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Compact spaces with a $\mathbb{P}$-diagonal

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Abstract

We prove that compact Hausdorff spaces with a $\mathbb{P}$-diagonal are metrizable. This answers problem 4.1 (and the equivalent problem 4.12) from Cascales et al. (2011).

Keywords: Compact space; Cantor cube; $\mathfrak{b}$; $\mathfrak{d}$; Diagonal; $\mathbb{P}$-diagonal; $\mathbb{P}$-dominated space; Metrizability

1. Introduction

The purpose of this note is to show that a compact space with a $\mathbb{P}$-diagonal is metrizable.

To explain the meaning of this statement we need to introduce a bit of notation and define a few notions. For a space M (always assumed to be at least completely regular) we let $\mathcal{K}(M)$ denote the family of compact subsets of M. Following [4] we say that a space X is *M-dominated* if there is a cover $\{C_K : K \in \mathcal{K}(M)\}$ of X by compact subsets with the property that $K \subseteq L$ implies $C_K \subseteq C_L$.

In the case that we deal with, namely where M is the space of irrational numbers, we can simplify the cover a bit and make it more amenable to combinatorial treatment. The space of irrationals is homeomorphic to the product space ω^ω, where ω carries the discrete topology. We shall reserve the letter $\mathbb{P}$ for this space.

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The set $\mathbb{P}$ is ordered coordinatewise: $f \leqslant g$ means $(\forall n)(f(n) \leqslant g(n))$. Using this order we simplify the formulation of $\mathbb{P}$-dominated as follows. If K is a compact subset of $\mathbb{P}$ then the function f_K, given by $f_K(n) = \max\{g(n) : g \in K\}$, is well-defined. Using this one can easily verify that a space X is $\mathbb{P}$-dominated iff there is a cover $\langle K_f : f \in \mathbb{P}\rangle$ of X by compact sets such that $f \leqslant g$ implies $K_f \subseteq K_g$. We shall call such a cover *an order-preserving cover by compact sets*.

Finally then we say that a space X has a $\mathbb{P}$-diagonal if the complement of the diagonal, Δ, in X^2 is $\mathbb{P}$-dominated. Problem 4.1 from [2] asks whether a compact space with a $\mathbb{P}$-diagonal is metrizable. The authors of that paper proved that the answer is positive if X is assumed to have countable tightness, and in general if $\mathsf{MA}(\aleph_1)$ is assumed. The latter proof used that assumption to show that X has a small diagonal, which in turn implies that X has countable tightness so that the first result applies. Thus, Problem 4.12 from [2], which asks if a compact space with a $\mathbb{P}$-diagonal has a small diagonal, is a natural reformulation of Problem 4.1.

The property of $\mathbb{P}$-domination arose in the study of the geometry of topological vector spaces; in [1] it was shown that if a locally convex space has a form of $\mathbb{P}$-domination then its compact sets are metrizable. The paper [2] contains more information and results leading up to its Problem 4.1.

The main result of [3] states that compact spaces with a $\mathbb{P}$-diagonal are metrizable under the assumption of the Continuum Hypothesis. The proof establishes that a compact space with a $\mathbb{P}$-diagonal that has *uncountable* tightness maps onto the Tychonoff cube $[0, 1]^{\omega_1}$ and no compact space with a $\mathbb{P}$-diagonal maps onto the cube $[0, 1]^{\mathfrak{c}}$.

The principal result of this paper closes the gap between $\aleph_1$ and $\mathfrak{c}$ by establishing that no compact space with a $\mathbb{P}$-diagonal maps onto $[0, 1]^{\omega_1}$.

Furthermore we would like to point out that Lemma 3 establishes a Baire category type property of 2^{ω_1}: in an order-preserving cover by compact sets there are many members with non-empty interior in the G_δ-topology.

2. Some preliminaries

In the proof of the main lemma, Lemma 3, we need to consider three cases, depending on the values of the familiar cardinals $\mathfrak{b}$ and $\mathfrak{d}$. These are defined in terms of the mod finite order on $\mathbb{P}$: we say $f \leqslant^* g$ if $\{n : g(n) < f(n)\}$ is finite. Then $\mathfrak{b}$ is the minimum size of a subset of $\mathbb{P}$ that is unbounded with respect to $\leqslant^*$, and $\mathfrak{d}$ is the minimum size of a dominating (i.e., cofinal) set with respect to $\leqslant^*$. Interestingly, $\mathfrak{d}$ is also the minimum size of a dominating set with respect to the coordinatewise order $\leqslant$; we shall use this in the proof of the main lemma. We refer to Van Douwen's [6] for more information.

Since we shall be working with the Cantor cube 2^{ω_1} we fix a bit of notation. If I is some subset of ω_1 then $\mathrm{Fn}(I, 2)$ denotes the set of finite partial functions from I to 2. We let $2^{<\omega_1}$ denote the binary tree of countable sequences of zeros and ones. If $s \in \mathrm{Fn}(\omega_1, 2)$ then $[s]$ denotes $\{x \in 2^{\omega_1} : s \subseteq x\}$; the family $\{[s] : s \in \mathrm{Fn}(\omega_1, 2)\}$ is the standard base for the product topology of 2^{ω_1}. Similarly, if $\rho \in 2^{<\omega_1}$ then $[\rho] = \{x \in 2^{\omega_1} : \rho \subseteq x\}$, and the family $\{[\rho] : \rho \in 2^{<\omega_1}\}$ is the standard base for what is called the G_δ-topology on 2^{ω_1}; a set dense with respect to this topology will be called G_δ-dense.

When working with powers of the form I^{ω_1}, where $I = \omega$ or $I = 2$, we use π_δ to denote the projection of I^{ω_1} onto $I^{\omega_1 \setminus \delta}$.

In the proof of Lemma 3 we shall need the following result, due to Todorčević.

Lemma 1 *([5, Theorem 1.3]). If $\mathfrak{b} = \aleph_1$ then ω^{ω_1} has a subset, X, of cardinality $\aleph_1$ such that for every $A \in [X]^{\aleph_1}$ there are $D \in [A]^{\aleph_0}$ and $\delta \in \omega_1$ such that $\pi_\delta[D] = \{d \restriction (\omega_1 \setminus \delta) : d \in D\}$ is dense in $\omega^{\omega_1 \setminus \delta}$.* $\square$

Theorem 1.3 of [5] is actually formulated as a theorem about $\mathfrak{b}$: drop the assumption $\mathfrak{b} = \aleph_1$ and replace every ω_1 and $\aleph_1$ by $\mathfrak{b}$. As explained in [5] this shows that there are natural versions of the S-space problem that do have ZFC solutions.

The lemma also holds with ω replaced by 2, simply map ω^{ω_1} onto 2^{ω_1} by taking all coordinates modulo 2. In that case the density of $\pi_\delta[D]$ can be expressed by saying that for every $s \in \mathrm{Fn}(\omega_1 \setminus \delta, 2)$ the intersection $D \cap [s]$ is nonempty.

3. BIG sets in 2^{ω_1}

Let us call a subset, Y, of 2^{ω_1} BIG if it is compact and projects onto some final product, that is, there is a $\delta \in \omega_1$ such that $\pi_\delta[Y] = 2^{\omega_1 \setminus \delta}$. The latter condition can be expressed without mentioning projections as follows: there is a $\delta \in \omega_1$ such that for every $s \in \mathrm{Fn}(\omega_1 \setminus \delta, 2)$ the intersection $Y \cap [s]$ is nonempty (and a dense set that is closed is equal to the whole space).

BIG sets are also big combinatorially, in the following sense.

Lemma 2. *If Y is a BIG subset of 2^{ω_1} then there is a node ρ in the tree $2^{<\omega_1}$ such that $[\rho] \subseteq Y$.*

Proof. Let Y be BIG and fix a δ witnessing this. After reindexing we can assume $\delta = \omega$ and we let $B_t = \{x \in 2^{\omega_1} : t \subset x\}$ and $Y_t = Y \cap B_t$ for $t \in 2^{<\omega}$.

Starting from $t_0 = \langle\,\rangle$ and $s_0 = \varnothing$ we build a sequence $\langle t_n : n \in \omega\rangle$ in $2^{<\omega}$ and a sequence $\langle s_n : n \in \omega\rangle$ in $\mathrm{Fn}(\omega_1 \setminus \omega, \omega)$ such that $[s_n] \subseteq \pi_\delta[Y_{t_n}]$ for all n.

Given t_n we can choose $i_n < 2$, and set $t_{n+1} = t_n * i_n$, such that $[s_n] \cap \pi_\delta[Y_{t_{n+1}}]$ has nonempty interior. Then choose an extension s_{n+1} of s_n such that $[s_{n+1}] \subseteq \pi_\delta[Y_{t_{n+1}}]$. With a bit of bookkeeping one can ensure that $\bigcup_n \mathrm{dom}\, s_n$ is an initial segment of $\omega_1 \setminus \omega$. We let ρ be the concatenation of $\bigcup_n t_n$ and $\bigcup_n s_n$.

To see that ρ is as required let $x \in [\rho]$. By construction we have $x \in [s_n]$ for all n, so that, again for all n, there is $y_n \in Y_{t_n}$ such that y_n and x agree above $\mathrm{dom}\,\rho$. If $s \in \mathrm{Fn}(\omega_1, 2)$ determines a basic neighborhood of x then there is an m such that $\mathrm{dom}\, s \cap \mathrm{dom}\,\rho$ is a subset of $\mathrm{dom}\, t_m \cup \mathrm{dom}\, s_m$. Then $y_n \in [s]$ for all $n \geqslant m$, so that the sequence $\langle y_n : n \in \omega\rangle$ converges to x, which shows that $x \in Y$. □

4. Existence of BIG sets

It is clear that a compact space is $\mathbb{P}$-dominated: simply let K_f be the whole space for all f. However, in our proof we shall encounter $\mathbb{P}$-dominating covers that may consist of proper subsets. Our next result shows that such a cover of 2^{ω_1} by compact sets must contain a BIG subset.

Lemma 3. *If $\langle K_f : f \in \mathbb{P}\rangle$ is an order-preserving cover of 2^{ω_1} by compact sets then there is an f such that K_f is BIG.*

Proof. We consider three cases.

First we assume $\mathfrak{d} = \aleph_1$. In this case we show outright that there are $\rho \in 2^{<\omega_1}$ and $f \in \mathbb{P}$ such that $[\rho] \subseteq K_f$. Let $\langle f_\alpha : \alpha \in \omega_1\rangle$ be a sequence that is $\leqslant$-dominating.

Working toward a contradiction we assume no ρ and f, as desired, can be found. This implies that for every ρ and every f the intersection $K_f \cap [\rho]$ is nowhere dense in $[\rho]$. Indeed, if such an intersection has interior then there is $s \in \mathrm{Fn}(\omega_1, 2)$ such that $[s] \cap [\rho]$ is nonempty and contained in K_f. It would then be an easy matter to find $\sigma \in 2^{<\omega_1}$ that extends both ρ and s, and then $[\sigma] \subseteq K_f$.

This allows us to choose an increasing sequence $\langle \rho_\alpha : \alpha \in \omega_1 \rangle$ in $2^{<\omega_1}$ such that $[\rho_\alpha] \cap K_{f_\alpha} = \varnothing$ for all α. Then the point $x = \bigcup_\alpha \rho_\alpha$ does not belong to any K_f because the K_{f_α} are cofinal in the whole family.

Next we assume $\mathfrak{d} > \mathfrak{b} = \aleph_1$. We apply $\mathfrak{b} = \aleph_1$ to find a special subset X of 2^{ω_1} as in the comment after Lemma 1. In what follows, when $t \in \omega^{<\omega}$ we let $K(t)$ denote the union $\bigcup\{K_f : t \subseteq f\}$.

We choose an increasing sequence $\langle t_n : n \in \omega \rangle$ in $\omega^{<\omega}$, together with, for each n, an uncountable subset A_n of X, a countable subset D_n of A_n, and $\delta_n \in \omega_1$ such that $A_n \subseteq K(t_n)$ and for all $s \in \mathrm{Fn}(\omega_1 \setminus \delta_n, 2)$ the intersection $D_n \cap [s]$ is nonempty. Simply use that $K(t) = \bigcup_k K(t * k)$ for all t.

Let $\delta = \sup_n \delta_n$ and enumerate each D_n as $\langle d(n, m) : m \in \omega \rangle$.

For each $s \in \mathrm{Fn}(\omega_1 \setminus \delta, 2)$ each D_n intersects $[s]$ so that we can define $h_s \in \omega^\omega$ by $h_s(n) = \min\{m : d(n, m) \in [s]\}$.

By $\mathfrak{d} > \aleph_1$ there is $g \in \omega^\omega$ such that $\{n : h_s(n) < g(n)\}$ is infinite for all s.

Now let $E = \{d(n, m) : m < g(n), n \in \omega\}$ and observe that E meets $[s]$ for every $s \in \mathrm{Fn}(\omega_1 \setminus \delta, 2)$, so that $\pi_\delta[E]$ is dense in $2^{\omega_1 \setminus \delta}$.

For each n there is $f_n \in \mathbb{P}$ that extends t_n and is such that $\{d(n, m) : m < g(n)\}$ is a subset of K_{f_n}. As $f_m(n) = t_{n+1}(n)$ if $m > n$ we may define $f \in \mathbb{P}$ by $f(n) = \max\{f_m(n) : m \in \omega\}$ for all n. Thus we find a single f such that $E \subseteq K_f$, which immediately implies that K_f is BIG.

Our last case is when $\mathfrak{b} > \aleph_1$. We let A be the set of members, t, of $\omega^{<\omega}$ for which there is a $\rho \in 2^{<\omega_1}$ such that $K(t) \cap [\rho]$ is G_δ-dense in $[\rho]$.

As $K(\langle\rangle) = 2^{\omega_1}$ we have $\langle\rangle \in A$.

We show that if $t \in A$, as witnessed by ρ, then there is an m_t such that $t * n \in A$ whenever $n \geqslant m_t$; as $K(t * m) \subseteq K(t * n)$ whenever $m \leqslant n$ it follows that we need to find just one n such that $t * n \in A$. Build, recursively, an increasing sequence $\rho = \rho_0 \subseteq \rho_1 \subseteq \rho_2 \subseteq \cdots$ in $2^{<\omega_1}$ such that $\rho_0 = \rho$ and, if possible, $[\rho_{n+1}] \cap K(t * n) = \varnothing$; if such a ρ_{n+1} cannot be found then $K(t * n) \cap [\rho_n]$ is G_δ-dense in $[\rho_n]$ and we are done. So assume that the recursion does not stop and set $\varrho = \bigcup_n \rho_n$; then $[\varrho]$ is disjoint from $\bigcup_n K(t * n)$, which is equal to $K(t)$. This would contradict G_δ-density of $K(t)$ in $[\rho]$.

We can define $h \in \mathbb{P}$ recursively by $h(n) = m_{h \restriction n}$, together with an increasing sequence $\langle \rho_n : n \in \omega \rangle$ in $2^{<\omega_1}$ such that $K(h \restriction n) \cap [\rho_n]$ is G_δ-dense in $[\rho_n]$. Let $\rho = \bigcup_n \rho_n$, then $K(h \restriction n) \cap [\rho]$ is G_δ-dense in $[\rho]$ for all n.

Let $\delta = \operatorname{dom} \rho$ and let $s \in \mathrm{Fn}(\omega_1 \setminus \delta, 2)$. We know that $K(h \restriction n) \cap [\rho] \cap [s] \neq \varnothing$ for all n. So for every n we can take $h_{s,n} \in \mathbb{P}$ that extends $h \restriction n$ and is such that $K_{h_{s,n}} \cap [\rho] \cap [s] \neq \varnothing$. Because $h_{s,n}(m) = h(m)$ if $n > m$ we can define $h_s \in \mathbb{P}$ by $h_s(m) = \max_n h_{s,n}(m)$.

As $\mathfrak{b} > \aleph_1$ we can find $f \geqslant h$ such that $h_s \leqslant^* f$ for all s. We claim that $K_f \cap [\rho] \cap [s] \neq \varnothing$ for all s, so that $[\rho] \subseteq K_f$ (the closed set $K_f \cap [\rho]$ is dense in $[\rho]$).

To see this take an s and let n be such that $f(m) \geqslant h_s(m)$ for $m \geqslant n$. It follows that $f(m) \geqslant h(m) = h_{s,n}(m)$ for $m \leqslant n$ and $f(m) \geqslant h(m) \geqslant h_{s,n}(m)$ for $m \geqslant n$. This implies that K_f meets $[\rho] \cap [s]$. $\square$

Remark 4. The previous result is valid for all BIG sets: simply work inside $[\rho]$, where ρ is as in the conclusion of Lemma 2.

Remark 5. Lemma 3 generalizes itself to the following situation: let X be compact, let $\varphi : X \to 2^{\omega_1}$ be continuous and onto, and let $\langle K_f : f \in \mathbb{P} \rangle$ be an order-preserving cover of X by compact sets. Then there is an f such that $\varphi[K_f]$ is BIG.

One can go one step further: take a closed subset Y of X such that $\varphi[Y]$ is BIG and conclude that for some $f \in \mathbb{P}$ the image $\varphi[Y \cap K_f]$ is BIG. Simply take ρ such that $[\rho] \subseteq \varphi[Y]$ and work in the compact space $Y \cap \varphi^{\leftarrow}\big[[\rho]\big]$.

Remark 6. The reader may have pondered the need to consider three cases in the proof of Lemma 3. The cases $\mathfrak{d} = \aleph_1$ and $\mathfrak{b} > \aleph_1$ lead to fairly straightforward arguments because each gives one a definite handle on things, be it a cofinal set of size $\aleph_1$ or the knowledge that *all* $\aleph_1$-sized sets are bounded. The intermediate case, with just one unbounded set of size $\aleph_1$, is saved by Todorčević's non-trivial translation of such a set into a subset of 2^{ω_1} that is already quite big.

It would be interesting to see if Lemma 3 can be proved using just one argument.

5. The main result

Now we show that a compact space with a $\mathbb{P}$-diagonal does not admit a continuous map onto $[0, 1]^{\omega_1}$ and deduce our main result.

Theorem 7. *Assume X is a compact space that maps onto 2^{ω_1}. Then X does not have a $\mathbb{P}$-diagonal.*

Proof. Let $\varphi : X \to 2^{\omega_1}$ be continuous and onto. We use Remark 5 and say that a closed subset, Y, of X is BIG if its image $\varphi[Y]$ is. That is, Y is BIG if there is a $\delta \in \omega_1$ such that $Y \cap \varphi^{\leftarrow}\big[[s]\big] \neq \varnothing$ for all $s \in \mathrm{Fn}(\omega_1 \setminus \delta, 2)$.

We observe the following: if Y is BIG, as witnessed by δ, then for every $s \in \mathrm{Fn}(\omega_1 \setminus \delta, 2)$ the intersection $Y \cap \varphi^{\leftarrow}\big[[s]\big]$ is BIG as well; this will be witnessed by any γ that contains the domain of s.

In order to prove our theorem we assume that X does have a $\mathbb{P}$-diagonal, witnessed by $\langle K_f : F \in \mathbb{P}\rangle$, and reach a contradiction.

In order for the final recursion in the proof to succeed we need some preparation. Enumerate $\omega^{<\omega}$ in a one-to-one fashion as $\langle t_n : n \in \omega\rangle$, say in such a way that $t_m \subseteq t_n$ implies $m \leqslant n$ (so that $t_0 = \langle\,\rangle$). We set $Z_0 = X$ and given a BIG set Z_n we determine a BIG set Z_{n+1} as follows. We check if there is a BIG subset Z of Z_n with the property that for *no* point z in Z are there a BIG subset Y of Z and an $f \in \mathbb{P}$ with $t_n \subset f$ such that $\{z\} \times Y \subseteq K_f$. If there is such a Z then every BIG subset of it also has this property so we can pick one that is a proper subset of Z_n and let it be Z_{n+1}; if there is no such Z then $Z_{n+1} = Z_n$. In the end we set $Y = \bigcap_n Z_n$. The set Y is BIG: for each n we have $\gamma_n \in \omega_1$ witnessing BIGness of Z_n, then $\delta_0 = \sup_n \gamma_n$ will witness BIGness of Y.

Pick $y_0 \in Y$, take $i_0 \in 2$ distinct from $\varphi(y_0)(\delta_0)$, let $s_0 = \{\langle \delta_0, i_0\rangle\}$, and set $Y_0 = Y \cap \varphi^{\leftarrow}\big[[s_0]\big]$. By the observation above, Y_0 is BIG. Also: $\varphi(y_0) \notin \varphi[Y_0]$, so that $\{y_0\} \times Y_0$ is disjoint from the diagonal, Δ, of X. By Remark 5 we can find a BIG subset Y_1 of Y_0 and $f_0 \in \mathbb{P}$ such that $\{y_0\} \times Y_1 \subseteq K_{f_0}$.

The point y_0 belongs to all Z_n and for any n such that $t_n \geqslant f_0$ (meaning that $t_n(i) \geqslant f_0(i)$ for $i \in \operatorname{dom} t_n$) it, the point y_0, witnesses that $Z_{n+1} = Z_n$ in the following sense. The reason for having Z_{n+1} be a proper subset of Z_n would be that for all $z \in Z$ and all BIG $Z' \subseteq Z$ and all $f \in \mathbb{P}$ with $t_n \subseteq f$ we would have $\{z\} \times Z' \nsubseteq K_f$. However, y_0 and Y_1 and f_0 show that this did not happen.

The conclusion therefore is that for every such t_n we know that every BIG $Z \subseteq Y$ does have an element z and a BIG subset Z' such that $\{z\} \times Z' \subseteq K_f$ for some $f \in \mathbb{P}$ that extends t_n.

This allows us to construct sequences $\langle y_n : n \in \omega\rangle$ (points in Y), $\langle Y_n : n \in \omega\rangle$ (BIG subsets of Y), and $\langle f_n : n \in \omega\rangle$ (in $\mathbb{P}$) such that

(1) $y_n \in Y_n$, except for $n = 0$,
(2) $Y_{n+1} \subseteq Y_n$,
(3) $\{y_n\} \times Y_{n+1} \subseteq K_{f_n}$,
(4) $f_{n+1} \geqslant f_n$ and $f_{n+1} \supseteq f_n \restriction (n+1)$.

As before we note that $f_m(n) = f_n(n)$ whenever $m \geqslant n$, so we can define a function $f \in \mathbb{P}$ by $f(n) = \max\{f_m(n) : m \in \omega\}$. Note that $f \geqslant f_n$ for all n so that

$$\{y_n\} \times Y_{n+1} \subseteq K_{f_n} \subseteq K_f$$

for all n.

It follows that $\langle y_m, y_n\rangle \in K_f$ whenever $m < n$. This shows that $\langle y_m, y\rangle \in K_f$ whenever $m \in \omega$ and y is a cluster point of $\langle y_n : n \in \omega\rangle$. But then $\langle y, y\rangle \in K_f$ for every cluster point y of $\langle y_n : n \in \omega\rangle$. However, K_f was assumed to be disjoint from the diagonal of X. □

We collect all previous results in the proof of our main theorem.

Theorem 8. *Every compact space with a $\mathbb{P}$-diagonal is metrizable.*

Proof. As noted in the introduction the authors of [3] proved that a non-metrizable compact space with a $\mathbb{P}$-diagonal will map onto the Tychonoff cube $[0, 1]^{\omega_1}$ or, equivalently, that it has a closed subset that maps onto 2^{ω_1}.

However that closed subset would be a compact space with a $\mathbb{P}$-diagonal that *does* map onto 2^{ω_1}. Theorem 7 says that this is impossible. □

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