f : G \to [0,1]$ that extends g. The complement, C, of G in $[0,1]$ is a countable union of closed sets, each of which is countable because closed sets that are disjoint from a Bernstein set are countable; and A is a Bernstein set. It follows that f belongs to the family $\mathcal{F}$.

The set A is dense in $[0,1]$, and the maps f and $t \circ s$ agree on A. This implies that f determines much of the behaviour of s on G, in the following way.

- If $x \in G \cap X$ then x is not split and $(t \circ s)(x) = f(x)$ and this implies that $s(x) \in \bigl\{\langle f(x), 0\rangle, \langle f(x), 1\rangle\bigr\}$.
- If $x \in G \setminus X$ then x is split and the continuity of f implies that $(t \circ s)(x,0) = (t \circ s)(x,1) = f(x)$ and so $\bigl\{s(x,0), s(x,1)\bigr\} \subseteq \bigl\{\langle f(x), 0\rangle, \langle f(x), 1\rangle\bigr\}$.

It follows that the range of s is contained in the union of $f[G] \times \{0,1\}$ and the image of the countable set of points whose first coordinates are in the countable set C.

We finish the proof by showing that $f[G]$ is countable.

Let $x \in E(f) \cap A$, then $f(x) = x$ and so $s(x) = \langle x, 0\rangle$ or $s(x) = \langle x, 1\rangle$. We divide $E(f) \cap A$ into two sets: $E_0 = \{x : s(x) = \langle x, 0\rangle\}$ and $E_1 = \{x : s(x) = \langle x, 1\rangle\}$.

If $x \in E_0$ then by continuity of s there is an interval (p_x, q_x) with rational end points such that $s[(p_x, q_x)] \subseteq [0, \langle x, 0\rangle]$. Here we use that $\mathbb{Q} \subseteq X$: we can talk without ambiguity about intervals with rational end points. It is clear that when $x < y$ in E_0 we have $y \in (p_y, q_y) \setminus (p_y, q_y)$, and it follows that $x \mapsto (p_x, q_x)$ is injective. We deduce that E_0 is countable. Likewise one shows that E_1 is countable.

We see that $E(f) \cap A$ is countable, and because A is a Bernstein set it follows that $E(f)$ itself is countable

Next we let $x \in C(f) \cap A$ such that $f(x) \in V$. Then $f(x) \notin Y$ and so $f(x)$ is split in $\mathbb{A}_Y$, and $s(x) \in \{\langle f(x), 0\rangle, \langle f(x), 1\rangle\bigr\}$; we split $\{x \in C(f) \cap A : f(x) \in V\}$ into $C_0 = \bigl\{x : s(x) = \langle f(x), 0\rangle\bigr\}$ and $C_1 = \bigl\{x : s(x) = \langle f(x), 1\rangle\bigr\}$.

As above we take for $x \in C_0$ an interval (p_x, q_x) with rational end points such that $s[(p_x, q_x)] \subseteq [0, \langle f(x), 0\rangle]$. If $x \neq y$ in C_0 then $f(x) \neq f(y)$ because f is injective on $C(f)$; if, say, $f(x) < f(y)$ then $y \in (p_y, q_y) \setminus (p_y, q_y)$ and it follows that $x \mapsto (p_x, q_x)$ is injective. We conclude, as above, that C_0 is countable, as is C_1.

We see that $f[C(f) \cap A] \cap V$ is countable and hence, by the properties of the family $\{V\} \cup \{A_\beta : \beta \in \mathfrak{c}\}$ in Proposition 3.1, that $f\bigl[S(f)\bigr]$ does not have cardinality $\mathfrak{c}$. But as noted in the remarks before that proposition this means that $f\bigl[S(f)\bigr]$ is countable.

Thus we see that $f[G] = f[E(f)] \cup f\bigl[S(f)\bigr]$ is countable. $\square$

Now let X and Y be subsets of $\mathfrak{c}$ such that $X \nsubseteq Y$ and take $\alpha \in X \setminus Y$. Then S_X and S_Y satisfy the conditions of Lemma 4.1. Indeed, by definition we have $\mathbb{Q} \cup A_\alpha \subseteq S_X$ and $(A_\alpha \cup V) \cap S_Y = \emptyset$.

The lemma then tells us that every continuous map $s : K_X \to K_Y$ has a countable range, so that K_Y is not a continuous image of K_X.

References

[1] F. Bernstein, *Zur Theorie der trigonometrischen Reihe.*, Leipz. Ber. **60** (1908), 325–338 (German). zbMATH 39.0474.02

[2] Ryszard Engelking, *General topology*, 2nd ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989. Translated from the Polish by the author. MR1039321

[3] Rafał Gruszczyński, *A strictly descending chain of subalgebras of $\mathcal{P}(\omega)/\mathit{fin}$* (March 5, 2023), `https://mathoverflow.net/questions/442111`.

[4] Klaas Pieter Hart and Jan van Mill, *A method for constructing ordered continua*, Topology Appl. **21** (1985), no. 1, 35–49, DOI 10.1016/0166-8641(85)90056-2. MR808722

[5] Sabine Koppelberg, *Handbook of Boolean algebras. Vol. 1*, North-Holland Publishing Co., Amsterdam, 1989. Edited by J. Donald Monk and Robert Bonnet. MR991565

[6] Jan van Mill, *Sierpiński's technique and subsets of* **R**, Topology Appl. **44** (1992), no. 1-3, 241–261, DOI 10.1016/0166-8641(92)90099-L. Proceedings of the Symposium on General Topology and Applications (Oxford, 1989). MR1173263

[7] Wacław Sierpiński, *Sur un problème concernant les types de dimensions*, Fundam. Math. **19** (1932), 65–71, DOI 10.4064/fm-19-1-65-71. zbMATH 0005.19702

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