Topology and its Applications 213 (2016) 230–237

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The Katowice problem and autohomeomorphisms of ω_0^*

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A R T I C L E   I N F O

Article history:
Received 14 August 2015
Accepted 2 September 2015
Available online 17 August 2016

The other authors dedicate this paper to Alan, who doesn't look a year over 59

MSC:
primary 54D40
secondary 03E05, 03E35, 06E05, 06E15, 54A35, 54D80

Keywords:
Katowice problem
Homeomorphism
Non-trivial autohomeomorphism
Čech–Stone remainder
ω_0^*
ω_1^*
Isomorphism
Non-trivial automorphism
Quotient algebra
$\mathcal{P}(\omega_0)/\mathit{fin}$
$\mathcal{P}(\omega_1)/\mathit{fin}$

A B S T R A C T

We show that the existence of a homeomorphism between ω_0^* and ω_1^* entails the existence of a non-trivial autohomeomorphism of ω_0^*. This answers Problem 441 in [8].

We also discuss the joint consistency of various consequences of ω_0^* and ω_1^* being homeomorphic.

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[1] Research of this author was supported by the GACR project 15-34700L and RVO: 67985840.
[2] Research of this author was supported by NSF grant No. NSF-DMS-0901168.

http://dx.doi.org/10.1016/j.topol.2016.08.006
0166-8641/© 2016 Published by Elsevier B.V.

0. Introduction

The Katowice problem, posed by Marian Turzański, is about Čech–Stone remainders of discrete spaces. Let κ and λ be two infinite cardinals, endowed with the discrete topology. The Katowice problem asks:

If the remainders κ^* and λ^* are homeomorphic must the cardinals κ and λ be equal?

Since the weight of κ^* is equal to 2^κ it is immediate that the Generalized Continuum Hypothesis implies a yes answer. In joint work Balcar and Frankiewicz established that the answer is actually positive without any additional assumptions, *except possibly for the first two infinite cardinals*. More precisely

Theorem *([1,5]). If $\langle\kappa,\lambda\rangle \neq \langle\aleph_0,\aleph_1\rangle$ and $\kappa < \lambda$ then the remainders κ^* and λ^* are not homeomorphic.*

This leaves open the following problem.

Question. Is it consistent that ω_0^* and ω_1^* are homeomorphic?

Through the years various consequences of "ω_0^* and ω_1^* are homeomorphic" were collected, in the hope that their conjunction would imply $0 = 1$ and thus yield a full positive answer to the Katowice problem.

In the present paper we add another consequence, namely that there is a non-trivial autohomeomorphism of ω_0^*. Whether this is a consequence was asked by Nyikos in [7] (as Problem 441 in the whole volume [8]), right after he mentioned the relatively easy fact that ω_1^* has a non-trivial autohomeomorphism if ω_0^* and ω_1^* are homeomorphic, see the end of Section 1.

After some preliminaries in Section 1 we construct our non-trivial autohomeomorphism of ω_0^* in Section 2. In Section 3 we shall discuss the consequences alluded to above and formulate a structural question related to them; Section 4 contains some consistency results regarding that structural question.

1. Preliminaries

We deal with Čech–Stone compactifications of discrete spaces exclusively. Probably the most direct way of defining $\beta\kappa$, for a cardinal κ with the discrete topology, is as the space of ultrafilters of the Boolean algebra $\mathcal{P}(\kappa)$, as explained in [6] for example.

The remainder $\beta\kappa \setminus \kappa$ is denoted κ^* and we extend the notation A^* to denote $\operatorname{cl} A \cap \kappa^*$ for all subsets of κ. It is well known that $\{A^* : A \subseteq \kappa\}$ is exactly the family of clopen subsets of κ^*.

All relations between sets of the form A^* translate back to the original sets by adding the modifier "modulo finite sets". Thus, $A^* = \emptyset$ iff A is finite, $A^* \subseteq B^*$ iff $A \setminus B$ is finite and so on.

This means that we can also look at our question as an algebraic problem:

Question. Is it consistent that the Boolean algebras $\mathcal{P}(\omega_0)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$ are isomorphic?

Here *fin* denotes the ideal of finite sets. Indeed, the algebraically inclined reader can interpret A^* as the equivalence class of A in the quotient algebra and read the proof in Section 2 below as establishing that there is a non-trivial automorphism of the Boolean algebra $\mathcal{P}(\omega_0)/\mathit{fin}$.

1.1. Auto(homeo)morphisms

It is straightforward to define autohomeomorphisms of spaces of the form κ^*: take a bijection $\sigma : \kappa \to \kappa$ and let it act in the obvious way on the set of ultrafilters to get an autohomeomorphism of $\beta\kappa$ that leaves

κ^* invariant. In fact, if we want to induce an autohomeomorphism on κ^* then it suffices to take a bijection between cofinite subsets of κ.

We shall call an autohomeomorphism of κ^* *trivial* if it is induced in the above way, otherwise we shall call it non-trivial. For example the simple shift $s : n \mapsto n + 1$ on ω_0 determines an autohomeomorphism s^* of ω_0^*.

A non-trivial autohomeomorphism for ω_1^*. For the reader's edification and to give the flavour of the arguments in the next section we prove that the autohomeomorphism s^* of ω_0^*, introduced above, has no non-trivial invariant clopen sets. From this we shall deduce that if ω_0^* and ω_1^* are homeomorphic then ω_1^* must have a non-trivial autohomeomorphism.

Assume $A \subseteq \omega_0$ is such that $s^*[A^*] = A^*$; translated back to ω_0 this means that the symmetric difference of $s[A]$ and A is finite. Let $K \in \omega$ be so large that this symmetric difference is contained in K.

If $k \geqslant K$ and $k \in A$ then $k + 1 \in s[A]$ and hence $k + 1 \in A$, and likewise if $k \geqslant K$ and $k \notin A$ then $k + 1 \notin s[A]$ and hence $k + 1 \notin A$. It follows that if $K \in A$ then $\omega_0 \setminus K \subseteq A$ and so $A^* = \omega_0^*$, and if $K \notin A$ then $A \cap (\omega_0 \setminus K) = \emptyset$ and so $A^* = \emptyset$.

It is an elementary fact about ω_1 that for every subset A of ω_1 and every map $f : A \to \omega_1$ there are uncountably many $\alpha \in \omega_1$ such that $f[A \cap \alpha] \subseteq \alpha$; in particular, if f is a bijection between cofinite sets A and B one has $f[A \cap \alpha] = B \cap \alpha$ for arbitrarily large α. This then implies that trivial autohomeomorphisms of ω_1^* have many non-trivial clopen invariant sets.

And so, if ω_0^* and ω_1^* are homeomorphic then ω_1^* must have a non-trivial autohomeomorphism. This result can be found as Corollary 1 to Theorem 4.1 in [7], where the latter result is credited to [4]. The present argument is probably folklore.

2. A non-trivial auto(homeo)morphism

In this section we prove our main result. We let $\gamma : \omega_0^* \to \omega_1^*$ be a homeomorphism and use it to construct a non-trivial autohomeomorphism of ω_0^*.

We consider the discrete space of cardinality $\aleph_1$ in the guise of $\mathbb{Z} \times \omega_1$. A natural bijection of this set to itself is the shift to the right, defined by $\sigma(n, \alpha) = \langle n + 1, \alpha\rangle$. The restriction, σ^*, of its Čech–Stone extension, $\beta\sigma$, to $(\mathbb{Z} \times \omega_1)^*$ is an autohomeomorphism. We prove that $\rho = \gamma^{-1} \circ \sigma^* \circ \gamma$ is a non-trivial autohomeomorphism of ω_0^*. To this end we assume there is a bijection $g : A \to B$ between cofinite sets that induces ρ and establish a contradiction.

2.1. *Properties of σ^* and $(\mathbb{Z} \times \omega_1)^*$*

We define three types of sets that will be useful in the proof: vertical lines $V_n = \{n\} \times \omega_1$, horizontal lines $H_\alpha = \mathbb{Z} \times \{\alpha\}$ and end sets $E_\alpha = \mathbb{Z} \times [\alpha, \omega_1)$.

These have the following properties.

Claim 2.1.1. *$\sigma^*[V_n^*] = V_{n+1}^*$ for all n.* □

Claim 2.1.2. *$\{H_\alpha^* : \alpha < \omega_1\}$ is a maximal disjoint family of σ^*-invariant clopen sets.*

Proof. It is clear that $\sigma^*[H_\alpha^*] = H_\alpha^*$ for all α.

To establish maximality of the family let $C \subseteq \mathbb{Z} \times \omega_1$ be infinite and such that $C \cap H_\alpha =^* \emptyset$ for all α; then $A = \{\alpha : C \cap H_\alpha \neq \emptyset\}$ is infinite.

For each $\alpha \in A$ let $n_\alpha = \max\{n : \langle n, \alpha\rangle \in C\}$; then $\{\langle n_\alpha + 1, \alpha\rangle : \alpha \in A\}$ is an infinite subset of $\sigma[C] \setminus C$, and hence $\sigma^*[C^*] \neq C^*$. □

Claim 2.1.3. *If $C \subseteq \mathbb{Z} \times \omega_1$ is such that $H^*_\alpha \subseteq C^*$ for uncountably many α then there is a subset S of V_0 such that $S^* \cap E^*_\alpha \neq \emptyset$ for all α and $(\sigma^*)^n[S^*] \subseteq C^*$ for all but finitely many n in $\mathbb{Z}$.*

Proof. For each α such that $H^*_\alpha \subseteq C^*$ let F_α be the finite set $\{n : \langle n, \alpha\rangle \notin C\}$. There are a fixed finite set F and an uncountable subset A of ω_1 such that $F_\alpha = F$ for all $\alpha \in A$; $S = \{0\} \times A$ is as required. □

2.2. Translation to ω_0 and ω_0^*

We choose infinite subsets v_n (for $n \in \mathbb{Z}$), and h_α and e_α (for $\alpha \in \omega_1$) such that for all n and α we have $v^*_n = \gamma^\leftarrow[V^*_n]$, $h^*_\alpha = \gamma^\leftarrow[H^*_\alpha]$, and $e^*_\alpha = \gamma^\leftarrow[E^*_\alpha]$.

Thus we obtain an almost disjoint family $\{v_n : n \in \mathbb{Z}\} \cup \{h_\alpha : \alpha \in \omega_1\}$ with properties analogous to those of the family $\{V_n : n \in \mathbb{Z}\} \cup \{H_\alpha : \alpha \in \omega_1\}$, these are

Claim 2.2.1. *$g[v_n] =^* v_{n+1}$ for all n.* □

Claim 2.2.2. *$\{h^*_\alpha : \alpha < \omega_1\}$ is a maximal disjoint family of g^*-invariant clopen sets.* □

Claim 2.2.3. *If c is such that $h_\alpha \subseteq^* c$ for uncountably many α then there is a subset s of v_0 such that $s \cap e_\alpha$ is infinite for all α and such that $g^n[s] \subseteq^* c$ for all but finitely many n in $\mathbb{Z}$.* □

2.3. Orbits of g

By defining finitely many extra values we can assume that at least one of A and B is equal to ω and, upon replacing σ by its inverse, we may as well assume that $A = \omega$.

For $k \in \omega$ we let $I_k = \{n \in \mathbb{Z} : g^n(k)$ is defined$\}$ and $O_k = \{g^n(k) : n \in I_k\}$ (the orbit of k).

We shall say that a set a splits a set b if both $b \cap a$ and $b \setminus a$ are nonempty.

Claim 2.3.1. *Each h_α splits only finitely many orbits.*

Proof. If h_α splits O_k then there is an $n \in I_k$ such that $g^n(k) \in h_\alpha$ but (at least) one of $g^{n+1}(k)$ and $g^{n-1}(k)$ is not in h_α. So either $g^{n+1}(k) \in g[h_\alpha] \setminus h_\alpha$ or $g^n(k) \in h_\alpha \setminus g[h_\alpha]$.

It follows that each orbit split by h_α meets the symmetric difference of $g[h_\alpha]$ and h_α; as the latter set is finite and orbits are disjoint only finitely many orbits can intersect it. □

We divide ω into two sets: F, the union of all finite g-orbits, and G, the union of all infinite g-orbits.

Claim 2.3.2. *If O_k is infinite then there are at most two αs for which $O_k \cap h_\alpha$ is infinite.*

Proof. First we let $k \in \omega \setminus B$; in this case $I_k = \omega$. The set O^*_k is g^*-invariant, hence $O_k \cap h_\alpha$ is infinite for some α. In fact: $O_k \subseteq^* h_\alpha$ (and so α is unique); for let $J = \{n : g^n(k) \in h_\alpha$ and $g^{n+1}(k) \notin h_\alpha\}$, then $\{g^{n+1}(k) : n \in J\} \subseteq g[h_\alpha] \setminus h_\alpha$ so that J is finite.

It follows that the set $X = \bigcup\{O_k : k \in \omega \setminus B\}$ is, save for a finite set, covered by finitely many of the h_α.

Next let $k \in \omega \setminus X$; in this case $I_k = \mathbb{Z}$ and both sets $\{g^n(k) : n < 0\}^*$ and $\{g^n(k) : n \geqslant 0\}^*$ are g^*-invariant. The argument above applied to both sets yields α_1 and α_2 (possibly identical) such that $\{g^n(k) : n < 0\} \subseteq^* h_{\alpha_1}$ and $\{g^n(k) : n \geqslant 0\} \subseteq^* h_{\alpha_2}$. □

The following claim is the last step towards our final contradiction.

Claim 2.3.3. *For all but countably many α we have $h_\alpha \subseteq^* F$.*

Proof. By Claim 2.3.2 the set D of those α for which h_α meets an infinite orbit in an infinite set is countable: each such orbit meets at most two h_αs and there are only countably many orbits of course.

If $\alpha \notin D$ then h_α meets every infinite orbit in a finite set and it splits only finitely many of these, which means that it intersects only finitely many infinite orbits, and hence that it meets G in a finite set. □

2.4. *The final contradiction*

We now apply Claim 2.2.3 to F. It follows that there is an infinite subset s of v_0 such that $g^n[s] \subseteq^* F$ for all but finitely many n. In fact, as F is g-invariant one n_0 suffices: we can then first assume that $g^{n_0}[s] \subseteq F$ (drop finitely many points from s) and then use g-invariance of F to deduce that $g^n[s] \subseteq F$ for all n.

Let $E = \bigcup_{k\in s} O_k$; as a union of orbits this set is g-invariant. There must therefore be an α such that $E \cap h_\alpha$ is infinite. Now there are infinitely many $k \in E$ such that h_α intersects O_k; by Claim 2.3.1 h_α must contain all but finitely many of these. This means that $O_k \subset h_\alpha$ for infinitely many $k \in s$ and hence that $h_\alpha \cap v_0$ is infinite, which is a contradiction because h_α and v_0 were assumed to be almost disjoint.

2.5. *An alternative contradiction*

For each α the set H_α^* splits into two minimal σ^*-invariant clopen sets, to wit $\{\langle n, \alpha\rangle : n < 0\}^*$ and $\{\langle n, \alpha\rangle : n \geqslant 0\}^*$ (apply the argument in subsection 1.1). Therefore the same is true for each h_α^* with respect to ρ. However, with the notation as above we find infinitely many ρ-invariant clopen subsets of h_α^*, for every infinite subset t of s we can take $(\bigcup_{k\in t} O_k)^*$. Now split s into infinitely many infinite subsets.

3. A question

Our result does not settle the Katowice problem but it may point toward a final solution. We list the known consequences of the existence of a homeomorphism between ω_0^* and ω_1^*.

(1) $2^{\aleph_0} = 2^{\aleph_1}$
(2) $\mathfrak{d} = \aleph_1$
(3) there is a strong-Q-sequence
(4) there is a strictly increasing ω_1-sequence $\mathcal{O}$ of clopen sets in ω_0^* such that $\bigcup \mathcal{O}$ is dense and $\omega_0^* \setminus \bigcup \mathcal{O}$ contains no P-points

The first consequence simply says that the weights of ω_0^* and ω_1^* are equal. Equality (2) was established in [1] as a major step in the proof of the theorem in the Introduction and statement (4) is [7, Theorem 3.5].

To explain (3) we need to define what a strong-Q-sequence is: a sequence $\langle A_\alpha : \alpha \in \omega_1\rangle$ of infinite subsets of ω with the property that for every choice $\langle x_\alpha : \alpha \in \omega_1\rangle$ of subsets ($x_\alpha \subseteq A_\alpha$) there is a single subset x of ω such that $x_\alpha =^* A_\alpha \cap x$ for all α. In [9] Steprāns showed the consistency of the existence of strong-Q-sequences with ZFC.

Not only is each of these consequences consistent with ZFC but in [2] Chodounský provides a model where these consequences hold simultaneously.

We shall now reprove the three structural consequences using the same sets that we employed in the construction of the non-trivial autohomeomorphism. We use the sets v_n to make ω resemble $\mathbb{Z} \times \omega$: first make them pairwise disjoint and then identify v_n with $\{n\} \times \omega$ via some bijection between ω and $\mathbb{Z} \times \omega$.

Our consequences are now obtained as follows:

(2) For every $\alpha < \omega_1$ define $f_\alpha : \mathbb{Z} \to \omega$ by $f_\alpha(m) = \min\{n : \langle m, n\rangle \in e_\alpha\}$; the family $\{f_\alpha : \alpha < \omega_1\}$ witnesses $\mathfrak{d} = \aleph_1$: for every $f : \mathbb{Z} \to \omega$ there is an α such that $\{n : f(n) \geqslant f_\alpha(n)\}$ is finite. Indeed, take A such that $A^* = \gamma\big[\{\langle m, n\rangle : n \leqslant f(m)\}^*\big]$ and observe that there is an α such that $E_\alpha \cap A = \emptyset$.

(3) The family $\{h_\alpha : \alpha \in \omega_1\}$ is a strong-Q-sequence: assume a subset x_α of h_α is given for all α; then there is a single subset x of ω such that $x^* \cap h_\alpha^* = x_\alpha^*$ for all α. To see this take $X_\alpha \subseteq H_\alpha$ such that $X_\alpha^* = \gamma[x_\alpha^*]$ and put $X = \bigcup_\alpha X_\alpha$ then $X \cap H_\alpha = X_\alpha$ and hence $\gamma^{\leftarrow}[X^*] \cap h_\alpha^* = x_\alpha^*$ for all α.
(4) Let b_α be the complement of e_α and let B_α be the complement of E_α. Then $\langle b_\alpha^* : \alpha < \omega_1\rangle$ is the required sequence: in ω_1^* the complement of $\bigcup_\alpha B_\alpha^*$ consists of the uniform ultrafilters on ω_1; none of these is a P-point.

To this list we can now add the existence of a non-trivial auto(homeo)morphism ρ and a disjoint family $\{v_n : n \in \mathbb{Z}\}$ of infinite subsets of ω_0 such that

(5) $\{v_n : n \in \mathbb{Z}\} \cup \{h_\alpha : \alpha < \omega_1\}$ is almost disjoint,
(6) $\rho[v_n^*] = v_{n+1}^*$ for all n,
(7) $\{h_\alpha^* : \alpha < \omega_1\}$ is a maximal disjoint family of ρ-invariant sets, and
(8) for each α the sets $(h_\alpha \cap \bigcup_{n<0} v_n)^*$ and $(h_\alpha \cap \bigcup_{n\geqslant 0} v_n)^*$ are minimal clopen ρ-invariant sets.

Since the family $\{h_\alpha : \alpha < \omega_1\}$ is a strong-Q-sequence one can find for any (uncountable) subset A of ω_1 an infinite set X_A such that $h_\alpha \subseteq^* X_A$ if $\alpha \in A$ and $h_\alpha \cap X_A =^* \emptyset$ if $\alpha \notin A$.

Our proof shows that ρ is in fact non-trivial on every such set X_A whenever A is uncountable.

Remark 3.1. Consequence (1) above is the equality $2^{\aleph_0} = 2^{\aleph_1}$; it does not specify the common value any further. We can actually assume, without loss of generality, that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. Indeed, one can collapse $2^{\aleph_1}$ to $\aleph_2$ by adding a Cohen subset of ω_2; this forcing adds no new subsets of ω_1 of cardinality $\aleph_1$ or less, so any isomorphism between $\mathcal{P}(\omega_0)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$ will survive.

Remark 3.2. It is straightforward to show that the completions of $\mathcal{P}(\omega_0)/\mathit{fin}$ and $\mathcal{P}(\omega_1)/\mathit{fin}$ *are* isomorphic, e.g., by taking maximal almost disjoint families of countable sets in both $\mathcal{P}(\omega_0)$ and $\mathcal{P}(\omega_1)$ of cardinality $\mathfrak{c}$. These represent maximal antichains in the completions consisting of mutually isomorphic elements and a global isomorphism will be the result of combining the local isomorphisms. This argument works for all cardinals κ that satisfy $\kappa^{\aleph_0} = \mathfrak{c}$, that is, for every cardinal κ in the interval $[\aleph_0, \mathfrak{c}]$ the completion of $\mathcal{P}(\kappa)/\mathit{fin}$ is isomorphic to the completion of $\mathcal{P}(\omega_0)/\mathit{fin}$, see [2, Corollary 1.2.7].

Thus, it will most likely be the incompleteness properties of the algebras that decide the outcome of the Katowice problem.

4. Some consistency

To see what is possible consistency-wise we indicate how some of the features of the edifice that we erected, based on the assumption that ω_0^* and ω_1^* are homeomorphic, can occur simultaneously. For this we consider the ideal $\mathcal{I}$ generated by the finite sets together with the sets b_α (the complements of the sets e_α). This ideal satisfies the following properties:

(1) $\mathcal{I}$ is non-meager,
(2) $\mathcal{I}$ intersects every P-point,
(3) $\mathcal{I}$ is generated by the increasing tower $\{b_\alpha : \alpha < \omega_1\}$, and
(4) the differences $b_{\alpha+1} \setminus b_\alpha$ form a strong-Q-sequence.

We have already established properties (2), (3) and (4).

We are left with property (1); that $\mathcal{I}$ must be non-meager was already known to B. Balcar and P. Simon.

We recall that a family of subsets of ω_0 is said to be *meager* if, upon identifying sets with their characteristic functions, it is meager in the product space 2^ω.

Lemma 4.1. *$\mathcal{I}$ is not meager.*

Proof. We assume $\mathcal{I}$ is meager and apply [10, Théorème 21] to find a sequence $\langle F_n : n \in \omega\rangle$ of pairwise disjoint finite sets with the property that $\{n : I \cap F_n \neq \emptyset\}$ is finite whenever $I \in \mathcal{I}$. By contraposition we find that whenever X is an infinite subset of ω the set $F_X = \bigcup_{n\in X} F_n$ does not belong to $\mathcal{I}$; this means that if $\gamma[F_X^*] = G_X^*$ then G_X must be an uncountable subset of $\mathbb{Z} \times \omega_1$.

Fix a family $\{X_s : s \in {}^{<\omega}2\}$ of infinite subsets of ω such that $X_s \supseteq X_t$, and hence $G_{X_s} \supseteq^* G_{X_t}$, whenever $s \subseteq t$, and $X_s \cap X_t = \emptyset$, and hence $G_{X_s} \cap G_{X_t} =^* \emptyset$, whenever s and t are incompatible. Using this we can fix $\alpha \in \omega_1$ such that all exceptions in the previous sentence occur in $\mathbb{Z} \times \alpha$.

So, the family $\{G_{X_s} \cap E_\alpha : s \in {}^{<\omega}2\}$ satisfies the relations above without the modifier 'modulo finite sets'. This implies that if $n \in \mathbb{Z}$ and $\beta \geqslant \alpha$ then there is at most one branch $y_{n,\beta}$ in the binary tree ${}^{<\omega}2$ such that $\langle n, \beta\rangle \in G_{X_s}$ for all $s \in y_{n,\beta}$.

Now, since $2^{\aleph_0} = 2^{\aleph_1}$ there is a branch, y, different from all $y_{n,\beta}$. We can take an infinite set X such that $X \subseteq^* X_s$ for all $s \in y$. This means of course that G_X is uncountable and that $G_X \subseteq^* G_{X_s}$ for all $s \in y$, and hence that there is $\beta \geqslant \alpha$ such that $G_X \setminus G_{X_s} \subseteq \mathbb{Z} \times \beta$ for all s. However, if $\langle n, \gamma\rangle \in G_X$ and $\gamma \geqslant \beta$ then we should have both $\langle n, \gamma\rangle \in \bigcap_{s\in y} G_{X_s}$ by the above and $\langle n, \gamma\rangle \notin \bigcap_{s\in y} G_{X_s}$ because $y \neq y_{n,\gamma}$. □

The methods from [2] and [3] can be used to establish the consistency of $\mathfrak{d} = \aleph_1$ with the existence of an ideal with the properties (1) through (4) of $\mathcal{I}$ — let us call such an ideal countable-like.

Sacrificing completeness for brevity we shall only give a sketch of the proof of the following result, which is Theorem 4.5.1 from [2]. The sketch should be comprehensible to anyone familiar with the various terms employed, such as ω^ω-bounding, Grigorieff forcing etc. We refer the reader in search for definitions and more details to [2].

Theorem 4.2. *It is consistent with* ZFC *that $\mathfrak{d} = \aleph_1$ and there is countable-like ideal $\mathcal{I}$ on ω.*

Proof. We start with a model of $\mathsf{ZFC} + \mathsf{GCH}$ and take an increasing tower $\mathcal{T} = \{T_\alpha : \alpha \in \omega_1\}$ in $\mathcal{P}(\omega)$ that generates a non-meager ideal and let $\mathcal{A}$ denote the almost disjoint family of differences $\{T_{\alpha+1} \setminus T_\alpha : \alpha \in \omega_1\}$ — we write $A_\alpha = T_{\alpha+1} \setminus T_\alpha$. Because of the GCH we can arrange that $\{\omega \setminus T_\alpha : \alpha \in \omega_1\}$ generates a P-point, which more than suffices for our purposes.

We set up an iterated forcing construction, with countable supports, of proper ${}^\omega\omega$-bounding partial orders that will produce a model in which $\mathfrak{d} = \aleph_1$ and the ideal $\mathcal{I}$ generated by $\mathcal{T}$ is countable-like. By the ${}^\omega\omega$-bounding property we get $\mathfrak{d} = \aleph_1$ and the non-meagerness of $\mathcal{I}$ for free.

To turn $\mathcal{A}$ into a strong-Q-sequence we use guided Grigorieff forcing, as in [3]: given a choice $F = \langle F_\alpha : \alpha \in \omega_1\rangle$, where each F_α is a subset of A_α, we let $\mathrm{Gr}(\mathcal{T}, F)$ be the partial order whose elements are functions of the form $p : T_\alpha \to 2$, with the property that $p^{\leftarrow}(1) \cap A_\beta =^* F_\beta$ for all $\beta \leqslant \alpha$. The ordering is by extension: $p \leqslant q$ if $p \supseteq q$. This partial order is proper and ${}^\omega\omega$-bounding and if G is generic on $\mathrm{Gr}(\mathcal{T}, F)$ then $X = (\bigcup G)^{\leftarrow}(1)$ is such that $X \cap A_\alpha =^* F_\alpha$ for all α. As indicated in [3], by appropriate bookkeeping one can set up an iteration that turns $\mathcal{A}$ into a strong-Q-sequence.

One can interleave this iteration with one that destroys all P-points; this establishes property (2) of countable-like ideals in a particularly strong way. For every ideal $\mathcal{I}$ that is dual to a non-meager P-filter one considers the 'normal' Grigorieff partial order $\mathrm{Gr}(\mathcal{I})$ associated to $\mathcal{I}$, which consists of functions with domain in $\mathcal{I}$ and $\{0, 1\}$ as codomain. The power $\mathrm{Gr}(\mathcal{I})^\omega$ is proper and ${}^\omega\omega$-bounding and forcing with it creates countably many sets that prevent the filter dual to $\mathcal{I}$ from being extended to a P-point, even in further extensions by proper ${}^\omega\omega$-bounding partial orders.

All bookkeeping can be arranged so that all potential choices for $\mathcal{A}$ and all potential non-meager P-filters can be dealt with. □

The question arises naturally whether this argument can be adapted so as to include an automorphism of $\mathcal{P}(\omega_0)/fin$ that acts in the same way as our non-trivial automorphism ρ. On the surface this seems unlikely.

The construction has the tendency of going completely in the wrong direction as regards autohomeomorphisms of ω_0^*. As explained in Chapter 5 of [2], if one has an autohomeomorphism φ that is not trivial on any element of the filter dual to $\mathcal{I}$ then the generic filter on $\mathrm{Gr}(\mathcal{I})$ destroys φ in the following sense: there is no possible value for $\varphi(X^*)$, where $X = (\bigcup G)^{\leftarrow}(1)$. The reason is that this value should satisfy $\varphi(p^{\leftarrow}(1)^*) \subseteq \varphi(X^*)$ and $\varphi(p^{\leftarrow}(0)^*) \cap \varphi(X^*) = \emptyset$ for all $p \in G$ and a density argument shows that no such set exists in $V[G]$.

Thus, if things go really wrong, one ends up with a model in which for every non-meager P-filter $\mathcal{F}$ and every autohomeomorphism there is a member of $\mathcal{F}$ on which the autohomeomorphism must be trivial. This would be in contradiction with the last sentence just before Remark 3.1, which says that our ρ is non-trivial on any set that contains uncountably many h_αs.

In fact, Theorem 5.3.12 in [2] shows that by interleaving some extra partial orders in the iteration this ubiquity of triviality can actually be made to happen.

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