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On subcontinua and continuous images of $\beta\mathbb{R} \setminus \mathbb{R}$

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Dedicated with great fondness to the memory of Mary Ellen Rudin

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We prove that the Čech–Stone remainder of the real line has a family of $2^{\mathfrak{c}}$ mutually non-homeomorphic subcontinua.
We also exhibit a consistent example of a first-countable continuum that is not a continuous image of $\mathbb{H}^*$.

0. Introduction

This paper contains two disparate results on $\mathbb{H}^*$, the Čech–Stone remainder of the half line $\mathbb{H} = [0, \infty)$.

We prove that $\mathbb{H}^*$ has a family of $2^{\mathfrak{c}}$ many mutually non-homeomorphic subcontinua. This completes the proof of this fact begun in [4]; in that paper the first-named author showed that $\neg\mathsf{CH}$, the negation of the Continuum Hypothesis, implies that such a family exists, consisting of *decomposable* continua.

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We prove that CH also implies the existence of a family of $2^{\mathfrak{c}}$ many mutually nonhomeomorphic subcontinua as well; in fact, we construct, in one fell swoop, two families: one consisting of indecomposable, the other of decomposable continua.

This suggests the obvious question whether one can construct from ZFC, or even ZFC + ¬CH, a family of $2^{\mathfrak{c}}$ many mutually non-homeomorphic indecomposable subcontinua of $\mathbb{H}^*$.

Our second result concerns continuous images of $\mathbb{H}^*$. There are various parallels between $\mathbb{H}^*$ and ω^* as regards their continuous images. Some of these can be found in [7]: every continuum of weight $\aleph_1$ or less is a continuous image of $\mathbb{H}^*$ and the Continuum Hypothesis implies that the continuous images of $\mathbb{H}^*$ are exactly the continua of weight $\mathfrak{c}$ or less (parallel to Parovičenko's results from [12] on continuous images of ω^*). That not all results carry over was shown in [8]: there is a continuum that is a continuous image of ω^* (it is even separable) that is consistently not a continuous image of $\mathbb{H}^*$. Also, the Open Coloring Axiom implies that $\mathbb{H}^*$ itself is not a continuous image of ω^*, see [6].

We present another parallel, this one of Bell's result from [3] that, consistently, not every first-countable compact space is a continuous image of ω^*. We give a consistent example of a first-countable continuum that is neither a continuous image of ω^* nor one of $\mathbb{H}^*$. The interest in such examples stems from Arhangel'skiĭ's theorem in [1] that compact first-countable spaces have cardinality and hence weight at most $\mathfrak{c}$ and thus are continuous images of ω^* if one assumes CH.

1. Preliminaries

In this section we collect the necessary results on the subcontinua of $\mathbb{H}^*$ that we shall need. We refer to [10] for the necessary proofs and further information.

1.1. An auxiliary space

A useful space to have is the product $\omega \times \mathbb{I}$, which we denote by $\mathbb{M}$. Its Čech–Stone compactification, $\beta\mathbb{M}$, and its remainder, $\mathbb{M}^*$, are very useful in the study of $\beta\mathbb{H}$ and $\mathbb{H}^*$ because there are many continuous maps from both onto their respective counterparts.

The natural projection $\pi : \mathbb{M} \to \omega$ extends to a surjection $\beta\pi : \beta\mathbb{M} \to \beta\omega$; because π is monotone the extension $\beta\pi$ is monotone as well. For $u \in \beta\omega$ we denote the preimage $\beta\pi^{\leftarrow}(u)$ by $\mathbb{I}_u$. For $n \in \omega$ we simply have $\mathbb{I}_n = \{n\} \times \mathbb{I}$ but if $u \in \omega^*$ then $\mathbb{I}_u$ is a continuum that has a few properties that make it resemble $\mathbb{I}$ somewhat.

It has two end points, 0_u and 1_u; these are obtained by intersecting $\mathbb{I}_u$ with the closures of $\omega \times \{0\}$ and $\omega \times \{1\}$ respectively. The continuum $\mathbb{I}_u$ is irreducible between these end points and thus it is divided into layers by the following quasi-order: $x \preccurlyeq y$ iff every subcontinuum of $\mathbb{I}_u$ that contains 0_u and y also contains x. These layers are the equivalence classes under the equivalence relation '$x \preccurlyeq y$ and $y \preccurlyeq x$' and they form an upper semicontinuous decomposition of $\mathbb{I}_u$ with an ordered continuum as its decomposition space.

Many of these layers are one-point sets, for instance: every sequence $\langle x_n : n \in \omega \rangle$ in $\mathbb{I}$ determines a point x_u: the unique point of $\mathbb{I}_u$ that is in the closure of the set $\{\langle n, x_n \rangle : n \in \omega\}$. Each such point is a cut point and the set of these is dense in $\mathbb{I}_u$, and linearly ordered by $\preccurlyeq$. If $\langle x_n : n \in \omega \rangle$ is an increasing sequence in $\mathbb{I}_u$ then its 'supremum' is a single layer that is non-trivial since it contains the accumulation points of $\langle x_n : n \in \omega \rangle$ and these form a set that is homeomorphic to ω^*, because $\mathbb{H}^*$ is an F-space. Also, every layer is an indecomposable continuum; this fact will make some verifications in our construction relatively painless.

1.2. Subcontinua of $\mathbb{H}^*$

We now describe a general construction of subcontinua of $\mathbb{H}^*$. To this end let $\langle [a_n, b_n] : n \in \omega \rangle$ be a sequence of closed intervals in $\mathbb{H}$ such that $b_n = a_{n+1}$ for all n and $\lim_{n\to\infty} a_n = \infty$. Take the map $q : \mathbb{M} \to \mathbb{H}$ defined by $q(n,t) = a_n + t(b_n - a_n)$ for all n and t. This map is almost everywhere one-to-one; the exceptions are at the end points: we always have $q(n,1) = q(n+1,0)$. This behavior persists when we take βq; this map is also almost injective, the exceptions are that $\beta q(1_u) = \beta q(0_{u+1})$ for all u, where $u+1$ is the image of u under the extension of the shift map $n \mapsto n+1$.

For every $u \in \omega^*$ the restriction of βq to $\mathbb{I}_u$ is injective and hence an embedding. We shall denote the image by $[a_u, b_u]$ and refer to such a continuum as a standard subcontinuum of $\mathbb{H}^*$.

These continua determine the structure of the other continua completely: every subcontinuum of $\mathbb{H}^*$ is both the intersection and the union of families of standard subcontinua.

Some work is needed to establish the following fundamental facts:

Lemma 1.1 *([10, Theorem 5.8]). Every decomposable subcontinuum of $\mathbb{H}^*$ is a non-trivial interval in some standard subcontinuum.* $\square$

Lemma 1.2 *([10, Theorem 5.9]). If K and L are subcontinua of $\mathbb{H}^*$ that intersect and if one of these is indecomposable then $K \subseteq L$ or $L \subseteq K$.* $\square$

In particular: if a standard subcontinuum K intersects an indecomposable subcontinuum L then either $K \subseteq L$ and K is nowhere dense in L, or L is contained in a layer of K and hence nowhere dense in K.

Lemma 1.3 *([10, Theorem 5.10]). If K and L are subcontinua of $\mathbb{H}^*$ such that K is a proper subset of L and L is indecomposable then there is a standard subcontinuum M such that $K \subseteq M \subseteq L$.* $\square$

2. Getting the continua

In this section we describe a general construction of indecomposable continua in $\mathbb{H}^*$; in the next section we show that we can actually find $2^{\mathfrak{c}}$ many such continua.

We let Γ denote the collection of all sequences $\langle [a_n, b_n] : n \in \omega \rangle$ of closed intervals in $\mathbb{H}$ with integer end points and such that $b_n = a_{n+1}$ for all n.

As we have seen above, if $A = \langle [a_n, b_n] : n \in \omega \rangle$ is such a sequence then for every free ultrafilter, u, on ω we obtain the standard subcontinuum $[a_u, b_u]$.

We can also associate another subcontinuum to A and an ultrafilter u, as follows. If q is the map from $\mathbb{M}$ to $\mathbb{H}$ associated to A as above then the restriction $\beta q \restriction \mathbb{M}^*$ maps $\mathbb{M}^*$ onto $\mathbb{H}^*$. Therefore there is an ultrafilter v on ω such that $u \in [a_v, b_v]$; this continuum we shall denote by A_u.

Thus each ultrafilter u determines a whole family of continua in $\mathbb{H}^*$, to wit $\mathcal{S}_u = \{A_u : A \in \Gamma\}$.

We shall find $2^{\mathfrak{c}}$ many ultrafilters on ω and for each such ultrafilter u a chain $\mathcal{C}_u$ in $\mathcal{S}_u$. Each chain $\mathcal{C}_u$ gives us an indecomposable continuum, $K_u = \operatorname{cl} \bigcup \mathcal{C}_u$, and our ulterior motive is to have all K_u be mutually non-homeomorphic.

To this end we shall find for each linear order $\langle T, \prec \rangle$ of cardinality $\aleph_1$ an ultrafilter u_T, in fact a P-point, such that T embeds in $\mathcal{S}_{u_T}$ in a special way: there will be a family $\{A^t : t \in T\}$ in Γ such that

(1) $t \prec s$ iff $A^t_{u_T}$ is contained in a layer of $A^s_{u_T}$
(2) every $A \in \Gamma$ is equivalent to some A^t, in a manner to be specified presently.

These two conditions will ensure that a homeomorphism between K_{u_T} and K_{u_S} will give rise to an isomorphism between final segments of T and S. Thus the proof will be finished once we exhibit $2^{\mathfrak{c}}$ many linearly ordered sets without isomorphic final segments.

As mentioned before, the construction proceeds under the assumption of the Continuum Hypothesis.

2.1. Bad triple

The central notion will be that of a bad triple.[2]

A bad triple has three coordinates:

- a free filter base $\mathcal{F}$ on ω,
- a linear order $\langle T, \prec\rangle$, and
- a subset $\mathcal{A}_T = \{A^t : t \in T\}$ of Γ.

These should satisfy the following properties, where, in the interest of readability we write $A(t,n)$ for $[a^t_n, b^t_n]$.

(1) If $s \prec t$ in T then there is $F \in \mathcal{F}$ such that for every k there is an l for which $A(s,k) \cap F \subseteq A(t,l)$
(2) For every decreasing sequence $\langle t_i : i < l\rangle$ in T, for every $m \in \omega$ and every $F \in \mathcal{F}$ there is a function $\varphi : {}^{\leqslant l}m \to \omega$ such that
  (a) if $\rho \in {}^{l}m$ then $\varphi(\rho) \in F$,
  (b) if $\rho \in {}^{<l}m$ then $i \mapsto \varphi(\rho ^\frown i)$ is increasing
  (c) if $k < l$ and $\rho \in {}^{k}m$ then $A\bigl(t_{k+1}, \varphi(\rho ^\frown i)\bigr) \subseteq A\bigl(t_k, \varphi(\rho)\bigr)$ for all $i < m$.

If $\mathcal{F}$ is an ultrafilter then property (1) translates into $A^s_{\mathcal{F}} \subseteq A^t_{\mathcal{F}}$ and property (2) implies that the inclusion is as described above: the (possibly partial) function ψ that satisfies $\psi(k) = l$ iff $A(s,k) \subseteq A(t,l)$ is finite to one, but its fibers have unbounded cardinality, even when restricted to an arbitrary element of $\mathcal{F}$ and this implies that $A^s_{\mathcal{F}}$ is a subset of a layer of $A^t_{\mathcal{F}}$.

Condition (2) will also be seen to keep our recursive constructions alive. To be able to keep our formulations readable we shall say that the function φ in this condition is m-dense for F and $\langle t_i : i < l\rangle$, or for F and $\{t_i : i < l\}$ (set rather than sequence). We shall abbreviate $\{\varphi(\rho) : \rho \in {}^{l}m\}$ as $\operatorname{Im}\varphi$ and refer to it as the image of φ.

The following is a sketch of the construction. Let $\langle T, \prec\rangle$ be a linear order of cardinality $\aleph_1$ and let $\langle t_\alpha : \alpha \in \omega_1\rangle$ be an enumeration of T. By transfinite recursion we construct a sequence $\langle F_\alpha : \alpha \in \omega_1\rangle$ of infinite subsets of ω and a map $t \mapsto A^t$ from T to Γ such that

(1) $F_\beta \subseteq^* F_\alpha$ whenever $\alpha < \beta$
(2) $\langle \mathcal{F}_\alpha, T_\alpha, \mathcal{A}_\alpha\rangle$ is a bad triple, where $\mathcal{F}_\alpha = \{F_\beta : \beta < \alpha\}$, $T_\alpha = \{t_\beta : \beta < \alpha\}$, and $\mathcal{A}_\alpha = \{A^{t_\beta} : \beta < \alpha\}$
(3) $\{F_\alpha : \alpha \in \omega_1\}$ generates an ultrafilter on ω that by (1) will be a P-point.

For technical reasons we add a minimum and a maximum to T, if not already present.

We will formulate and prove a series of lemmas about bad triples that will facilitate such a construction; the standing assumptions in the lemmas will be

(1) $\mathcal{F}$ and T are countable, and $\mathcal{F}$ extends the cofinite filter,
(2) T has a minimum and a maximum, denoted 0 and 1 respectively, and
(3) $\bigl\langle [a^0_n, b^0_n] : n \in \omega\bigr\rangle = \bigl\langle [n, n+1] : n \in \omega\bigr\rangle$.

[2] The word ‘good’ seems overused and, especially in the vernacular, ‘bad’ may carry a positive connotation

To begin we show that at any time during our construction we can assume that $\mathcal{F}$ is a principal filter, or rather, the restriction of the cofinite filter to a single set; in the next lemma cof_G denotes the filter $\{F : G \subseteq^* F\}$.

Lemma 2.1. *If $\langle \mathcal{F}, T, \mathcal{A}_T \rangle$ is a bad triple then there is a single infinite G such that $G \subseteq^* F$ for all $F \in \mathcal{F}$ and such that $\langle cof_G, T, \mathcal{A}_T \rangle$ is a bad triple.*

Proof. Let $\langle T_n : n \in \omega \rangle$ be an increasing sequence of finite sets whose union is T and let $\langle F_n : n \in \omega \rangle$ be a decreasing sequence in $\mathcal{F}$ such that for every $F \in \mathcal{F}$ there is an n such that $F_n \subseteq F$. Recursively let φ_m be m-dense for F_m and T_m and such that $\operatorname{Im} \varphi_m$ is disjoint from $\operatorname{Im} \varphi_i$ for $i < m$. Then $G = \bigcup_{m \in \omega} \operatorname{Im} \varphi_m$ is as required. □

This lemma is used at limit steps of our construction, basically to make them look like successor steps. For the remainder of the paper we shall write $\langle G, T, \mathcal{A}_T \rangle$ for $\langle cof_G, T, \mathcal{A}_T \rangle$.

At some steps in the construction the following technical fact will be useful.

Lemma 2.2. *A triple $\langle F, T, \mathcal{A}_T \rangle$ satisfies (2) in the definition of a bad triple if and only if for every (some) increasing sequence $\langle m_n : n \in \omega \rangle$ in ω and every (some) increasing sequence $\langle T_n : n \in \omega \rangle$ finite subsets of T such that $T = \bigcup_{n \in \omega} T_n$ there is a sequence $\langle \varphi_n : n \in \omega \rangle$ of functions such that φ_n is m_n-dense for F and T_n, and* $\max \operatorname{Im} \varphi_n < \min \operatorname{Im} \varphi_{n+1}$ *for all n.*

Proof. For the non-trivial implication we find the functions φ_n by recursion: φ_0 exists by assumption and if φ_n is found then we let $M = \max \operatorname{Im} \varphi_n$ and we choose a function φ that is $M + m_{n+1} + 1$-dense for F and T_{n+1}. By condition (2b) in the definition of a bad triple we have $\varphi(M+1+\rho) > M$ whenever $\rho \in {}^i m_{n+1}$ for some $i \leqslant |T_{n+1}|$ (here $M + 1 + \rho$ denotes the sequence obtained by adding $M + 1$ to all values of ρ). Thus defining $\varphi_{n+1}(\rho) = \varphi(M + 1 + \rho)$ gives us our next function. □

The next lemma ensures that we can make our final filter an ultrafilter.

Lemma 2.3. *Let $\langle F, T, \mathcal{A}_T \rangle$ be a bad triple and assume $F = F_0 \cup F_1$; then at least one of $\langle F_0, T, \mathcal{A}_T \rangle$ and $\langle F_1, T, \mathcal{A}_T \rangle$ is a bad triple.*

Proof. We show by induction on l: if $\langle t_i : i < l \rangle$ is decreasing and φ is $2m$-dense for F and $\langle t_i : i < l \rangle$ then φ induces an m-dense function for F_0 or F_1 and $\langle t_i : i < l \rangle$.

If $l = 1$ then $\operatorname{Im} \varphi$ is just a $2m$-element subset of F and its intersection with one of F_0 and F_1 has at least m elements; the increasing enumeration of that intersection is m-dense.

In the step from l to $l + 1$ we let $\langle t_i : i \leqslant l \rangle$ and a $2m$-dense φ be given. For each $j < 2m$ the function $\varphi_j : {}^{\leqslant l}2m \to \omega$, defined by $\varphi_j(\rho) = \varphi(j \frown \rho)$, is $2m$-dense for F and $\langle t_i : 1 \leqslant i \leqslant l \rangle$ and so induces an m-dense function φ'_j for F_{ϵ_j} and $\langle t_i : 1 \leqslant i \leqslant l \rangle$, where $\epsilon_j \in \{0, 1\}$. Take ϵ such that $A = \{j : \epsilon_j = \epsilon\}$ has size at least m and define $\varphi' : {}^{\leqslant l+1}m \to \omega$ by '$\varphi'(\langle j \rangle)$ is the jth element of A' and $\varphi'(j \frown \rho) = \varphi'_{\varphi'(\langle j \rangle)}(\rho)$ for $\rho \in {}^{\leqslant l}m$.

Now enumerate T as $\langle t_n : n \in \omega \rangle$ and apply the above for each m to the pair $\langle t_i : i < m \rangle$ and m. Whichever of F_0 and F_1 appears infinitely often in the conclusion is the set that we seek. □

Now we show how to extend the ordered set T by one element.

Lemma 2.4. *Let $\langle F, T, \mathcal{A}_T \rangle$ be a bad triple and let t^* be a point not in T. Assume $T \cup \{t^*\}$ is ordered so that T retains its original order and $0 \prec t^* \prec 1$. Then there are $G \subseteq F$ and $A^{t^*} \in \Gamma$ such that $\langle G, \mathcal{A}_T \cup \{A^{t^*}\}, T \cup \{t^*\} \rangle$ is a bad triple.*

Proof. We write T as an increasing union of finite sets T_m, with $0, 1 \in T_0$ and we construct G and A^{t^*} as follows. We apply Lemma 2.2 to find a sequence $\langle \varphi_m : m \in \omega \rangle$ such that φ_m is m^2-dense for F and T_m, and $\max \operatorname{Im} \varphi_m < \min \operatorname{Im} \varphi_{m+1}$ for all m.

We fix m for the moment and let $\langle t_i : i < l \rangle$ enumerate T_m in decreasing order and let i be such that $t_{i+1} \prec t^* \prec t_i$. Our task is to convert φ_m into an m-dense function, ψ_m, for our future G and $T_m \cup \{t^*\}$. The idea is simple — we use level $i+1$ in $\operatorname{dom} \varphi_m$ to create two levels in $\operatorname{dom} \psi_m$ — but the notation is a bit messy: we take the following subset of the domain of φ_m:

$$D = \{\rho \in \operatorname{dom} \varphi_m : (\forall j \in \operatorname{dom} \rho)(j \neq i \Rightarrow \rho(j) < m)\}$$

Using the m^2 values for all $\rho(i)$ we transform D into the tree ${}^{\leqslant l+1}m$:

- if $\operatorname{dom} \rho \leqslant i$ then ρ does not change;
- if $\operatorname{dom} \rho = i + 1$ then $\rho = \rho' \frown (km + j)$ for some $\rho' \in {}^i m$ and $k, j < m$; in this case ρ determines two nodes: $\rho^+ = \rho' \frown k$ and $\rho^{++} = \rho' \frown k \frown j$
- if $i+1 < \operatorname{dom} \rho$ then $\rho = \rho' \frown (km + j) \frown \sigma$ for some $\rho' \in {}^i m$, some $k, j < m$ and some sequence σ; then ρ determines $\rho^+ = \rho' \frown k \frown j \frown \sigma$.

We define $\psi_m : {}^{\leqslant l+1}m \to \omega$ by

$$\psi_m(\varrho) = \begin{cases} \varphi_m(\varrho) & \text{if } \operatorname{dom} \varrho \leqslant i \\ \varphi_m(\rho) & \text{if } \varrho = \rho^{++} \text{ for some } \rho \in {}^{i+1}m \\ \varphi_m(\rho) & \text{if } \varrho = \rho^{+} \text{ for some } \rho \text{ with } \operatorname{dom} \rho > i+1 \end{cases}$$

This leaves $\psi_m(\varrho)$ undefined in case $\operatorname{dom} \varrho = i + 1$, that is, if $\varrho = \rho \frown k$ for some $\rho \in {}^i m$ and $k < m$, and it is here that we build and insert part of A^{t^*}.

In words: for each $\rho \in {}^i m$ we bundle the m^2 intervals $[a^{t_{i+1}}_{\varphi_m(\rho \frown j)}, b^{t_{i+1}}_{\varphi_m(\rho \frown j)}]$ into groups of m consecutive ones and for each group take the smallest interval that surrounds its members.

In symbols: for each $k < m$ the interval $[a^{t_{i+1}}_{\varphi_m(\rho \frown (km))}, b^{t_{i+1}}_{\varphi_m(\rho \frown ((k+1)m-1))}]$ will be a term of A^{t^*} and its index will be the value of ψ_m at $\rho \frown k$.

In the end we set $G = \bigcup_m \psi_m$ and thus ensure that for every m the function ψ_m will be m-dense for G and $T_m \cup \{t^*\}$. $\square$

We now turn to the task of avoiding having to add points to our linear order when we do not want to, that is, we want ensure that we can achieve property (2) (on page 95) of the embedding. It is here that we define the notion of equivalence, promised in that property.

We introduce some notation: let $F \subseteq \omega$ and let $A, B \in \Gamma$.

We say that A refines B modulo F, and we write $A \preccurlyeq_F B$, if for every term, $[a, b]$, of A with $[a, b] \cap F \neq \emptyset$ there is a term $[c, d]$ of B such that $[a, b] \cap F \subseteq [c, d]$.

We say that A and B are equivalent modulo F, written $A \equiv_F B$, if for every $n \in F$ there are terms $[a, b]$ of A and $[c, d]$ of B such that $n \in [a, b] \cap F$ and $[a, b] \cap F = [c, d] \cap F$.

Lemma 2.5. *Let $\langle F, T, \mathcal{A}_T \rangle$ be a bad triple, let $t \in T$ and $A \in \Gamma$. Then there is $F_t \subseteq F$ such that $\langle F_t, T, \mathcal{A}_T \rangle$ is a bad triple and $A \preccurlyeq_{F_t} A^t$ or $A^t \preccurlyeq_{F_t} A$; in addition if t has a direct $\prec$-predecessor s then we can even achieve "$A \preccurlyeq_{F_t} A^s$ or $A^t \preccurlyeq_{F_t} A$".*

Proof. Write T as the union of an increasing sequence $\langle T_m : m \in \omega \rangle$ of finite sets such that $0, 1, t \in T_0$ (and also $s \in T_0$ if present). Upon applying Lemmas 2.3 and 2.2 we may assume that F does not meet consecutive intervals of A^t, and that we have a sequence $\langle \varphi_m : m \in \omega \rangle$ of functions such that φ_m is $(m+1)(m+2)$-dense for F and T_m, and $\max \operatorname{Im} \varphi_m < \min \operatorname{Im} \varphi_{m+1}$ for all m. We also assume $F = \bigcup_{m\in\omega} \operatorname{Im} \varphi_m$.

Enumerate T_m in decreasing order as $\langle t_i^m < l_m \rangle$, and for every m let i_m be the index of t. Abbreviate $t_{i_m}^m$ as t_m and $t_{i_m+1}^m$ as s_m (so $s_m = s$ for all m if s is present).

We fix m for a moment and for every $\rho \in {}^{i_m}((m+1)(m+2))$ we take a term $[a_\rho^m, b_\rho^m]$ of A such that

$$J_\rho^m = \{ j < (m+1)(m+2) : A\big(s_m, \varphi_m(\rho \frown j)\big) \subseteq [a_\rho^m, b_\rho^m] \}$$

has maximum cardinality. Divide ${}^{i_m}((m+1)(m+2))$ into two parts: $R_m = \{\rho : |J_\rho^m| \geqslant m\}$ and its complement S_m.

The proof of Lemma 2.3 gives us a subfunction ϕ_m of $\varphi_m \restriction {}^{\leqslant i_m}((m+1)(m+2))$ whose domain is $(m+1)(m+2)/2$-branching and such that $X_m = \operatorname{dom} \phi_m \cap {}^{i_m}((m+1)(m+2))$ is a subset of R_m or of S_m.

In case $X_m \subseteq R_m$ we define a set F_t^m as follows:

$$F_t^m = F \cap \bigcup \{ A\big(s_m, \varphi(\rho \frown j)\big) : \rho \in X_m \text{ and } j \in J_\rho^m \}$$

We extend ϕ_m to a subfunction ψ_m of φ_m by adding

$$\{ \rho \in \operatorname{dom} \varphi_m : (\exists \sigma \in X_m)(\exists j \in J_\sigma^m)(\sigma \frown j \subseteq \rho) \}$$

to its domain and using the values of φ_m at those points. The resulting function is (more than) m-dense for F_t^m and T_m. Also, if $n \in F_t^m$ then there are $\rho \in X_m$ and $j \in J_\rho^m$ such that $n \in A\big(s_m, \varphi(\rho \frown j)\big) \subseteq A\big(t, \varphi(\rho)\big)$ and, by definition, $F_t^m \cap A\big(t, \varphi(\rho)\big) \subseteq [a_\rho^m, b_\rho^m]$. This shows that if F_t^m were to contribute to F_t it would also witness $A^t \preccurlyeq_{F_t} A$.

Thus, if the situation $X_m \subseteq R_m$ occurs infinitely often then we can build an F_t such that $A^t \preccurlyeq_{F_t} A$.

In the other case we get $X_m \subseteq S_m$ infinitely (even cofinitely) often. We shall build an F_t that will satisfy $A \preccurlyeq_{F_t} A^t$ and even $A \preccurlyeq_{F_t} A^s$ if s is present.

Consider an m such that $X_m \subseteq S_m$ and fix $\rho \in X_m$. For each term $[a, b]$ of A the set $\{j : A\big(s_m, \varphi(\rho \frown j)\big) \subseteq [a, b]\}$ has at most $m - 1$ elements; as $[a, b]$ is an interval these are consecutive elements. This means that $[a, b]$ can intersect at most $m + 1$ of these intervals: at most $m - 1$ in the interior and possibly two more that merely overlap at the ends. We use the intervals indexed by X_m and $I = \{(m + 2)(j + 1) : j < m\}$ to define F_t^m:

$$F_t^m = F \cap \bigcup \{ A\big(s_m, \varphi(\rho \frown i)\big) : \rho \in X_m \text{ and } i \in I \}$$

the same formula as in the case '$X_m \subseteq R_m$' with J_ρ^m replaced by I. Now if $[a, b]$ is a term of A and $n \in F_t^m \cap [a, b]$ then there are one $\rho \in X_m$ and one $i \in I$ such that $n \in A\big(s_m, \varphi(\rho \frown i)\big)$ and the latter is also the only interval of that form that $[a, b]$ intersects. It follows automatically that

$$F_t^m \cap [a, b] \subseteq A\big(s_m, \varphi(\rho \frown i)\big) \subseteq A\big(t, \varphi_m(\rho)\big).$$

Thus, if we let F_t be the union of these F_t^m then we achieve $A \preccurlyeq_{F_t} A^t$ and even $A \preccurlyeq_{F_t} A^s$ if s is present. $\square$

Lemma 2.6. *Let $\langle F, T, \mathcal{A}_T \rangle$ be a bad triple and $A \in \Gamma$. Then there are $G \subseteq F$ and an extension T^* of T by at most one point t^* such that $\langle G, T^*, \mathcal{A}_{T^*} \rangle$ is a bad triple and $A \equiv_G A^t$ for some $t \in T^*$.*

Proof. We apply Lemma 2.5 countably many times and Lemma 2.1 once so that we can assume that for every $t \in T$ there is a cofinite subset F_t of F such that $A \preccurlyeq_{F_t} A^t$ or $A^t \preccurlyeq_{F_t} A$ and even $A \preccurlyeq_{F_t} A^s$ or $A^t \preccurlyeq_{F_t} A$ if t has a direct $\prec$-predecessor s.

We divide T into $S_0 = \{t : A^t \preccurlyeq_{F_t} A\}$ and $S_1 = \{t : A \preccurlyeq_{F_t} A^t\}$. Note that $0 \in S_0$ by default.

We need to consider several cases.

Case 1: *S_0 has a maximum and S_1 has a minimum.* Note that by the condition on direct predecessors these must be identical, say $t = \max S_0 = \min S_1$. Then one verifies that $A \equiv_{F_t} A^t$.

Case 2: *S_1 is empty.* In this case we have $A^1 \preccurlyeq_F A$ and we can thin out F to a set G such that $A^1 \equiv_G A$; then $\langle G, T, \mathcal{A}_T\rangle$ is a bad triple.

For the other cases we write T as the union of an increasing sequence $\langle T_m : m \in \omega\rangle$ of finite sets such that $0, 1 \in T_0$; as before we take the decreasing enumeration $\langle t_i^m : I < l_m\rangle$ of T_m. For each m we let i_m be such that $t_{i_m}^m \in S_1$ and $t_{i_m+1}^m \in S_0$; we denote these two points by t_m and s_m respectively.

Furthermore we choose $\langle \varphi_m : m \in \omega\rangle$ as in Lemma 2.2 so that φ_m is m-dense for F and T_m and such that $\operatorname{Im}\varphi_m \subseteq F_{t_m} \cap F_{s_m}$.

Fix m for a moment. We know that $A^{s_m} \preccurlyeq_{F_{s_m}} A \preccurlyeq_{F_{t_m}} A^{t_m}$; this implies that for every $\rho \in {}^{i_m}m$ and every $j < m$ there is a term $[a, b]$ of A such that

$$A\bigl(s_m, \varphi(\rho ^\frown j)\bigr) \cap F \subseteq [a, b] \cap F \subseteq A\bigl(t_m, \varphi(\rho)\bigr) \cap F \tag{*}$$

indeed, $[a, b]$ is found by an application of $A^{s_m} \preccurlyeq_{F_{s_m}} A$ and $A\bigl(t_m, \varphi(\rho)\bigr)$ is the only possible term of A^{t_m} that can help witness $A \preccurlyeq_{F_{t_m}} A^{t_m}$.

We put $G_m = F \cap \bigcup_\rho A\bigl(s_m, \varphi(\rho ^\frown 0)\bigr)$, where ρ runs through ${}^{i_m}m$. We can define two functions ϕ_m and ψ_m on ${}^{\leqslant l_m - 1}m$, as follows.

(1) If $|\rho| < i_m$ then $\phi_m(\rho) = \psi_m(\rho) = \varphi_m(\rho)$.
(2) If $|\rho| = i_m$ then $\phi_m(\rho) = \varphi_m(\rho)$ and $\psi_m(\rho) = \varphi_m(\rho ^\frown 0)$.
(3) If $|\rho| > i_m$, say $\rho = \varrho ^\frown \sigma$, with $|\varrho| = i_m$, then $\phi_m(\rho) = \psi_m(\rho) = \varphi(\varrho ^\frown 0 ^\frown \sigma)$.

So, in ϕ_m we skip level $i_m + 1$ of the domain of φ_m and in ψ_m we skip level i_m. The effect is that ϕ_m is m-dense for $T_m \setminus \{s_m\}$ and G_m, whereas ψ_m is m-dense for $T_m \setminus \{t_m\}$ and G_m.

In addition we have made sure that $A^{s_m} \equiv_{G_m} A \equiv_{G_m} A^{t_m}$.

We let $G = \bigcup_m G_m$ and consider the remaining cases in turn.

Case 3: *S_0 has no maximum and S_1 has a minimum, say $s = \min S_1$.* In this case we know that $s_m = s$ cofinitely often. If we drop the finitely many G_m for which $s_m \neq s$ then we achieve $A \equiv_G A^t$. Moreover $\langle G, T, \mathcal{A}_T\rangle$ is a bad triple, as witnessed by the functions ϕ_m.

Case 4: *S_0 has a maximum and S_1 has no minimum, say $t = \max S_0$.* In this case we know that $t_m = s$ cofinitely often. If we drop the finitely many G_m for which $s \neq t_m$ then we achieve $A^s \equiv_G A$. Moreover $\langle G, T, \mathcal{A}_T\rangle$ is a bad triple, as witnessed by the functions ψ_m.

Case 5: *S_0 has no maximum and S_1 has no minimum.* This case necessitates adding a new point, t^*, to T and inserting it into the gap formed by S_0 and S_1 to form T^*. We then redefine ϕ_m on level i_m so that its value at ρ becomes the index of the term of A that was chosen to satisfy inclusions (*). The new ϕ_m is m-dense for $\{t^*\} \cup T_m \setminus \{s_m, t_m\}$ and G_m; this establishes that $\langle G, T^*, \mathcal{A}_{T^*}\rangle$ is a bad triple. □

Repeated application of these lemmas will prove the following theorem, where we extend the notion of equivalence to (ultra)filters: if p is an (ultra)filter on ω then $A \equiv_p B$ means that $A \equiv_F B$ for some $F \in p$.

Theorem 2.7 (CH). *Let T be a linear order of cardinality at most $\aleph_1$ that has a maximum and no $\langle\omega, \omega\rangle$-gaps. Then one can find a subcollection $\mathcal{A}_T = \{A^t : t \in T\}$ of Γ and a P-point ultrafilter p on ω such that*

(1) *$\langle p, T, \mathcal{A}_T \rangle$ is a bad triple*
(2) *for all $A \in \Gamma$, there is a $t \in T$ such that $A \equiv_p A^t$.* □

3. Finding many different continua

In this section we shall use Theorem 2.7 (and hence the Continuum Hypothesis) to find $2^{\mathfrak{c}}$ many different subcontinua of $\mathbb{H}^*$.

We shall apply the theorem to the following type of linearly ordered sets

(1) cardinality at most $\aleph_1$
(2) no $\langle \omega, \omega \rangle$-gaps
(3) cofinality $\aleph_0$ (in particular: no maximum)

In keeping with our use of the vernacular we shall call this a *mean* linear order.

3.1. One continuum

Let T be a mean linear order. We order $T^+ = T \cup \{T\}$ ordered by stipulating that $t \prec T$ for all $t \in T$. We apply Theorem 2.7 to T^+ to obtain a family $\mathcal{A}_T = \{A^t_p : t \in T^+\}$ and a P-point p satisfying the conditions of that theorem. We define

$$K_T = \operatorname{cl} \bigcup_{t \in T} A^t_p,$$

as announced in the beginning of Section 2.

We list some properties of K_T and the individual continua A^t_p.

Lemma 3.1. *For every $t \neq \min T$ there is a layer L^t_p of A^t_p such that $\bigcup_{s \prec t} A^s_p \subseteq L^t_p$.*

Proof. Lemma 6.2 of [10] establishes that A^s_p is contained in a layer of A^t_p whenever $s \prec t$; because $\mathcal{A}_T$ is a chain this layer is independent of s. We need the assumption $t \neq \min T$ to ensure that we actually have points below t. □

Lemma 3.2. *Every A^t_p is nowhere dense in K_T and $\bigcup_{t \in T} L^t_p = \bigcup_{t \in T} A^t_p$.*

Proof. Given $t \in T$ there is $s \in T$ such that $t \prec s$. Then $A^t_p \subseteq L_s$, which establishes the equality of the two unions.

Because L_s is nowhere dense in A^s_p this also implies that A^t_p is nowhere dense in K_T. □

Lemma 3.3. *K_T is indecomposable.*

Proof. The proof is implicit in [14] and [10] as part of a construction of an indecomposable subcontinuum of $\mathbb{H}^*$ called K_9 in the latter paper.

Let L be a proper subcontinuum of K_T. Note that because each L^t_p is indecomposable we know that $L^t_p \subseteq L$ or $L \subseteq L^t_p$ for all t such that $L \cap L^t_p$ is nonempty. Since it is impossible that $L^t_p \subseteq L$ for all t (otherwise $L = K_T$) it follows that $L \cap \bigcup_{t \in T} L^t_p = \emptyset$ or $L \subseteq L^t_p$ for some t. In either case L is nowhere dense in K_T. □

Lemma 3.4. *Every A^t_p is a P-set in $\mathbb{H}^*$ as is every L^t_p, for $t \neq \min T$.*

Proof. The preimage of A_p^t under the parametrizing map $q : \mathbb{M}^* \to \mathbb{H}^*$ consists of $\mathbb{I}_v$, the point 1_{v-1} and the point 0_{v+1}, where v is such that $A_p^t = [a_v^t, b_v^t]$. This makes the preimage a P-set, as π is closed this implies that A_p^t is a P-set as well.

It suffices to show that L_p^t is not a countable cofinality layer in A_p^t if $t \neq \min T$. If L_p^t were such a layer then one of the open intervals with L_p^t as its end layer, call it I, would be an F_σ-set such that $I \cap L_p^t = \emptyset$ and $L_p^t \subseteq \operatorname{cl} I$. Now let $s \prec t$; then A_p^s is a P-set and $A_p^s \cap I = \emptyset$. It follows that $A_p^s \cap \operatorname{cl} I = \emptyset$ as well, which contradicts $L_p^t \subseteq \operatorname{cl} I$. $\square$

3.2. Consequences of homeomorphy

Let T and S be two mean linear orders. We assume we have families $\mathcal{A}_T$ and $\mathcal{A}_S$ and P-points p and q respectively as in Theorem 2.7. We write $F_T = \bigcup_{t\in T} A_p^t$ and $F_S = \bigcup_{s\in S} A_q^s$ and let $K_T = \operatorname{cl} F_T$ and $K_S = \operatorname{cl} F_S$. We retain the notations L_p^t and L_q^t respectively for the layers from Lemma 3.1. We assume that K_T and K_S are homeomorphic and let $f : K_T \to K_S$ be a homeomorphism.

Lemma 3.5. $f[F_T] = F_S$.

Proof. Let $t \in T$. Because the P-set $f[A_p^t]$ is in the closure of the F_σ-set F_S it must actually intersect that set. Thus there is an $s \in S$ such that $f[A_p^t] \cap A_q^s \neq \emptyset$ and hence $f[A_p^t] \cap L_q^r \neq \emptyset$ whenever $s \prec r$ in S. It follows that $f[A_p^t] \subseteq L_q^r$ or $L_q^r \subseteq f[A_p^t]$ for all $r \succ s$ and because $f[A_p^t]$ is nowhere dense in K_S we must have $f[A_p^t] \subseteq L_q^r$ for a final segment of r in S.

This shows that $f[F_T] \subseteq F_S$ and, using f^{-1} instead of f, we can also deduce that $F_S \subseteq f[F_T]$. Thus we find that F_T is mapped onto F_S by f. $\square$

Our aim is now to show that T and S have isomorphic final segments.

Let $T' = \{t \in T : (\exists s \in S)(A_q^s \subseteq f[L_p^t])\}$ and, symmetrically, let $S' = \{s \in S : (\exists t \in T)(f[A_p^t] \subseteq L_q^s)\}$. We shall show that T' and S' are isomorphic by showing that f induces an isomorphism between the families $\{L_p^t : t \in T'\}$ and $\{L_q^s : s \in S'\}$ (ordered by inclusion).

Let $t \in T'$ and consider $f[A_p^t]$; this is a decomposable continuum and hence it is an interval of some standard subcontinuum. We shall find $A \in \Gamma$ such that $f[A_p^t]$ is in fact an interval of A_q. To this end let $\langle [c_n, d_n] : n \in \omega \rangle$ be a sequence of closed intervals with $d_n = c_{n+1}$ for all n and let $r \in \omega^*$ be such that $f[A_p^t]$ is an interval of $[c_r, d_r]$. For every n let $i_n = \lfloor c_n \rfloor$ and $j_n = \lceil d_n \rceil$.

There is a member R of r such that if $n < m$ in R then $j_n < i_m$ and in this case we can assume that $\langle [i_n, j_n] : n \in R \rangle$ is a subsequence of some $A \in \Gamma$. It is clear that $[c_r, d_r] \subseteq [i_r, j_r]$ and it is also true that $q \in f[A_p^t] \subseteq [c_r, d_r]$; together these statements imply that $A_q = [i_r, j_r]$, so that $f[A_p^t]$ is indeed an interval of A_q.

Now let $s_t \in S$ be such that $A \equiv_q A^{s_t}$ and fix some $s \in S$ such that $A_q^s \subseteq f[L_p^t]$. We claim that $s \prec s_t$. Indeed, if $s_t \preccurlyeq s$ then we find that $A_q^{s_t} \subseteq A_q^s \subseteq f[L_p^t]$ and hence that $A_q^{s_t}$ is nowhere dense in $f[A_p^t]$ and hence in A_q, which contradicts $A \equiv_q A^{s_t}$. Thus we find that $A_q^s \subseteq L_q^{s_t}$ and hence that $f[L_p^t] \cap L_q^{s_t} \neq \emptyset$. But $f[L_p^t]$ is a layer of $f[A_p^t]$ and hence of $A_q \cup A_q^{s_t}$, as is $L_q^{s_t}$ of course. But then we must have $f[L_p^t] = L_q^{s_t}$.

Since $L_p^{t_1}$ is nowhere dense in $L_p^{t_2}$, whenever $t_1 \prec t_2$ in T, the map $t \mapsto s_t$ from T' to S' is strictly increasing; that it is surjective follows by interchanging S' and T' and considering f^{-1}.

This shows that T' and S' are isomorphic.

3.3. Many ordered sets

We define a family of $2^{\aleph_1}$ many linear orders of countable cofinality and without isomorphic final segments.

For a set X of countable limit ordinals we define a linear order L_X by inserting upside–down copies of ω into ω_1, one between α and $\alpha + 1$ for every $\alpha \in X$. More formally we let

$$L_X = \{\langle\alpha, m\rangle \in \omega_1 \times \omega : \alpha \notin X \to m = 0\}$$

ordered by $\langle\alpha, m\rangle \prec \langle\beta, n\rangle$ if 1) $\alpha \in \beta$, or 2) $\alpha = \beta$ and $m = 0 < n$, or 3) $\alpha = \beta$ and $m > n > 0$.

Proposition 3.6. *L_X and L_Y are isomorphic iff $X = Y$.*

Proof. Let $f : L_X \to L_Y$ be an isomorphism. We show by induction that $f(\langle\alpha, 0\rangle) = \langle\alpha, 0\rangle$ for every limit ordinal α as well as $\alpha \in X$ iff $\alpha \in Y$.

In both L_X and L_Y the point $\langle\omega, 0\rangle$ has $\omega \times \{0\}$ as its set of predecessors and so $f(\langle\omega, 0\rangle) = \langle\omega, 0\rangle$. Assume α is a limit and that $f(\langle\beta, 0\rangle) = \langle\beta, 0\rangle$ for all limits below α. If α is a limit of limits then in both ordered sets we have $\langle\alpha, 0\rangle = \sup\{\langle\beta, 0\rangle : \beta \in \alpha \text{ and } \beta \text{ is a limit}\}$ and hence $f(\langle\alpha, 0\rangle) = \langle\alpha, 0\rangle$.

Next assume $\alpha = \beta + \omega$ for a limit β. If $\beta \notin X$ then $\langle\beta + 1, 0\rangle$ is the direct successor in L_X of $\langle\beta, 0\rangle$, hence $\langle\beta, 0\rangle$ must have a direct successor in L_Y as well. From this it follows that $\beta \notin Y$ and $f(\langle\beta + n, 0\rangle) = \langle\beta + n, 0\rangle$ for all $n \in \omega$ and hence also $f(\langle\alpha, 0\rangle) = \langle\alpha, 0\rangle$.

If $\beta \in X$ then the interval $\bigl(\langle\beta, 0\rangle, \langle\alpha, 0\rangle\bigr)$ has the same order type as $\mathbb{Z}$, the set of integers. Now the interval $\bigl(\langle\beta, 0\rangle, \langle\beta, 1\rangle\bigr]$ is infinite and every point in it has a direct predecessor. This means that $f(\langle\beta, 1\rangle) \prec \langle\alpha, 0\rangle$ and hence that $\langle\beta, 0\rangle$ does not have a direct successor in L_Y and hence that $\beta \in Y$. It follows that f maps the interval $\bigl(\langle\beta, 0\rangle, \langle\alpha, 0\rangle\bigr)$ isomorphically onto the corresponding interval of L_Y and that $f(\langle\alpha, 0\rangle) = \langle\alpha, 0\rangle$. $\square$

From L_X we define T_X to be the ordered sum of ω copies of L_X:

$$T_X = \omega \times L_X$$

ordered lexicographically. Now note that the points $\bigl\langle n, \langle 0, 0\rangle\bigr\rangle$ are the only ones in T_X whose sets of predecessors have cofinality $\aleph_1$.

Thus, if f is an isomorphism between final segments of some T_X and T_Y then there an isomorphism g between final segments of ω such that $f(\langle n, \langle 0, 0\rangle\rangle) = \bigl\langle g(n), \langle 0, 0\rangle\bigr\rangle$ for all n in the final segment on the T_X-side. For each such n the map f then maps $\{n\} \times L_X$ isomorphically onto $\{g(n)\} \times L_Y$. It follows that $X = Y$.

This then provides us with our family of $2^{\aleph_1}$ many linear orders, indexed by the family of sets of countable limit ordinals.

This proves the following theorem and with it the existence of a family of $2^{\mathfrak{c}}$ many mutually nonhomeomorphic subcontinua of $\mathbb{H}^*$.

Theorem 3.7 (CH). *There is a family of $2^{\mathfrak{c}}$ mean linear orders such that no two members have isomorphic final segments.* $\square$

3.4. Summary: two families of continua

The combination of Subsection 3.2 and Theorem 3.7 tells us that $\{K_{T_X} : X \text{ a set of countable limit ordinals}\}$ is a family of $2^{\mathfrak{c}}$ many indecomposable subcontinua of $\mathbb{H}^*$ that are mutually non-homeomorphic.

To get a family of $2^{\mathfrak{c}}$ many decomposable continua use Lemma 1.3 to deduce that in our construction the continuum K_T is actually a layer of the 'top continuum' A_{T^+}. Indeed, K_T is a subset of some layer L of A_{T^+}; if it were a proper subset then there would be a standard subcontinuum M with $K_T \subseteq M \subseteq L$. As in Subsection 3.2 we could then find $A \in \Gamma$ such that M is an interval of A; yet there would be no $t \in T^+$ such that $A \equiv_p A^t$.

Our second family is now obtained by taking for every set X of countable limit ordinals the interval $[a_X, K_{T_X}]$ of the standard subcontinuum $A_{T_X^+}$, where a_X is the initial point of $A_{T_X^+}$ as described in Subsection 1.2. These decomposable continua are mutually non-homeomorphic because a homeomorphism between

$[a_X, K_{T_X}]$ and $[a_Y, K_{T_Y}]$ will have to map a_X to a_Y (as these are the unique end points) and K_{T_X} onto K_{T_Y}; the latter is not possible if $X \neq Y$.

Remark 3.1. The family in [4] consists of standard subcontinua. By one of the results in [5] CH implies that all standard subcontinua are homeomorphic. Thus there is a striking difference between the effects of CH and ¬CH on the structure of family of standard subcontinua.

Our result shows that under CH each standard subcontinuum has a rich variety of layers and intervals. We leave as an open question how rich this variety is in ZFC alone.

4. A first-countable continuum

As mentioned in the introduction CH implies that the continuous images of $\mathbb{H}^*$ are precisely the continua of weight $\mathfrak{c}$ or less — in particular every first-countable continuum is such an image. In this section we show that in the absence of CH there may be a first-countable continuum that is neither an image of ω^* nor of $\mathbb{H}^*$.

4.1. Bell's graph

A major ingredient in our construction is Bell's graph, constructed in [2]. It is a graph on the ordinal ω_2, represented by a symmetric subset E of $(\omega_2)^2$. The crucial property of this graph is that there is *no* map $\varphi : \omega_2 \to \mathcal{P}(\omega)$ that represents this graph, where φ *represents* E if $\langle \alpha, \beta \rangle \in E$ if and only if $\varphi(\alpha) \cap \varphi(\beta)$ is infinite.

Bell's graph exists in any forcing extension in which $\aleph_2$ Cohen reals are added; for the reader's convenience we shall, in Subsection 4.5 below, describe the construction of E and adapt Bell's proof so that it applies to continuous maps defined on $\mathbb{H}^*$. The proof shows that a similar graph also exists in the extension by $\aleph_2$ random reals.

4.2. Building C_E

Our starting point is a connected version of the Alexandroff double of the unit interval, devised by Saalfrank [13]. We topologize the unit square as follows.

(1) A local base at points of the form $\langle x, 0 \rangle$ consists of the sets

$$U(x, 0, n) = (x - 2^{-n}, x + 2^{-n}) \times [0, 1] \setminus \{x\} \times [2^{-n}, 1]$$

(2) A local base at points of the form $\langle x, y \rangle$, with $y > 0$ consists of the sets

$$U(x, y, n) = \{x\} \times (y - 2^{-n}, y + 2^{-n})$$

We call the resulting space the *connected comb* and denote it by C. It is straightforward to verify that C is compact, Hausdorff and connected; it is first-countable by definition.

For each $x \in [0, 1]$ and positive a we define the following cross-shaped closed subset of C^2:

$$D_{x,a} = \big(\{x\} \times [a, 1] \times C\big) \cup \big(C \times \{x\} \times [a, 1]\big)$$

We note the following two properties of the sets $D_{x,a}$

(1) if $a < b$ then $D_{x,b}$ is in the interior of $D_{x,a}$, and
(2) if $x \neq y$ then $D_{x,a} \cap D_{y,a}$ is the union of two squares: $\{x\} \times [a, 1] \times \{y\} \times [a, 1]$ and $\{y\} \times [a, 1] \times \{x\} \times [a, 1]$.

Next take any $\aleph_2$-sized subset of $[0,1]$ and index it (faithfully) as $\{x_\alpha : \alpha < \omega_2\}$. We use this indexing to identify E with the subset $\{\langle x_\alpha, x_\beta\rangle : \langle\alpha,\beta\rangle \in E\}$ of the unit square. We remove from C^2 the following open set:

$$\bigcup_{\langle x,y\rangle \notin E} \Big(\big(\{x\} \times (0,1] \times \{y\} \times (0,1]\big) \cup \big(\{y\} \times (0,1] \times \{x\} \times (0,1]\big) \Big)$$

The resulting compact space we denote by C_E. Observe that the intersections $D_{x_\alpha,a} \cap C_E$ represent E in the sense that $D_{x_\alpha,a} \cap D_{x_\beta,a} \cap C_E$ is nonempty if and only if $\langle\alpha,\beta\rangle \in E$. We write $D^E_{x,a} = D_{x,a} \cap C_E$.

4.3. C_E is (arcwise) connected

To begin: the square S of the base line of C is a subset of C_E and homeomorphic to the unit square so that it is (arcwise) connected.

Let $\langle x,a,y,b\rangle$ be a point of C_E not in S. If, say, $a = 0$ then $\big\{\langle x,0\rangle\big\} \times \big(\{y\} \times [0,b]\big)$ is an arc in C_E that connects $\langle x,0,y,b\rangle$ to the point $\langle x,0,y,0\rangle$ in S. If $a,b > 0$ then $\langle x,y\rangle \in E$, so the whole square $\{x\} \times [0,1] \times \{y\} \times [0,1]$ is in C_E and it provides us with an arc in C_E from $\langle x,a,y,b\rangle$ to $\langle x,0,y,0\rangle$.

We find that C_E is a first-countable continuum.

4.4. C_E is not an $\mathbb{H}^*$-image

Assume $h : \mathbb{H}^* \to C_E$ is a continuous surjection and consider, for each α, the sets $D^E_{x_\alpha,\frac{3}{4}}$ and $D^E_{x_\alpha,\frac{1}{2}}$.

Using standard properties of $\beta\mathbb{H}$, see [10, Proposition 3.2], we find for each α a sequence $\big\langle (a_{\alpha,n}, b_{\alpha,n}) : n \in \omega\big\rangle$ of open intervals with rational endpoints, and with $b_{\alpha,n} < a_{\alpha,n+1}$ for all n, such that $h^{\leftarrow}[D^E_{x_\alpha,\frac{3}{4}}] \subseteq \operatorname{Ex} O_\alpha \cap \mathbb{H}^* \subseteq h^{\leftarrow}[D^E_{x_\alpha,\frac{1}{2}}]$, where $O_\alpha = \bigcup_n (a_{\alpha,n}, b_{\alpha,n})$ and $\operatorname{Ex} O_\alpha = \beta\mathbb{H} \setminus \operatorname{cl}(\mathbb{H} \setminus O_\alpha)$.

Because the intersections of the sets $D^E_{x_\alpha,a}$ represent E the intersections of the O_α will do this as well: the conditions '$O_\alpha \cap O_\beta$ is unbounded' and '$\langle\alpha,\beta\rangle \in E$' are equivalent.

In the next subsection we show that for (many) $\langle\alpha,\beta\rangle$ this equivalence does not hold and that therefore C_E is not a continuous image of $\mathbb{H}^*$.

Note also that our continuum is not an ω^*-image either: if $g : \omega^* \to C_E$ were continuous and onto we could use clopen subsets of ω^* and their representing infinite subsets of ω to contradict the unrepresentability property of E.

4.5. Building the graph

We follow the argument from [2] and we rely on Kunen's book [11, Chapter VII] for basic facts on forcing. We let $L = \{\langle\alpha,\beta\rangle \in (\omega_2)^2 : \alpha \leqslant \beta\}$ and we force with the partial order $\operatorname{Fn}(L,2)$ of finite partial functions with domain in L and range in $\{0,1\}$. If G is a generic filter on $\operatorname{Fn}(L,2)$ then we let $E = \{\langle\alpha,\beta\rangle : \bigcup G(\alpha,\beta) = 1 \text{ or } \bigcup G(\beta,\alpha) = 1\}$.

To show that E is as required we take a nice name $\dot{F}$ for a function, F, from ω_2 to $(\mathbb{Q}^2)^\omega$ that represents a choice of open sets $\alpha \mapsto O_\alpha$ as above in that $F(\alpha) = \big\langle \langle a_{\alpha,n}, b_{\alpha,n}\rangle : n \in \omega\big\rangle$ for all α. As a nice name $\dot{F}$ is a subset of $\omega_2 \times \omega \times \mathbb{Q}^2 \times \operatorname{Fn}(L,2)$, where for each point $\langle\alpha,n,a,b\rangle$ the set $\{p : \langle\alpha,n,a,b,p\rangle \in \dot{F}\}$ is a maximal antichain in the set of conditions that forces the nth term of $\dot{F}(\alpha)$ to be $\langle a,b\rangle$.

For each α we let I_α be the set of ordinals that occur in the domains of the conditions that appear as a fifth coordinate in the elements of $\dot{F}$ with first coordinate α. The sets I_α are countable, by the ccc of $\operatorname{Fn}(L,2)$. We may therefore apply the Free-Set Lemma, see [9, Corollary 44.2], and find a subset A of ω_2 of cardinality $\aleph_2$ such that $\alpha \notin I_\beta$ and $\beta \notin I_\alpha$ whenever $\alpha,\beta \in A$ and $\alpha \neq \beta$.

Let $p \in \mathrm{Fn}(L, 2)$ be arbitrary and take α and β in A with $\alpha < \beta$ and such that $\alpha > \eta$ whenever η occurs in p. Consider the condition $q = p \cup \{\langle\alpha, \beta, 1\rangle\}$. If q forces $O_\alpha \cap O_\beta$ to be bounded in $[0, \infty)$ then we are done: q forces that the equivalence fails at $\langle\alpha, \beta\rangle$.

If q does not force the intersection to be bounded we can extend q to a condition r that forces $O_\alpha \cap O_\beta$ to be unbounded. We define an automorphism h of $\mathrm{Fn}(L, 2)$ by changing the value of the conditions only at $\langle\alpha, \beta\rangle$: from 0 to 1 and vice versa. The condition p as well as the values $\dot{F}(\alpha)$ and $\dot{F}(\beta)$ are invariant under h. It follows that $h(r)$ extends p and

$$h(r) \Vdash \bigcup \dot{G}(\alpha, \beta) = 0 \text{ and } O_\alpha \cap O_\beta \text{ is unbounded}$$

so again the equivalence is forced to fail at $\langle\alpha, \beta\rangle$.

Remark 4.1. The argument above goes through almost verbatim to show that Bell's graph can also be obtained adding $\aleph_2$ random reals. When forcing with the random real algebra one needs only consider conditions that belong to the σ-algebra generated by the clopen sets of the product $\{0, 1\}^L$; these all have countable supports so that, again by the ccc, one can define the sets I_α as before. The rest of the argument remains virtually unchanged.

Remark 4.2. Bell's original example from [2] was not easily made connected. One obtains an essentially equivalent example by taking the square of the Alexandroff double of the unit interval (the subspace $\{\langle x, i\rangle : x \in [0, 1], i \in \{0, 1\}\}$ of C) and removing the points $\langle\langle x, 1\rangle, \langle y, 1\rangle\rangle$ with $\langle x, y\rangle \notin E$.

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