A problem from Katowice

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This article is about a problem from Set Theory. To understand what it is about you need to know what a bijection is and it helps if you know what groups and rings are. If you want to solve you probably need to learn quite a lot of Set Theory.

A (seemingly) simple problem

Here is an exercise you could find in almost any first-year book on basic mathematics.

Exercise. Given two sets X and Y such that there is a bijection between X and Y. Show that there is a bijection between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

This exercise is not very difficult, as long as you know what a bijection is: a map $f : X \to Y$ with the following two properties:

injective if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, and

surjective for every $y \in Y$ there is an $x \in X$ with $f(x) = y$.

These properties are notorious because many students have a hard time getting them straight in their heads. They are very important in mathematics, however, because many problems can be formulated as searches for injective, surjective or even bijective maps between sets, and where often the maps should have or respect various structural properties.

It is a good exercise to prove that the given bijection $f : X \to Y$ determines a bijection $F : \mathcal{P}(X) \to \mathcal{P}(Y)$, via $F(A) = f[A]$.

If you, as a mathematician, meet a statement as in the exercise then you wonder almost automatically whether the converse is also true; you would expect to see the following exercise as well:

Exercise. Given two sets X and Y such that there is a bijection between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$. Show that there is a bijection between X and Y.

It sounds reasonable that you should be able to take a bijection $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ and manufacture a bijection $f : X \to Y$ out of it. If X and Y are finite that even seems to work.

Indeed, assume X and Y have m and n elements respectively, with $m, n \in \mathbb{N}$. Then we know that $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ have 2^m and 2^n elements respectively. Because F is a bijection we have $2^m = 2^n$ and so $m = n$. Conclusion: there is a bijection between X and Y.

The argument given above does not do what we wanted: the bijection between X and Y is found in a roundabout way; we used extra information: first about the number of elements of the family of subsets of a finite set and second the knowledge that $n \mapsto 2^n$ is strictly increasing. We did not really use the given bijection F.

The reason you will *not* find this exercise in the books is that it is impossible to do: if X and Y are infinite sets then you *can not* outright prove what is asked in the exercise.

This does not mean that the exercise is wrong; there is a proof that there is no proof that the existence of a bijection $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ entails the existence of a bijection $f : X \to Y$. If you want to know how to prove that something is not provable: take a course in *Mathematical Logic*. And in this particular case you need to take a fair amount of Set Theory too.

More structure

Let us give the bijection $F : \mathcal{P}(X) \to \mathcal{P}(Y)$ a bit more structure.

On $\mathcal{P}(X)$ (and $\mathcal{P}(Y)$) we have binary operations: intersection and union. Let us assume that F respects those operations, that is

$$F(A \cap B) = F(A) \cap F(B) \text{ and } F(A \cup B) = F(A) \cup F(B)$$

for any two subsets of X.

Then you can verify quote quickly that the inverse of F also respects intersection and union. In addition you will find that for every $x \in X$ there is $y \in Y$ such that $F(\{x\}) = \{y\}$. That follows using the following property of one-point sets: a set A consists of one point if and only if for every set B we have $A \cap B = A$ or $A \cap B = \emptyset$.

Here then is a bijection $f : X \to Y$: for every $x \in X$ we let $f(x)$ be that y for which $F(\{x\}) = \{y\}$.

Exercise (Doable). Check all the claims made above; especially that f is a bijection.

We see that a bijection-with-structure enables us to draw stronger conclusions.

The problem from Katowice

Here then is the real problem that I want to discuss. It is a problem that arose in General Topology but it can be formulated purely algebraically and there is also a Functional Analytic version. The question was posed in the 1970's in the Topology seminar at the Silezian University in Katowice in Poland, hence the name of this article.

In many first course in Algebra you will learn the theory (and applications) of groups. A notion that is used a lot is that of a quotient group; you make one by taking a subgroup H of a group G and turn the set of *cosets* into a group. This works always if G is Abelian, otherwise we need to take a *normal* subgroup.

We can put a group structure on $\mathcal{P}(X)$ (Abelian even) by defining an addition as follows:

$$A + B = (A \setminus B) \cup (B \setminus A)$$

this is the symmetric difference of course. In this way we do get a group: $\emptyset$ serves as neutral element and for every A we have $A + A = \emptyset$. The *finite* subsets form a subgroup, which we denote *fin*.

Those who know some ring theory: by taking intersection as *multiplication* we turn $\mathcal{P}(X)$ even into a ring and the finite subsets form an *ideal* in that ring. We now consider the quotient group/ring $\mathcal{P}(X)/\mathit{fin}$ and ask the following.

The Katowice problem. Suppose $F : \mathcal{P}(X)/\mathit{fin} \to \mathcal{P}(Y)/\mathit{fin}$ is an isomorphism of rings. Must there be a bijection between X and Y?

This question could have appeared in a book on ring theory if it wasn't for the fact that answering it requires a fair amount of Set Theory, in particular cardinal and ordinal numbers, and some infinitary combinatorics. Oh, and there no complete solution to the problem yet.

What we know

First note that the *nature* of the sets X and Y is not important; we look for bijections only so we can restrict our attention to some standard sets. You can learn what these standard sets look like in just about any course in Set Theory. It suffices for now to know that we can put these standard sets in a (very long) sequence that starts

$$\omega_0, \omega_1, \omega_2, \omega_3, \ldots$$

This sequence is way longer than sequence you meat in Analysis and how it continues is for our problem not important any more, because in 1978 the (then) Czechoslovak mathematician Bohuslav Balcar and the Polish mathematician Ryszard Frankiewicz showed that for any two standard sets the answer to the question is "yes", *except* for the first two sets in our sequence. That does not mean that the answer is "no" ω_0 and ω_1, their proof does not say anything about these two.

This is what I mean by "the solution is not complete": there is one pair of sets for which the problem is not solved, yet. Between distinct standard sets there is no bijection and Balcar and Frankiewicz have proved that for each pair ω_α and ω_β there is no ring isomorphism between $\mathcal{P}(\omega_\alpha)/\mathit{fin}$ and $\mathcal{P}(\omega_\beta)/\mathit{fin}$, except for that one pair $\langle\omega_0, \omega_1\rangle$.

To make things a bit more interesting: the status of the problem is a bit like that of the simple problem from the beginning. What we want to prove is this

$$\text{there is no ring isomorphism between } \mathcal{P}(\omega_0)/\mathit{fin} \text{ and } \mathcal{P}(\omega_1)/\mathit{fin} \qquad (*)$$

Then the problem from Katowice is solved: the answer is an unqualified "yes". What we know thus far is that we can not derive a contradiction from $(*)$; that is, however, not enough to be able to say that we have a proof of $(*)$.

To formulate this properly we need a few terms from *Mathematical Logic*. *Set Theory* is, like most every mathematical discipline a first-order theory, with its own language and axioms. You can find those axioms is every book on the subject, for example in Kunen's book [4].

What we know at present, in logical terms, is that $(*)$ is consistent with the axioms of Set Theory. And that means that the negation of $(*)$,

$$\text{there is a ring isomorphism between } \mathcal{P}(\omega_0)/\mathit{fin} \text{ and } \mathcal{P}(\omega_1)/\mathit{fin} \qquad (**)$$

is not provable.

What we want to know

What we want to know is whether $(*)$ is true, or rather: whether we can prove $(*)$. The attacks of this problem have thus far been of the following form: suppose $(**)$ is true and derive a contradiction. There are quite a few consequences of $(**)$, but none of these has given rise to contradictions and they are not even mutually contradictory.

There are two possibilities: somebody proves $(*)$ or somebody shows that $(**)$ is consistent with the axioms of Set Theory.

Who picks up the gauntlet?

Further reading

In the article [1] you will find the proof of Balcar and Frankiewicz; you need to know some Set Theory to be able to read it. Reference [3] is based on a master project at Leiden University and deals with one of the aforementioned consequences of $(**)$. In [2] (in Dutch) you will find a short history of Set Theory and of a statement that is consistent with its axioms, the Continuum Hypothesis, and that implies $(*)$ and thus shows that $(**)$ is not provable.

Referenties

[1] B. Balcar en R. Frankiewicz, *To distinguish topologically the spaces m^*. II*, Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques, **26** (1978), 521–523.

[2] K. P. Hart, *De ContinuümHypothese*, Nieuw Archief voor Wiskunde, **10** (2009), 33–39

[3] K. P. Hart en H. de Vries, *The Katowice problem and autohomeomorphisms of ω^**, `arXiv:1307.3930 [math.GN]`

[4] K. Kunen, *Set theory. An introduction to independence proofs*, Studies in Logic and the Foundations of Mathematics 102. North-Holland Publishing Co., Amsterdam.