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# ULTRAFILTERS OF CHARACTER $\omega_1$

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**Abstract.** Using side-by-side Sacks forcing, it is shown that it is consistent that $2^\omega$ be large and that there be many types of ultrafilters of character $\omega_1$.

**§0. Introduction.** The aim of this paper is to prove the relative consistency of "ZFC + $2^\omega$ is big + there are many types of ultrafilters of character $\omega_1$".

There are already quite a few ultrafilters of character less than $2^\omega$, for example the one constructed by Kunen [Ku2; VIII, A10] using iterated forcing, and also Shelah's [Sh] unique Ramsey ultrafilter. There is also the model in [BaLa] in which every selective ultrafilter is of character $\omega_1$. All of these ultrafilters have one thing in common: they are selectives or $P$-points, or constructed using selectives or $P$-points.

Inspired by a question of Bukovský: "Is it consistent that there are no $P$-points, yet there is an ultrafilter of character less than $2^\omega$", we found ultrafilters that are somewhat higher in the Rudin-Frolík order $\leq_{RF}$ on $\omega^*$. We find among others an unbounded $\omega_1$-chain consisting of ultrafilters of character $\omega_1$, a point with exactly $\omega$ predecessors and a weak $P$-point that is not a $P$-point.

Our strategy is to build these ultrafilters in such a way that after adding any number of Sacks reals side-by-side they will (i) still be ultrafilters and (ii) still have most of their pleasant properties. This paper owes much to Laver's paper [La] in which an indestructible selective ultrafilter on $\omega$ is constructed (i.e. Sacks reals do not destroy ultrafilterness).

The paper is organized as follows. §§1 and 2 contain definitions and preliminaries. In §3 we prove some simple results on preservation of properties of ultrafilters when forcing with various types of posets. In §§4, 5 and 6 we adapt some older constructions to our needs and produce many selectives, $P$-points, an $\omega_1$-OK point which is not a $P$-point and the promised $\omega_1$-chain. In §7 we give a new (we think) CH-construction of a point with $\omega_0$ $\leq_{RF}$-predecessors. In §8 we state the full consistency result. Finally, §9 contains some questions and remarks.

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I would like to thank Lev Bukovský for raising the abovementioned question, which is not solved here, but which did start the research for this paper.

**§1. Definitions, notation and preliminaries.** For set theory we refer to [Ku2]; for more information on $\beta\omega$ we refer to [vM].

As usual, if $X$ is a set then we define $[X]^{<\omega} = \{F \subseteq X: |F| < \omega\}$, $[X]^{\leq\omega} = \{C \subseteq X: |C| \leq \omega\}$ and $[X]^{\omega} = [X]^{\leq\omega}\backslash[X]^{<\omega}$.

For $A, B \subseteq \omega$, $A \subseteq^* B$ means that $A\backslash B$ is finite, $A \subset^* B$ means $A \subseteq^* B$ but not $B \subseteq^* A$, and $A =^* B$ means that $A \subseteq^* B$ and $B \subseteq^* A$.

All ultrafilters are assumed to be on $\omega$ and nonprincipal. For $A \subseteq \omega$, $A^* = \{u: u$ is an ultrafilter and $A \in u\}$. Then $\{A^*: A \in u\}$ is a local base at the point $u$ of $\omega^*$. Also it is easy to verify that $A^* \subseteq B^*$ iff $A \subseteq^* B$, $A^* = B^*$ iff $A =^* B$, etc.

An ultrafilter $u$ is said to be *selective* iff whenever $\mathscr{P}$ is a partition of $\omega$, either $\mathscr{P} \cap u \neq \varnothing$ or there is a $U \in u$ such that $|U \cap P| \leq 1$ for all $P \in \mathscr{P}$; we call $U$ a *selector* for $\mathscr{P}$ in this case. We call $u$ a *Q-point* iff it is selective for all partitions of $\omega$ into finite sets. We call $u$ a *P-point* iff whenever $\mathscr{P}$ is a partition of $\omega$, either $\mathscr{P} \cap u \neq \varnothing$ or there is a $U \in u$ such that $|U \cap P| < \omega$ for all $P \in \mathscr{P}$. Clearly $u$ is selective iff it is both a $P$-point and a $Q$-point. We call $u$ *$\omega_1$-OK* [Ku 1] iff whenever $\{V_n: n \in \omega\} \subseteq u$ there is $\{U_\alpha: \alpha \in \omega_1\} \subseteq u$ such that whenever $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ in $\omega_1$, $\bigcap_{i=1}^n U_{\alpha_i} \subseteq^* V_n$. We call $\{U_\alpha: \alpha \in \omega_1\}$ *OK for* $\{V_n: n \in \omega\}$. Finally, $u$ is a *weak P-point* iff there is no countable subset of $\omega^*$ having $u$ as an accumulation point. It is not too difficult to show [Ku1] that every $\omega_1$-OK point is a weak $P$-point. A *base* for an ultrafilter $u$ is a subset $\mathscr{B}$ of $u$ such that $\forall U \in u\, \exists B \in \mathscr{B}: B \subseteq U$. The *character* of $u$ is $\chi(u) = \min\{|\mathscr{B}|: \mathscr{B}$ is a base for $u\}$. It is well known that, for all $u$, $\omega_1 \leq \chi(u) \leq 2^\omega$, and that e.g. MA implies that $\chi(u) = 2^\omega$ for all $u \in \omega^*$.

Two sets $A, B \subseteq \omega$ are *almost disjoint* iff $A \cap B =^* \varnothing$. An *almost disjoint* (A.D.) *family* on $\omega$ is a family $\mathscr{A}$ of infinite subsets of $\omega$ such that any two of its members are almost disjoint. It is known that A.D. families of size $2^\omega$ exist.

Let $\mathbf{P}$ be a poset. We call $\mathbf{P}$ *$\langle\kappa, \lambda, \mu\rangle$-distributive*, where $\kappa$, $\lambda$ and $\mu$ are cardinals, iff whenever $p \in \mathbf{P}$ and $\tau$ are such that $p \Vdash$ "$\tau: \kappa \to \lambda$", there are $q \leq p$ and $F: \kappa \to [\lambda]^{<\mu}$ such that, for all $\alpha \in \kappa$, $q \Vdash$ "$\tau(\alpha) \in F(\alpha)$", i.e. $\tau$ can be approximated from the outside by a narrow (width $< \mu$) pipe from the ground model. We call $\mathbf{P}$ *${}^\omega\omega$-bounding* iff whenever $p \in \mathbf{P}$ and $\tau$ are such that $p \Vdash$ "$\tau: \omega \to \omega$" there are $q \leq p$ and $f: \omega \to \omega$ such that $q \Vdash$ "$\tau(n) \leq f(n)$" for all $n \in \omega$. One readily sees that $\mathbf{P}$ is ${}^\omega\omega$-bounding iff it is $\langle\omega, \omega, \omega\rangle$-distributive. Finally, let $u$ be an ultrafilter; we call $u$ **P**-*indestructible iff* $\Vdash_{\mathbf{P}}$ "$u$ generates an ultrafilter". It is an easy observation that $u$ is $\mathbf{P}$-indestructible iff whenever $p \in \mathbf{P}$ and $\tau$ are such that $p \Vdash$ "$\tau: \omega \to 2$" there are $q \leq p$ and $U \in u$ such that $q \Vdash$ "$\tau \restriction U$ is constant".

**§2. Side-by-side Sacks forcing.** A good introduction to this type of forcing can be found in [Ba]. We shall describe the poset used and derive some facts about new reals and old ultrafilters needed in this paper.

Let $\mathrm{Seq} = \bigcup\{{}^n2: n \in \omega\}$. A nonempty subset $p$ of Seq is a *perfect tree* iff

(i) $\forall s \in p\, \forall n \in \omega: s \restriction n \in p$, and

(ii) $\forall s \in p\, \exists t \in p: s \leq t$ and $t\frown 0, t\frown 1 \in p$.

Let $\mathbf{PF} = \{p: p$ is a perfect tree$\}$ ordered by inclusion: $p \leq q$ iff $p \subseteq q$. For a cardinal $\kappa$ let $\mathbf{P}(\kappa) = \{p: p$ is a function, $\mathrm{dom}(p) \in [\kappa]^{\leq\omega}$, $\mathrm{ran}(p) \subseteq \mathbf{PF}\}$ ordered by $p \leq q$ iff $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$ and $\forall \alpha \in \mathrm{dom}(q): p(\alpha) \leq q(\alpha)$.

Forcing with **PF** is usually called *Sacks forcing* or *perfect-set forcing*. If $G$ is generic on **PF** then $x_G = \bigcup \bigcap G$ is an element of ${}^{\omega}2$, usually called a *Sacks real.* Likewise a generic set $G$ on $\mathbf{P}(\kappa)$ determines $\kappa$ different reals: for each $\alpha$ we get $x_\alpha = \bigcup \bigcap \{p(\alpha): p \in G, \alpha \in \mathrm{dom}(p)\}$. In [Ba] it is shown that if the ground model satisfies CH then $\mathbf{P}(\kappa)$ preserves all cardinals and the new value of $2^\omega$ is the old value of $\kappa^\omega$. Thus, using side-by-side Sacks forcing, $2^\omega$ can be made as big as you want.

We introduce some notation. For $p \in \mathbf{PF}$ and $s \in p$ put

$$\mathrm{fl}(s,p) = |\{i \in \mathrm{dom}(s): s \restriction i^\frown\langle 1 - s(i)\rangle \in p\},$$

the forking level of $s$ in $p$; $\mathrm{fl}(s,p)$ is the number of forks below $s$ in the tree $p$. We put $l(p,n) = \{s \in p: \mathrm{fl}(s,p) = n+1 \text{ and } t < s \to \mathrm{fl}(t,p) \le n\}$; note that $|l(p,n)| = 2^{n+1}$.

For $p \in \mathbf{PF}$ and $s \in p$ we let $p \restriction s = \{t \in p: t \le s \vee s \le t\}$; note that $p \restriction s \in \mathbf{PF}$. We extend the above notions to $\mathbf{P}(\kappa)$. Let $p \in \mathbf{P}(\kappa)$, $F \subseteq \mathrm{dom}(p)$ finite and $\sigma: F \to \mathbf{PF}$ such that $\sigma(\alpha) \in p(\alpha)$ for $\alpha \in F$; then $q = p \restriction \sigma$ is the element of $\mathbf{P}(\kappa)$ satisfying $\mathrm{dom}(q) = \mathrm{dom}(p)$, for $\alpha \in F, q(\alpha) = p(\alpha) \restriction \sigma(\alpha)$ and for $\alpha \in \mathrm{dom}(p)\setminus F, q(\alpha) = p(\alpha)$. If, in addition, $n \in \omega$ then

$$l(p,F,n) = \{\sigma: \mathrm{dom}(\sigma) = F \text{ and for } \alpha \in F, \sigma(\alpha) \in l(p(\alpha),n)\};$$

note that $|l(p,F,n)| = 2^{|F|\cdot(n+1)}$.

From [Ba] we quote

**2.0.** Lemma. *If $p \in \mathbf{P}(\kappa)$ and $p \Vdash$ "$\tau: \omega \to A$", then there are a $q \le p$, a sequence $\langle F_n: n\in\omega\rangle$ of finite subsets of* $\mathrm{dom}(q)$ *and a function $f: \bigcup_{n\in\omega} l(q,F_n,n) \to A$ such that:*

(i) $F_0 \subseteq F_1 \subseteq \cdots$ *and* $\mathrm{dom}(q) = \bigcup_{n\in\omega} F_n$, *and*

(ii) *if $\sigma \in l(q,F_n,n)$ then $q \restriction \sigma \Vdash$ "$\tau(n) = f(\sigma)$".* □

**2.1.** Corollary. $\mathbf{P}(\kappa)$ *is $\langle\omega,\lambda,\omega\rangle$-distributive for every $\lambda$, and in particular it is ${}^{\omega}\omega$-bounding.*

□ If $p \Vdash$ "$\tau: \omega \to \lambda$" then in the terminology of 2.0 set

$$G_n = \{f(\sigma): \sigma \in l(q,F_n,n)\} \qquad (n \in \omega).$$

Then each $G_n$ is finite and, for every $n \in \omega$, $q \Vdash$ "$\tau(n) \in G_n$". □

The method of proof of Lemma 2.0 also establishes the fact that $\mathbf{P}(\kappa)$ is proper.

To be able to handle new reals and to formulate a convenient criterion for $\mathbf{P}(\kappa)$-indestructibility, we introduce some more notation. First we show that we can restrict our attention to $\mathbf{P}(\omega)$.

**2.2.** Lemma. *Let $u$ be an ultrafilter. The following are equivalent:*

(i) *$u$ is $\mathbf{P}(\kappa)$-indestructible for all infinite $\kappa$.*

(ii) *$u$ is $\mathbf{P}(\kappa)$-indestructible for some infinite $\kappa$.*

(iii) *$u$ is $\mathbf{P}(\omega)$-indestructible.*

□ (i) → (ii) is trivial, and (ii) → (iii) holds because $\mathbf{P}(\omega)$ is a complete suborder of $\mathbf{P}(\kappa)$.

For (iii) → (i) let $\kappa \ge \omega$ and assume $p \Vdash$ "$\tau: \omega \to 2$". Identifying $\mathrm{dom}(p)$ with $\omega$, we have $p \in \mathbf{P}(\omega)$. Find $q \in \mathbf{P}(\omega)$, $q \le p$, and $U \in u$ such that $q \Vdash$ "$\tau \restriction U$ is constant". Reversing the process, we get $q \le p$ in $\mathbf{P}(\kappa)$, forcing the same thing. □

We denote by **A** the set of pairs $\langle p, f\rangle$ where $p \in \mathbf{P}(\omega)$ and $f: \bigcup_{n\in\omega} l(p,n,n) \to 2$. Note that **A** has cardinality $2^\omega$. If $\langle p,f\rangle \in \mathbf{A}$ then $\langle p,f\rangle$ determines (a name for) a new real $\phi_{p,f}$ by requiring that

$$\forall\sigma \in l(p,n,n) \quad p \restriction \sigma \Vdash \text{“}\phi_{p,f}(n) = f(\sigma)\text{”}.$$

We get the following useful lemma.

**2.3.** LEMMA. *For an ultrafilter $u$ on $\omega$ the following are equivalent:*

(i) *$u$ is $\mathbf{P}(\omega)$-indestructible.*

(ii) *For all $\langle p,f\rangle \in \mathbf{A}$ there are a $q \leq p$ and a $U \in u$ such that $q \Vdash$ “$\phi_{p,f} \restriction U$ is constant”.*

□ (i) → (ii) is easy. For (ii) → (i) assume that $r \Vdash$ “$\tau\colon \omega \to 2$”. Applying Lemma 2.0 and noting that in case $\kappa = \omega$ we can take $F_n = n$ for all $n$, we can find $\langle p,f\rangle \in A$ such that $p = \tau$ and $\forall n \in \omega\, \forall\sigma \in l(p,n,n)\, p \restriction \sigma \Vdash$ “$\tau(n) = f(\sigma)$”. But then $p \restriction \sigma \Vdash$ “$\tau(n) = f(\sigma) = \phi(n)$” for all $n$ and $\sigma$. It follows that $p \Vdash$ “$\tau = \phi_{p,f}$”. Now find $q \leq p$ and $U \in u$ as in (ii). Then $q \leq r$ and $q \Vdash$ “$\tau \restriction U$ is constant”. □

It follows that when constructing $\mathbf{P}(\kappa)$-indestructible ultrafilters one has to take care of $2^\omega$ objects only. In fact if CH holds only $\omega_1$ tasks need to be done, and when studying $\omega^*$ that is always comforting [vM].

The ideas expressed in Lemmas 2.2 and 2.3 are implicit in Laver's construction of an indestructible selective ultrafilter [La]. I have spelled them out here for future reference.

I end this section with a statement of Laver's theorem [La], which is basic to this paper.

**2.4.** THEOREM. *If $p \in \mathbf{P}(\omega)$ and $\tau$ are such that $p \Vdash$ “$\tau\colon \omega \to 2$”, then for every infinite $A \subseteq \omega$ there are $q \leq p$ and an infinite $B \subseteq A$ such that $q \Vdash$ “$\tau \restriction B$ is constant”.* □

## §3. Preservation of properties.

In this section we collect a few easy results that guarantee the preservation of some of the properties that an ultrafilter may have. For the rest of this section $\mathbf{P}$ is a poset and $u$ is a $\mathbf{P}$-indestructible ultrafilter.

**3.0.** LEMMA. [BlSh]. *If $\mathbf{P}$ is proper and $u$ is a $P$-point then $\mathbf{1} \Vdash$ “$u$ is a $P$-point”.*

□ If in $M[G]$ $\{U_n\colon n \in \omega\}$ is a subfamily of $u$ then, because $\mathbf{P}$ is proper, there is in $M$ a subfamily $\{V_n\colon n \in \omega\}$ of $u$ such that $\{U_n\colon n \in \omega\} \subseteq \{V_n\colon n \in \omega\}$. Now pick $U \in u$ such that $\forall n \in \omega\; U \subseteq^* V_n$. Then surely $\forall n \in \omega\; U \subseteq^* U_n$. □

Our next lemma deals with $Q$-points. For this we need a criterion for $u$ to be a $Q$-point due to Copláková and Vojtáš [CoVo]. Let $f \in {}^\omega\omega$ be such that $\forall n \in \omega\; n < f(n) < f(n+1)$. Define $\bar{f} \in {}^\omega\omega$ by $\bar{f}(0) = f(0)$ and $\bar{f}(n+1) = f(\bar{f}(n))$ $(n > 0)$. Next let $P_f = \{[0,\bar{f}(0)), [\bar{f}(0),\bar{f}(1)), \ldots\}$, a partition of $\omega$ into finite sets. Also let $\mathscr{F} = \{f \in {}^\omega\omega\colon \forall n \in \omega,\, n < f(n) < f(n+1)\}$. Then the result from [CoVo] is as follows.

**3.1.** LEMMA. *For an ultrafilter $v$ the following are equivalent:*

(i) *$v$ is a $Q$-point.*

(ii) *For some (every) dominating subfamily $\mathscr{D}$ of $\mathscr{F}$, $v$ contains a selector for every $P_f$ $(f \in \mathscr{D})$.* □

**3.2.** LEMMA. *If $P$ is ${}^\omega\omega$-bounding and $u$ is a $Q$-point then $\mathbf{1} \Vdash$ “$u$ is a $Q$-point”.*

□ It suffices to note that $\mathscr{F} \cap M$ is dominating in $M[G]$. □

**3.3.** COROLLARY. *If $\mathbf{P}$ is proper and ${}^\omega\omega$-bounding and if $u$ is selective, then $\mathbf{1} \Vdash$ “$u$ is selective”.* □

Our next result deals with $\omega_1$-OK points.

**3.4.** LEMMA. *Let $u$ be $\omega_1$-OK.*

(i) *If* $\mathbf{P}$ *is* $\langle \omega, 2^\omega, \omega\rangle$*-distributive then* $\mathbf{1} \Vdash$ "*$u$ is $\omega_1$-OK*".

(ii) *If* $\mathbf{P}$ *is proper then* $\mathbf{1} \Vdash$ "*$u$ is a weak P-point*".

$\square$ (i) In $M[G]$ let $\langle u_n : n \in \omega\rangle$ be a sequence in $u$. Back in $M$ there is a sequence $\langle W_n : n \in \omega\rangle$ in $u$ such that $\forall n \in \omega\ W_n \subseteq U_n$: To see this first find $F$ such that $\operatorname{dom} F = \omega$ and, for all $n$, $F(n) \in [u]^{<\omega}$ and $U_n \in F(n)$. Then let $W_n = \bigcap F(n)$ $(n \in \omega)$. Then if $\langle V_\alpha : \alpha \in \omega_1\rangle$ is OK for $\langle W_n : n \in \omega\rangle$, it is also OK for $\langle U_n : n \in \omega\rangle$.

(ii) In $M[G]$ let $\{U_n : n \in \omega\} \subseteq \omega^* \backslash \{u\}$, and for each $n$ pick $U_n \in u$ with $U_n \notin u_n$. In $M$ find $\langle W_n : n \in \omega\rangle$ in $u$ such that $\langle U_n : n \in \omega\rangle$ is a subsequence of it. Let $\langle V_\alpha : \alpha \in \omega_1\rangle$ be OK for $\langle W_n : n \in \omega\rangle$. It follows readily that, for each $n \in \omega$, $\{\alpha : V_\alpha \in u_n\}$ is finite; hence, for some $\alpha$, $V_\alpha^* \cap \{u_n : n \in \omega\} = \varnothing$. $\square$

**§4. Selective and nonselective *P*-points.** In this section we construct $\mathbf{P}(\omega)$-indestructible selective and nonselective $P$-points. In [La] Laver constructed a $\mathbf{P}(\omega)$-indestructible selective ultrafilter $u$ such that

$$\mathbf{1} \Vdash \text{"}u \text{ is selective"}.$$

By Corollary 3.3 this last fact is automatic. In §6 we shall need many different selective ultrafilters, so we shall redo Laver's construction with some extra care. From now on we assume that CH holds and we fix an enumeration $\{\langle p_\alpha, f_\alpha\rangle : \alpha \in \omega_1\}$ of the set $\mathbf{A}$ from §2. In addition we let $\phi_\alpha$ be a name for the real determined by $\langle p_\alpha, f_\alpha\rangle$ for $\alpha \in \omega_1$, i.e. $\phi_\alpha = \phi_{p_\alpha, f_\alpha}$.

**4.0.** THEOREM. *There are* $2^{\omega_1}$ $\mathbf{P}(\omega)$*-indestructible selective ultrafilters on* $\omega$.

$\square$ Let $\{P_\alpha : \alpha \in \omega_1\}$ be an enumeration of the collection of partitions of $\omega$ into finite sets. Inductively we define families $\mathscr{A}_\alpha$ $(\alpha \in \omega_1)$ of infinite subsets of $\omega$ satisfying the following conditions:

(i) $\mathscr{A}_0 = \{\omega\}$.

(ii) Each $\mathscr{A}_\alpha$ is an almost disjoint family of size $\omega_1$ $(\alpha > 0)$.

(iii) If $\alpha < \beta$ then $\mathscr{A}_\beta$ refines $\mathscr{A}_\alpha$ and, $\forall A \in \mathscr{A}_\alpha$, $|\{B \in \mathscr{A}_\beta : B \subseteq^* A\}| = \omega_1$.

(iv) Every $A \in \mathscr{A}_{\alpha+1}$ is a selector for $P_\alpha$.

(v) For every $A \in \mathscr{A}_{\alpha+1}$ there is a $q_A \leq p_\alpha$ such that $q_A \Vdash$ "$\phi_\alpha$ is constant on $A_\alpha$".

At successor stages we use Laver's theorem to obtain for each $A \in \mathscr{A}_\alpha$ an almost disjoint family $\mathscr{B}_A$ of size $\omega_1$ of subsets of $A$ satisfying (iii)–(v); then we let $\mathscr{A}_{\alpha+1} = \bigcup\{\mathscr{B}_A : A \in \mathscr{A}_\alpha\}$. At limit stages we let $\mathscr{A}_\alpha$ be an almost disjoint family refining each $\mathscr{A}_\beta$ $(\beta \in \alpha)$ and such that for every sequence $\langle A_\beta : \beta \in \alpha\rangle$ with $\forall \beta \in \alpha\ A_\beta \in \mathscr{A}_\beta$ and $A_\gamma \subseteq^* A_\beta$ if $\beta \in \gamma \in \alpha$ there is an $A \in \mathscr{A}_\alpha$ such that $\forall \beta \in \alpha\ A \subseteq^* A_\beta$. Now let $\langle A_\alpha : \alpha \in \omega_1\rangle$ be a branch through the tree $\bigcup_{\alpha\in\omega_1} \mathscr{A}_\alpha$, i.e. $\forall \alpha \in \omega_1\ A_\alpha \in \mathscr{A}_\alpha$ and if $\beta \in \alpha \in \omega_1$ then $A_\alpha \subseteq^* A_\beta$. Then the filter $u$ generated by $\{A_\alpha : \alpha \in \omega_1\}$ is a $\mathbf{P}(\omega)$-indestructible selective ultrafilter. To see this, note that for every $\alpha \in \omega_1$

$$q_{A_{\alpha+1}} \Vdash \text{"}\phi_\alpha \text{ is constant on } A_{\alpha+1}\text{"}$$

whence $\mathbf{1} \Vdash$ "$u$ is an ultrafilter". Also, $u$ is selective: it is a $P$-point because it has a linearly ordered (by $\subseteq^*$) base, and it is a $Q$-point by construction. In this way we obtain $\omega_1^\omega = 2^\omega$ such ultrafilters, one for each branch through $\bigcup_{\alpha\in\omega_1} \mathscr{A}_\alpha$. $\square$

Our next aim is to show that there are many $\mathbf{P}(\omega)$-indestructible $P$-points which are not selective. We must make $P$-points that are not $Q$-points. To do this, for $m \in \omega$

we let $P_m = [2^m - 1, 2^{m+1} - 1)$, and $\mathscr{P} = \{P_m : m \in \omega\}$. Let

$$I = \left\{ A \subseteq \omega : \limsup_{m \to \omega} |A \cap P_m| < \omega \right\}$$

and $I^+ = \mathscr{P}(\omega) \backslash I$. We shall find an indestructible $P$-point $u$ such that $u \subseteq I^+$; in particular, $u$ will have no selector for $\mathscr{P}$. For this we need the following lemma.

**4.1.** LEMMA. *Let $p \in \mathbf{P}(\omega)$ and $f: \bigcup_{n \in \omega} l(p, n, n) \to 2$ determine the real $\phi$, and let $A \in I^+$. Then there are a $q \leq p$ and a $B \subseteq A$ with $B \in I^+$ such that $q \Vdash$ "$\phi \restriction B$ is constant".*

□ To begin, fix $m_0 < m_1 < m_2 < \cdots$ in $\omega$ such that $|A \cap P_{m_i}| \geq i \cdot 2^i$ $(i \in \omega)$. For $i \in \omega$ set $l_i = 2^{m_i + 1} - 1$. Now thin out $p$ to a condition $q$ in $\mathbf{P}(\omega)$ such that, for every $i$, $|F_i| \leq i$, where $F_i = \{\sigma \in l(p, l_i, l_i)$: if $j < l_i$ then $\sigma(j) \in q(j)\}$. Note that, for any $\sigma \in F_i$, $q \restriction \sigma$ decides the whole of $\phi \restriction l_i$, say $\phi \restriction l_i = \phi_\sigma$. Fix $i$. As $|A \cap P_{m_i}| \geq i \cdot 2^i$, $A \cap P_{m_i} \subseteq l_i$ and $|F_i| \leq i$, there is a set $A_i \subseteq A \cap P_{m_i}$ such that $|A_i| \geq i$ and, for each $\sigma \in F_i$, $\phi_\sigma \restriction A_i$ is constant. Then $q \Vdash$ "$\phi \restriction A_i$ is constant". Define a (name for a) real $\rho$ by requiring that for every $i$

$$q \Vdash \text{“}\rho(i) \text{ is the value of } \phi \restriction A_i\text{”}.$$

Then find $r \leq q$ and $C \subseteq \omega$ infinite such that $r \Vdash$ "$\rho \restriction C$ is constant"; let $B = \bigcup_{i \in C} A_i$. Then $B \subseteq A$, $B \in I^+$ $(i \in C \to |B \cap P_{m_i}| \geq i)$ and $r \Vdash$ "$\phi \restriction B$ is constant". □

It is now easy to prove:

**4.2.** THEOREM. *There are $2^{\omega_1}$ $\mathbf{P}(\omega)$-indestructible nonselective $P$-points.*

□ Much as in the proof of Theorem 4.0 we construct almost disjoint families $\mathscr{A}_\alpha$ $(\alpha \in \omega_1)$ of infinite subsets of $\omega$ satisfying the following conditions:

(i) $\mathscr{A}_0 = \{\omega\}$.

(ii) $\forall \alpha \in \omega_1\ \forall A \in \mathscr{A}_\alpha\ A \in I^+$.

(iii) If $\alpha < \beta$, then $\mathscr{A}_\beta$ refines $\mathscr{A}_\alpha$ and, for every $A \in \mathscr{A}_\alpha$, $|\{B \in \mathscr{A}_\beta : B \subseteq^* A\}| = \omega_1$.

(iv) For every $A \in \mathscr{A}_{\alpha+1}$ there is a $q_A \leq p_\alpha$ such that $q_A \Vdash$ "$\phi_\alpha$ is constant on $A$".

Fix an almost disjoint family $\mathscr{C}$ of size $\omega_1$ on $\omega$. At successor stages we use Lemma 4.1 to obtain for an $A \in \mathscr{A}_\alpha$ an $A' \subseteq A$ and $q \leq p_\alpha$ such that $A' \in I^+$ and $q \Vdash$ "$\phi_\alpha \restriction A'$ is constant". Then we pick $m_0 < m_1 < m_2 < \cdots$ in $\omega$ such that $|A' \cap P_{m_i}| \geq i$ for each $i$. For each $C \in \mathscr{C}$ set $B_C = \bigcup_{i \in C} (A' \cap P_{m_i})$. Then $B_C \in I^+$ and $q \Vdash$ "$\phi_\alpha \restriction B_C$ is constant". We let $q_B = q$ for each $B \in \mathscr{B}_A = \{B_C : C \in \mathscr{C}\}$ and we set $\mathscr{A}_{\alpha+1} = \bigcup \{\mathscr{B}_A : A \in \mathscr{A}_\alpha\}$. At limit stages we choose for each branch $\langle A_\beta : \beta \in \alpha \rangle$ through $\bigcup_{\beta \in \alpha} \mathscr{A}_\beta$ a set $A$ as follows. Fix a sequence $\langle \alpha_i : i \in \omega \rangle$ cofinal in $\alpha$. For each $i$ pick $m_i$ such that $|\bigcap_{j \leq i} A_{\alpha_j} \cap P_{m_i}| \geq i$; let $A = \bigcup_{i \in \omega} (\bigcap_{j \leq i} A_{\alpha_j} \cap P_{m_i})$. Then $A \in I^+$ and $A \subseteq^* A_\beta$ for $\beta \in \alpha$. We let $\mathscr{A}_\alpha$ be the collection of sets thus obtained. As in the proof of Theorem 4.0 this provides us with $\omega_1^{\omega_1} = 2^{\omega_1}$ nonselective $P$-points, each of which is indestructible. □

**§5. An $\omega_1$-OK point which is not a $P$-point.** In this section we describe a $\mathbf{P}(\omega)$-indestructible ultrafilter which is $\omega_1$-OK but not a $P$-point. We shall adapt a construction by M. E. Rudin [Ru; C2] to our needs.

We shall need another strengthening of Laver's theorem.

**5.0.** LEMMA. *Let $p \in \mathbf{P}(\omega)$ and let $\phi$ be a $\mathbf{P}(\omega)$-name for a real. Let $\{A_n : n \in \omega\}$ be a family of infinite subsets of $\omega$. Then there are a $q \leq p$ and a set $A \subseteq \omega$ such that*

*(i) $q \Vdash$ "$\phi \restriction A$ is constant", and*

*(ii) $A \cap A_n$ is infinite for infinitely many $n$.*

□ We construct a sequence $p = p_0 \geq p_1 \geq \cdots$ in $\mathbf{P}(\omega)$ and a sequence $B_0, B_1, \ldots$ of infinite subsets of $\omega$ as follows: set $p_0 = p$ and, given $p_n$, determine $p_{n+1} \leq p_n$ and $B_n \subseteq A_n$ as follows. Enumerate $l(p_n, n, n)$ as $\{\sigma_i : i < l_n\}$, set $r_0 = p_n$, set $B_{n,0} = A_n$ and, given $r_i$ with $l(r_i, n, n) = l(p_n, n, n)$ and $B_{n,i}$, find $r_{i+1} \leq r_i$ and $B_{n,i+1} \subseteq B_{n,i}$ as follows: first find $q_i \leq r_i \restriction \sigma_i$ and $B_{n,i+1} \subseteq B_{n,i}$ infinite such that

$$q_i \Vdash \text{“}\phi \restriction B_{n,i+1} \text{ is constant”},$$

and then define $r_{i+1}$ by

$$r_{i+1}(j) = \begin{cases} q_i(j) \cup \bigcup\{r_i(j) \restriction s : s \in l(r_i(j), n) \text{ and } s \neq \sigma_i(j)\} & \text{if } j < n, \\ q_i(j) & \text{if } j \geq n; \end{cases}$$

then $r_{i+1} \leq r_i$ and $l(r_{i+1}, n, n) = l(r_i, n, n)$. In the end set $p_{n+1} = r_{l_n}$ and $B_n = B_{n,l_n}$. Note that $p_{n+1} \restriction \sigma_i \leq r_{i+1} \restriction \sigma_i = q_i$ for $i < l_n$, so that

$$p_{n+1} \restriction \sigma \Vdash \text{“}\phi \restriction B_n \text{ is constant”}$$

for every $\sigma \in l(p_{n+1}, n, n)$, and hence $p_{n+1} \Vdash$ “$\phi \restriction B_n$ is constant”.

Now define $p_\omega$ by $p_\omega(j) = \bigcap_{n \in \omega} p_n(j)$ $(j \in \omega)$.

One readily checks that $p_\omega \in \mathbf{P}(\omega)$ and that $l(p_\omega, n, n) = l(p_n, n, n)$ for every $n \in \omega$; in the terminology of [Ba], $p_\omega$ is the fusion of the sequence $\langle\langle p_n, n\rangle : n \in \omega\rangle$. Now $p_\omega \leq p_{n+1}$ for every $n$, so that

$$p_\omega \Vdash \text{“}\phi \restriction B_n \text{ is constant”}.$$

Let $\psi$ be a name for the real determined by

$$p_\omega \Vdash \text{“}\psi(n) \text{ is the constant value of } \phi \restriction B_n\text{”}$$

for every $n$. Then find $q \leq p_\omega$ and $B \subseteq \omega$ infinite such that $q \Vdash$ “$\psi \restriction B$ is constant”. Let $A = \bigcup_{n \in B} B_n$; then $A \cap A_n \supseteq B_n$ is infinite for $n \in B$, and $q \Vdash$ “$\phi \restriction A$ is constant”. □

We recall that $\{\langle p_\alpha, f_\alpha\rangle : \alpha \in \omega_1\}$ enumerates $\mathbf{A}$ and that $\phi_\alpha$ is a name for the real determined by $\langle p_\alpha, f_\alpha\rangle$ $(\alpha \in \omega_1)$. Before we begin the construction let us outline the strategy. Our ultrafilter $u$ will be generated by a family $\{U_\alpha : \alpha \in \omega_1\}$. Now if $\{V_n : n \in \omega\} \subseteq u$ we can pick $\alpha_n \in \omega_1$ such that $U_{\alpha_n} \subseteq V_n$ for $n \in \omega$. It then suffices to find an uncountable subset $S$ of $\omega_1$ such that $\{U_\alpha : \alpha \in S\}$ is OK for $\{U_{\alpha_n} : n \in \omega\}$. To ensure that this is possible we enumerate ${}^\omega\omega_1$, the set of functions from $\omega$ to $\omega_1$, as $\{s_\delta : \delta \in \omega_1\}$ and we split $\mathrm{LIM}(\omega_1)$, the set of nonzero limit ordinals in $\omega_1$, into $\omega_1$ uncountable sets $\{S_\delta : \delta \in \omega_1\}$. In the construction we will make sure that, for every $\delta$, $\{U_\alpha : \alpha \in S_\delta\}$ is OK for $\{U_{s_\delta(n)} : n \in \omega\}$. We arrange things in such a way that, for every $\delta$, $\operatorname{ran} s_\delta \subseteq \min S_\delta$. We let $\mathscr{P} = \{P_m : m \in \omega\}$ be a partition of $\omega$ into infinite sets. $\mathscr{P}$ will witness the fact that $u$ is not a $P$-point; i.e. we make sure that $\forall \alpha \in \omega_1$ $|U_\alpha \cap P_m| = \omega$ for infinitely many $m$. Finally, if $\alpha > 0$ is a limit ordinal we fix an enumeration $\{\alpha_i : i \in \omega\}$ of the set $\alpha$ such that if $\alpha \in S_\delta$ then $\alpha_0 = s_\delta(0)$ and $\alpha_1 = s_\delta(1)$.

**5.1.** Proposition. *We can find sets $\{U_\alpha : \alpha \in \omega_1\}$ in $\omega$ such that*

1) *$\{U_\alpha : \alpha \in \omega_1\}$ generates a $\mathbf{P}(\omega)$-indestructible ultrafilter,*
2) *$\forall \alpha \in \omega_1$ $\{m : U_\alpha \cap P_m \neq^* \varnothing\}$ is infinite, and*
3) *$\forall \delta \in \omega_1$ $\{U_\alpha : \alpha \in S_\delta\}$ is OK for $\{U_{s_\delta(n)} : n \in \omega\}$.*

□ For $\alpha \in \omega_1$ we shall find $I_\alpha \subseteq \omega$ (infinite), $A(\alpha, m) \subseteq P_m$ (infinite; $m \in \omega$), and $B(\alpha, m, \gamma) \subseteq A(\alpha, m)$ (infinite; $\gamma \in \omega_1$ and $m \in \omega$). We shall set $U_\alpha = \bigcup_{m \in I_\alpha} A(\alpha, m)$

($\alpha \in \omega_1$). Then 2) is immediate. The sets will satisfy the following conditions:

(i) If $\beta \in \alpha$ then $I_\alpha \subseteq^* I_\beta$.

(ii) $\mathscr{B}(\alpha, m) = \{B(\alpha, m, \gamma): \gamma \in \omega_1\}$ is an almost disjoint family.

(iii) If $\beta \in \alpha$ then either $A(\alpha, m) \cap A(\beta, m) =^* \varnothing$, or, for some $\gamma \leq \alpha$, $A(\alpha, m) \subseteq^* B(\beta, m, \gamma)$.

(iv) For all $\alpha \neq 0$ and all $m$ there is a $\beta \in \alpha$ such that $A(\alpha, m) \subseteq^* B(\beta, m, \alpha)$.

(v) For some $q_\alpha \leq p_\alpha$, $q_\alpha \Vdash$ "$\phi_\alpha \restriction U_{\alpha+1}$ is constant".

(vi) For $m \in \omega \backslash I_\alpha$, $A(\alpha, m) = B(0, m, \alpha)$.

(vii) If $\alpha > 0$ is a limit and $\{m_i: i \in \omega\}$ is the monotone enumeration of $I_\alpha$, then

$$m_i \in \bigcap_{j \leq i+1} I_{\alpha_j} \quad \text{and} \quad A(\alpha, m_i) \subseteq \bigcap_{j \leq i+1} A(\alpha_j, m_i).$$

(viii) For every $\alpha$, $I_{\alpha+1} \subseteq I_\alpha$; and for $m \in I_{\alpha+1}$, $A(\alpha+1, m) \subseteq B(\alpha, m, \alpha+1)$.

It follows from (vii) and (viii) that $\{U_\alpha: \alpha \in \omega_1\}$ generates a filter $u$, and (v) guarantees that $u$ is in fact a $\mathbf{P}(\omega)$-indestructible ultrafilter. To ensure that, for every $\delta$, $\{U_\alpha: \alpha \in S_\delta\}$ is OK for $\{U_{s_\delta(n)}: n \in \omega\}$, we have to exercise some extra care. For $\alpha \in \omega_1$ and $m \in \omega$ set $K(\alpha, m) = \{\beta \in \alpha: A(\alpha, m) \subseteq^* A(\beta, m)\}$. The claim is that $K(\alpha, m)$ is finite. Given $\alpha$ and $m$, fix $\beta$ as given by (iv). Then $K(\alpha, m) = \{\beta\} \cup K(\beta, m)$. [The inclusion $\supseteq$ is immediate. For $\subseteq$ let $\gamma \in K(\alpha, m)$; then $A(\alpha, m) \subseteq^* A(\beta, m) \cap A(\gamma, m)$, so $A(\beta, m) \cap A(\gamma, m) \neq^* \varnothing$. If $\beta < \gamma < \alpha$ then $A(\gamma, m) \subseteq^* B(\beta, m, \varepsilon)$ for some $\varepsilon \leq \gamma$, so that $A(\alpha, m) \cap A(\gamma, m) \subseteq^* B(\beta, m, \alpha) \cap B(\beta, m, \varepsilon) =^* \varnothing$, which is a contradiction. So either $\gamma = \beta$ or $\gamma < \beta$, in which case $A(\beta, m) \subseteq^* B(\gamma, m, \varepsilon) \subseteq^* A(\gamma, m)$ for some $\varepsilon \leq \beta$, so that $\gamma \in K(\beta, m)$.] As $K(0, m) = \varnothing$ for every $m$, it now follows by induction that $K(\alpha, m)$ is finite for every $\alpha$ and $m$.

We define, for every $\alpha$, $m$ and $\delta$,

$$\begin{aligned} K_\delta(\alpha, m) &= K(\alpha, m) \cap S_\delta, \\ k_\delta(\alpha, m) &= |K_\delta(\alpha, m)|, \\ l_\delta(\alpha, m) &= \max\{l: \forall i \leq l, s_\delta(i) \in K(\alpha, m)\} \end{aligned}$$

(in case ran $s_\delta \subseteq K(\alpha, m)$ we put $l_\delta(\alpha, m) = \infty$). We make the following additional requirements:

(ix) If $\alpha \in S_\delta$ then

$$\forall m \in I_\alpha \quad l_\delta(\alpha, m) > k_\delta(\alpha, m).$$

(x) If $\alpha \in \omega_1$ and ran $s_\delta \subseteq \alpha$ then

$$\lim_{\substack{m \to \omega \\ m \in I_\alpha}} l_\delta(\alpha, m) - k_\delta(\alpha, m) = \infty.$$

Let us first check that this works. Let $\delta \in \omega_1$ and $\alpha_1 < \cdots < \alpha_n$ in $S_\delta$. We show first that for every $m \in I_{\alpha_n}$

$$\bigcap_{i=1}^{n} A(\alpha_i, m) \subseteq^* A(s_\delta(n), m).$$

If $\bigcap_{i=1}^n A(\alpha_i, m) =^* \varnothing$ this is clear; in the other case we conclude that $\{\alpha_1, \ldots, \alpha_{n-1}\} \subseteq K_\delta(\alpha_n, m)$ (by (iii)), so that $k_\delta(\alpha_n, m) \geq n-1$. But then by (ix) we have $l_\delta(\alpha_n, m) \geq n$, so that $s_\delta(n) \in K(\alpha_n, m)$ and hence

$$\bigcap_{i=1}^{n} A(\alpha_i, m) \subseteq A(\alpha_n, m) \subseteq^* A(s_\delta(n), m).$$

In addition (vii) implies that, for all but finitely many $m \in I_{\alpha_n}$, $A(\alpha_n, m) \subseteq A(s_\delta(n), m)$. We conclude that $\bigcap_{i=1}^n U_{\alpha_i} \subseteq^* U_{s_\sigma(n)}$, as required. It remains to perform the construction.

At every stage, once $A(\alpha, m)$ is found, the family $\mathscr{B}(\alpha, m)$ can be chosen arbitrarily; also, once $I_\alpha$ is found, we set $A(\alpha, m) = B(0, m, \alpha)$ for $m \in \omega \backslash I_\alpha$ to fulfill (vi). We begin by setting $A(0, m) = P_m (m \in \omega)$ and $I_0 = \omega$. Going from $\alpha$ to $\alpha + 1$, we apply Lemma 5.0 to the pair $\langle p_\alpha, f_\alpha \rangle$ and the family $\{B(\alpha, m, \alpha + 1): m \in I_\alpha\}$, to obtain an infinite set $I_{\alpha+1} \subseteq I_\alpha$, infinite sets $A(\alpha + 1, m) \subseteq B(\alpha, m, \alpha + 1)$ $(m \in I_{\alpha+1})$, and $q_\alpha \leq p_\alpha$ such that $q_\alpha \Vdash$ "$\phi \restriction U_{\alpha+1}$ is constant". One can readily check that (i), (iii) and (iv) are fulfilled. To check (x), note that $K_\delta(\alpha + 1, m) \subseteq K_\delta(\alpha, m) \cup \{\alpha\}$ for every $\delta$ and every $m$, so that always $k_\delta(\alpha + 1, m) \leq k_\delta(\alpha, m) + 1$; hence (x) is no problem. Next assume that $\alpha > 0$ is a limit ordinal. Enumerate the set $\{\delta: \min S_\delta \leq \alpha\}$ as $\{\delta_i: i \in \omega\}$; this is possible because these sets $S_\delta$ $(\delta \in \omega_1)$ are pairwise disjoint. Moreover, make sure that $\alpha \in S_{\delta_0}$. We determine $I_\alpha = \{m_i: i \in \omega\}$ as follows: assume that $m_j$ is found for $j < i$; to determine $m_i$, set $\varepsilon_i = \max\{\delta_j: j \leq i + 1\}$ and pick $m_i \in \bigcap_{j \leq i+1} I_{\alpha_j}$ so big that:

—$m_i > m_j$ for all $j < i$,

—for all $j \leq i$, $l_{\delta_j}(\varepsilon_i, m_i) - k_{\delta_j}(\varepsilon_i, m_i) \geq i + 2$, and

—$\bigcap_{j \leq i+1} A(\alpha_j, m_i) \neq^* \varnothing$ (an easy check using (vii) and (viii) shows that this is possible).

Now set $A(\alpha, m_i) = B(\varepsilon_i, m_i, \alpha) \cap \bigcap_{j \leq i+1} A(\alpha_j, m_i)$. It is straightforward to check (i), (iii) and (iv); condition (vii) is fulfilled by construction. For (ix) note that for every $m_i$

$$l_{\delta_0}(\alpha, m_i) - k_{\delta_0}(\alpha, m_i) \geq l_{\delta_0}(\varepsilon_i, m_i) - k_{\delta_0}(\varepsilon_i, m_i) + 1 \geq i + 1 \geq 1,$$

because $K_{\delta_0}(\alpha, m_i) \subseteq K_{\delta_0}(\varepsilon_i, m_i) \cup \{\varepsilon_i\}$ and $l_{\delta_0}(\alpha, m_i) \geq l_{\delta_0}(\varepsilon_{i, m_i})$. Likewise if ran $s_\delta \subseteq \alpha$ then $\lim_{m \to \omega, m \in I_\alpha} l_\delta(\alpha, m) = \infty$, so if $\min S_\delta > \alpha$ there is no problem. If $\min S_\delta \leq \alpha$, say $\delta = \delta_j$, then for $i \geq j$

$$l_{\delta_j}(\alpha, m_i) - k_{\delta_j}(\alpha, m_i) \geq l_{\delta_j}(\varepsilon_i, m_i) - (k_{\delta_j}(\varepsilon_i, m_i) + 1).$$

This finishes the construction and the proof of the proposition. □

To summarize we state

**5.2.** Theorem. *There is a* $\mathbf{P}(\omega)$*-indestructible ultrafilter which is an* $\omega_1$*-OK point but not a P-point.* □

In fact we can find $2^\omega$ such ultrafilters by taking, at successor stages, instead of one set $I_{\alpha+1}$ an almost disjoint family of size $\omega_1$ of such sets; at limit stages we would have to take care of all possible branches through the tree of $I_\alpha$'s much as in the proof of Theorem 4.0.

**§6. An unbounded $\omega_1$-chain in the Rudin Frolík order.** In this and the next section we shall consider the Rudin-Frolík order of $\omega^*$. We shall give here its definition and a few of its basic properties. For details we refer to [Ru] and [BuBu].

**6.0.** Definition. Let $u \in \omega^*$ and let $X = \{x_n: n \in \omega\} \subseteq \beta\omega$ be relatively discrete, so that $\bar{X} = \beta X \approx \beta\omega$. We denote by $\Sigma(X, u)$ the copy of $u$ in $\bar{X} \backslash X$, with respect to the given indexing of $X$. So, for $A \subseteq \omega$, $A \in \Sigma(x,u)$ iff $\{n\colon A \in x_n\} \in u$. Conversely if $v \in \bar{X}\backslash X$ then $\Omega(X,v)$ is the ultrafilter of which $v$ is the copy, i.e., for $A \subseteq \omega$, $A \in \Omega(X,v)$ iff $v \in \overline{\{x_n\colon n \in A\}}$. For $u, v \in \omega^*$ we define $u \leq_{\mathrm{RF}} v$ iff there is a discrete $X \subseteq \beta\omega$ such that $v = \Sigma(X,u)$. We call $\leq_{\mathrm{RF}}$ the *Rudin-Frolík order* on $\omega^*$. □

One checks readily that $\leq_{\mathrm{RF}}$ is reflexive and transitive; $\leq_{\mathrm{RF}}$ is not antisymmetric. We say $u <_{\mathrm{RF}} v$ iff $u \leq_{\mathrm{RF}} v$ but not $v \leq_{\mathrm{RF}} u$. Then $<_{\mathrm{RF}}$ is the strict version of $\leq_{\mathrm{RF}}$.

**6.1.** Definition. Two ultrafilters $u$ and $v$ on $\omega$ are *equivalent* (in symbols $u \sim v$) iff there is a permutation $f\colon \omega \leftrightarrow \omega$ such that $f(u) = v$. □

**6.2.** *Facts on* $\leq_{\mathrm{RF}}$. a) *If* $u \leq_{\mathrm{RF}} v$ *and* $v \leq_{\mathrm{RF}} u$ *then* $u \sim v$.

b) $u <_{\mathrm{RF}} v$ *iff there is a discrete* $X \subseteq \omega^*$ *such that* $v = \Sigma(X,u)$.

c) *If* $u \in \omega^*$ *and* $X, Y \subseteq \beta\omega$ *are discrete then* $\Sigma(X,u) \leq_{\mathrm{RF}} \Sigma(Y,u)$ *iff* $\{n\colon x_n \leq_{\mathrm{RF}} y_n\} \in u$. *Hence* $\Sigma(X,u) \sim \Sigma(Y,u)$ *iff* $\{n\colon x_n \sim y_n\} \in u$.

d) *If* $u, v, w \in \omega^*$ *and* $u, v \leq_{\mathrm{RF}} w$ *then* $u \leq_{\mathrm{RF}} v$ *or* $v \leq_{\mathrm{RF}} u$. □

In [Bu2] Butkovičová constructed an unbounded $\omega_1$-chain with respect to $\leq_{\mathrm{RF}}$ in $\omega^*$. In this section we shall see that this $\omega_1$-chain can be constructed so that it consists of $\mathbf{P}(\omega)$-indestructible ultrafilters; moreover in any generic extension by $\mathbf{P}(\kappa)$ the chain will still be unbounded.

We shall need the following lemmas.

**6.3.** Lemma. *Let* $\mathbf{P}$ *be an* $\langle \omega, 2^\omega, \omega\rangle$*-distributive poset and let* $X \cup \{u\}$ *be a set of* $\mathbf{P}$*-indestructible ultrafilters with* $X$ *discrete. Then* $v = \Sigma(X,u)$ *is also* $\mathbf{P}$*-indestructible.*

□ Let $p \in \mathbf{P}$ and let $\tau$ be a $\mathbf{P}$-name such that $p \Vdash \tau\colon \omega \to 2$. Each $x_n \in X$ is $\mathbf{P}$-indestructible, so there is a $\mathbf{P}$-name $\gamma$ such that

$$p \Vdash \text{“}\gamma \text{ is a function, dom } \phi = \omega, \forall n \in \omega(\gamma(n) \in x_n \cap \tau \restriction \gamma(n) \text{ is constant)”}.$$

Then there are a $q \leq p$ and a function $F$ such that dom $F = \omega$, $\forall n \in \omega\, F(n)$ is a finite subset of $x_n$, and $\forall n \in \omega\, q \Vdash$ “$\gamma(n) \in F(n)$”. For $n \in \omega$ let $U_n = \bigcap F(n)$; then $U_n \in x_n$ and $q \Vdash$ “$U_n \subseteq \gamma(n)$”, so that $q \Vdash$ “$\tau \restriction U_n$ is constant”. Let $\sigma$ be a $\mathbf{P}$-name for a real such that for every $n$

$$q \Vdash \text{“}\sigma(n) \text{ is the constant value of } \tau \restriction U_n\text{”}.$$

Then find $r \leq q$ and $U \in u$ such that $r \Vdash$ “$\sigma \restriction U$ is constant”. Set $V = \bigcup_{n \in U} U_n$; then $V \in v$ and $r \Vdash$ “$\tau \uparrow V$ is constant”. □

One can also check directly that $v$ in fact generates the $\Sigma(X,u)$ of the extension.

**6.4.** Lemma. *Let* $\mathbf{P}$ *be an* ${}^\omega\omega$*-bounding poset and let* $u, v \in \omega^*$ *be* $\mathbf{P}$*-indestructible Q-points. Then* $u \sim v$ *iff* $\mathbf{1} \Vdash$ “$u \sim v$”.

□ One direction is trivial. For the hard direction assume that $\gamma$ is a $\mathbf{P}$-name such that $\mathbf{1} \Vdash$ “$\gamma$ is a permutation of $\omega$ and $\gamma(u) = v$”. Find $p \in \mathbf{P}$ and $g\colon \omega \to \omega$ strictly increasing such that $\forall n \in \omega\;\; p \Vdash$ “$n + \gamma(n) + \gamma^{-1}(n) < g(n)$”. Let $P = \{[0, g^2(0)), [g^2(0), g^4(0)), \ldots\}$. $P$ is a partition of $\omega$ into finite sets. Let $U \in u$ be a selector for the partition. Let $U_0 = U \cap \bigcup_{m \in \omega}[g^{4m}(0), g^{4m+2}(0)]$ and $U_1 = U\backslash U_0$. For definiteness assume $U_0 \in u$. Let $n \in U_0$ and let $m = m_n$ be the unique number for which $g^{4m}(0) \leq n < g^{4m+2}(0)$. Then $p \Vdash$ “$g^{4m-1}(0) < \gamma(n) < g^{4m+3}(0)$”:

—$p \Vdash$ “$n < g^{4m+2}(0)$”, so $p \Vdash$ “$\gamma(n) < g(n) < g^{4m+3}(0)$”; and

—if, for some $q \leq p$, $q \Vdash$ “$\gamma(n) \leq g^{4m-1}(0)$”, then

$$q \Vdash \text{“}n = \gamma^{-1}\gamma(n) < g(\gamma(n)) \leq g^{4m}(0)\text{”},$$

a contradiction. Moreover if $n < n'$ then $m = m_n < m_{n'} = m'$ and so $4m + 3 \leq 4m' - 1$ and $g^{4m+3}(0) \leq g^{4m'-1}(0)$. For $m \in \omega$ let $Q_m = (g^{4m-1}(0), g^{4m+3}(0))$. Then $\{Q_m: m \in \omega\}$ is pairwise disjoint. Let $V = \bigcup_{n \in U_0} Q_{m_n}$. Then $q \Vdash$ "$\forall n \in U_0\, \gamma(n) \in Q_{m_n} \subseteq V$", i.e. $q \Vdash$ "$\gamma[U_0] \subseteq V$". But then $q \Vdash$ "$\exists W \in v: W \subseteq V$", and so $V \in v$. Now let $V_0 \in v$ be a selector for $\{Q_{m_n}: n \in Q_0\}$. Let $U_2 = \{n: V_0 \cap Q_{m_n} \neq \emptyset\}$. Then $q \Vdash$ "$U_2 = \gamma^{-1}[V_0]$", so that $U_2 \in u$. Now define $h: U_2 \to V_0$ by $h(n) =$ the point in $V_0 \cap Q_{m_n}$. Also, for $n \in U_2$, $q \Vdash$ "$\gamma(n)$ is the point in $V_0 \cap Q_{m_n}$", so that $q \Vdash$ "$h = \gamma \restriction U_2$" and so $h(u) = v$. □

Now we are ready for the main result of this section.

**6.5.** Theorem. *There exists an $\omega_1$-chain $\langle u_\alpha: \alpha \in \omega_1\rangle$, with respect to $\leq_{RF}$, of $\mathbf{P}(\omega)$-indestructible ultrafilters. Moreover, $\langle u_\alpha: \alpha \in \omega_1\rangle$ has no $\leq_{RF}$-upper bound, neither in the real world nor in any generic extension by $\mathbf{P}(\kappa)$ ($\kappa \geq \omega$).*

□ From Theorem 4.0 we obtain $2^{\omega_1}$ $\mathbf{P}(\omega)$-indestructible selective ultrafilters. By CH there are only $\omega_1$ permutations of $\omega$. Fix an almost disjoint family $\mathscr{A} = \{A_\alpha: \alpha \in \omega_1\}$ on $\omega$, and choose for each $\alpha \in \omega_1$ a $\mathbf{P}(\omega)$-indestructible selective ultrafilter $v_\alpha$ such that $A_\alpha \in v_\alpha$ and $\alpha \neq \beta \to v_\alpha \not\approx v_\beta$. Note that $V = \{v_\alpha: \alpha \in \omega_1\}$ is relatively discrete. Write $V = \bigcup\{D(\alpha, n): \alpha \in \omega_1, n \in \omega\}$ with $\langle \alpha, n\rangle \neq \langle \beta, m\rangle \to D(\alpha, n) \cap D(\beta, n) = \emptyset$. Moreover write $D(\alpha, n) = \{d(\alpha, n, i): i \in \omega\}$ (each $D(\alpha, n)$ is countably infinite). Now every selective ultrafilter is $\leq_{RF}$-minimal, so that for any $\langle \alpha, n\rangle$ and $\langle \beta, m\rangle$

$$\{i: d(\alpha, n, i) \leq_{RF} d(\beta, m, i)\} = \emptyset,$$

and by Lemma 6.4 this is also true in any generic extension by $\mathbf{P}(\kappa)$ ($\kappa \geq \omega$).

We construct for every $\alpha \in \omega_1$ a countable discrete set $X_\alpha = \{x(\alpha, n): n \in \omega\}$ of ultrafilters as follows:

Set $X_0 = D(0,0)$. Given $X_\alpha$, let $x(\alpha + 1, n) = \Sigma(D(\alpha + 1, n), x(\alpha, n))$ ($n \in \omega$) and let $X_{\alpha+1} = \{x(\alpha + 1, n): n \in \omega\}$. If $\alpha$ is a limit, fix a strictly increasing cofinal sequence $\langle \alpha_i: i \in \omega\rangle$ in $\alpha$ with $\alpha_0 = 0$. Set $Z_0 = \omega$ and, given $Z_i$, let

$$Z_{i+1} = \{n \in Z_i: n > i \text{ and } \forall j \leq i,\ x(\alpha_j, n) <_{RF} x(\alpha_{j+1}, n)\}.$$

Then for $n \in Z_i \backslash Z_{i+1}$ set $x(\alpha, n) = \Sigma(D(\alpha, n), x(\alpha_i, n))$, and $X_\alpha = \{x(\alpha, n): n \in \omega\}$ of course. By Lemma 6.3 every $x(\alpha, n)$ is $\mathbf{P}(\omega)$-indestructible. In the real world and in the generic extension(s) by $\mathbf{P}(\kappa)(\kappa \geq \omega)$, the set $\{x(\alpha, n): \alpha \in \omega_1,\ n \in \omega\}$ satisfies the following conditions:

(i) If $\beta \in \alpha$ then $\{n: x(\beta, n) <_{RF} x(\alpha, n)\}$ is cofinite.

(ii) If $\alpha$ is a limit, $i \in \omega$, $n \in Z_i \backslash Z_{i+1}$ and $\alpha_i < \beta < \alpha$, then $x(\alpha, n)$ and $x(\beta, n)$ are $\leq_{RF}$-incomparable.

(iii) For every $\alpha \in \omega_1$ and $n \in \omega$, the set $\{\beta \in \alpha: x(\beta, n) <_{RF} x(\alpha, n)\}$ is finite.

The proof of (i) is straightforward; one should note that if $v = \Sigma(X, u)$ and all ultrafilters are indestructible then $\mathbf{1} \Vdash v = \Sigma(X, u)$. The proof of (ii) uses 6.2 except in the case when $x(\beta, n) = \Sigma(D(\beta, n), x(\alpha_i, n))$, but then we know that in the real world and in the extension $\{i: d(\alpha, n, i)$ and $d(\beta, n, i)$ are $\leq_{RF}$-comparable$\} = \emptyset$, so that also $x(\beta, n)$ and $x(\alpha, n)$ are $\leq_{RF}$-incomparable, by 6.2. Finally, (iii) follows from (ii) by induction.

Now let $u$ be any $\mathbf{P}(\omega)$-indestructible ultrafilter, and set $u_\alpha = \Sigma(X_\alpha, u)$ for $\alpha \in \omega_1$. Each $u_\alpha$ is indestructible, and the following arguments work in the real world and the extension. By (i), if $\beta \in \alpha$ then $u_\beta <_{RF} u_\alpha$. Now assume $v = \Sigma(Y, u)$ is an upper bound for $\{u_\alpha: \alpha \in \omega_1\}$. Then, for every $\alpha \in \omega_1$, $\{n: x(\alpha, n) <_{RF} y_n\} \in u$. It follows that, for some $n$, $\{\alpha: x(\alpha, n) <_{RF} y_n\}$ is uncountable ($\omega_1$ is uncountable in the extension). Let $\alpha$ be the $\omega$th element of this set. It then follows that $\{\beta \in \alpha: x(\beta, n) <_{RF} x(\alpha, n)\}$ is infinite, contradicting (iii). Thus, $\{u_\alpha: \alpha \in \omega_1\}$ has all required properties. □

**§7. A point with $\omega$ predecessors.** An ultrafilter can have $0, 1, 2, \ldots, \omega$ or $2^\omega$ $\leq_{RF}$-predecessors [BuBu]. The previous sections give us ultrafilters of character $\omega_1$ with 0 and $2^\omega$ predecessors: for 0 they can be selective, $P$-point, or weak $P$-point; for $2^\omega$, take $u_\omega$ from §5. By [Bu1] there are $2^\omega$ ultrafilters between $\{u_n: n \in \omega\}$ and $u_\omega$. It is also easy to find ultrafilters with $1, 2, \ldots$ predecessors: pick one with 0 predecessors, call it $x$, take a countable discrete set $X$ of copies of $x$ and set $u_0 = x$ and $u_{n+1} = \Sigma(X, u_n)$ ($n \in \omega$); then $u_n$ has exactly $n$ predecessors. If we take $X$ to be $\mathbf{P}(\omega)$-indestructible and selective, then $u_n$ will also have $n$ predecessors in any generic extension by $\mathbf{P}(\kappa)$ $(\kappa \geq \omega)$.

We shall construct a $\mathbf{P}(\omega)$-indestructible ultrafilter with exactly $\omega$ predecessors, both in the real world and in the extension.

To begin, let $u$ be $\mathbf{P}(\omega)$-indestructible and selective (this to insure that $\mathbf{1} \Vdash$ "$u$ has no $\leq_{RF}$-predecessors"). Let $\mathscr{P}_0 = \{P_{0j}: j \in \omega\}$ be a partition of $\omega$ into infinite sets, and for each $j \in \omega$ let $x_{0j}$ be a copy of $u$ on $P_{0j}$. Inductively let

$$X_i = \{x_{ij}: j \in \omega\},$$
$$x_{i+1,j} = \Sigma(X_i, x_{0j}) = \Sigma(X_0, x_{ij}),$$
$$P_{i+1,j} = \bigcup\{P_{il}: l \in P_{0j}\} = \bigcup\{P_{0l}: l \in P_{ij}\}.$$

By Lemma 6.3 each $x_{ij}$ is an indestructible ultrafilter; moreover $P_{ij} \in x_{ij}$ and $\{P_{ij}: j \in \omega\}$ is pairwise disjoint for every $i$. We shall find an indestructible $v \in \bigcap_{i \in \omega} \bar{X}_i$ such that $\{\Omega(X_i, v): i \in \omega\}$ has no $\leq_{RF}$-lower bound in this and the other world.

It follows, since $u$ was $\leq_{RF}$-minimal to begin with, that $\{\Omega(X_i, v): i \in \omega\}$ is exactly the set of $\leq_{RF}$-predecessors of $v$. It suffices $[\mathrm{S}_0, \mathrm{E}]$ to ensure that whenever $Y \subseteq \bigcup_{i \in \omega} X_i$ is discrete and $v \in \bar{Y}$ there is an $i \in \omega$ such that $v \in \overline{Y \cap X_i}$. In $[\mathrm{S}_0]$ and [BuBu] this was accomplished by "simply" taking care of all such subsets of $\bigcup_{i \in \omega} X_i$. We have to take care of new subsets of $\bigcup_{i \in \omega} X_i$ too. To do this assume $Y \subseteq \bigcup_{i \in \omega} X_i$ is discrete and that $v \notin \overline{Y \cap X_i}$ for every $i$. Pick $V_i \in v$ such that $V_i^* \cap (Y \cap X_i) = \varnothing$ (for $i \in \omega$). If $\langle V_i: i \in \omega\rangle$ is not from the real world, we can find $\langle W_i; i \in \omega\rangle$ in the real world such that $\forall i \in \omega\ W_i \subseteq V_i$. We arrange it so that there is then one $W \in v$, from the real world, such that $W^* \cap X_i \subseteq W_i^* \cap X_i$. This $W$ then satisfies $W^* \cap Y = \varnothing$, so that $v \notin \bar{Y}$. We recall that $\langle\langle p_\alpha, f_\alpha\rangle: \alpha \in \omega_1\rangle$ enumerates $\mathbf{A}$ and that $\phi_\alpha$ is a name for the real determined by $\langle p_\alpha, f_\alpha\rangle$. We let $\{s_\alpha: \alpha \in \omega_1, \alpha \text{ a limit}\}$ count ${}^\omega\omega_1$ in such a way that always $\operatorname{ran}(s_\alpha) \subseteq \alpha$. Fix a bijection $\phi: \omega \leftrightarrow \omega \times \omega$.

We find $\{V_\alpha: \alpha \in \omega_1\}$ such that for every $\alpha \in \omega_1$ the following conditions hold:

(i) The filter $\mathscr{F}_\alpha$ generated by $\{V_\beta: \beta \leq \alpha\}$ satisfies

$$\forall F \in \mathscr{F}_\alpha \forall i \in \omega \quad \{j: F \in x_{ij}\} \text{ is infinite.}$$

(ii) There is a $q_\alpha \leq p_\alpha$ such that $q_\alpha \Vdash$ "$\phi_\alpha \restriction V_{\alpha+1}$ is constant".

(iii) If $\alpha$ is a limit then $\forall i \in \omega\ V_\alpha^* \cap X_i \subseteq V_{s_\alpha(i)}^* \cap X_i$.

By (ii) the filter $v$ generated by $\{V_\alpha: \alpha \in \omega_1\}$ is a $\mathbf{P}(\omega)$-indestructible ultrafilter; by (i) $v \in \bigcap_{i \in \omega} \bar{X}_i$; and by (iii) and the above argument $\{\Omega(X_i, v): i \in \omega\}$ has no $\leq_{RF}$-lower bound in this or the other world. We set $V_0 = \omega$. Given $\{V_\beta : \beta \le \alpha\}$, find $V_{\alpha+1}$ as follows. Much as in the proof of Lemma 5.0, we can find a $q \le p_\alpha$ and $U_{ij} \in x_{ij}$ for every $i, j$ such that $q \Vdash$ "$\phi_\alpha \restriction U_{ij}$ is constant" (when constructing $p_{i+1}$ from $p_i$, take $B_{ij+1} \in x_{\phi(i)}$, in the end $q = p_\omega$).

Next let $\{F_i : i \in \omega\}$ enumerate $\{V_\beta : \beta \le \alpha\}$. Let $J_i = \{j \in \omega : \bigcap_{l \le i} F_l \in X_{ij}\}$ $(i \in \omega)$. By assumption each $J_i$ is infinite. Define a real $\psi$ by $q \Vdash$ "$\psi(n) =$ the constant value of $\phi_\alpha$ on $U_{\phi(n)}$". Using Lemma 5.0, find $q_\alpha \le q$ and $A \le \omega$ such that $q_\alpha \Vdash$ "$\psi \restriction A$ is constant" and the set $\phi[A] \cap \{\langle i, j\rangle : j \in J_i\}$ is infinite for infinitely many $i \in \omega$. Set $V_{\alpha+1} = \bigcup\{U_{\phi(n)} : n \in A\}$; then $q_\alpha \Vdash$ "$\phi_\alpha \restriction V_{\alpha+1}$ is constant" and, for infinitely many (and hence for all) $i$, $\{j : V_{\alpha+1} \cap \bigcap_{l \le i} F_l \in x_{ij}\}$ is infinite. Finally, if $\alpha$ is a limit let $\{F_i : i \in \omega\}$ enumerate $\{V_\beta : \beta \in \alpha\}$ in such a way that always $F_{2n} = V_{s_\alpha(n)}$. Set $G_i = \bigcap_{l \le 2i} F_l$ $(n \in \omega)$. For $i \in \omega$ determine $W_i$ as follows. The set $\{j : G_i \in x_{ij}\}$ is infinite and $X_{i+1}$ is nowhere dense in $\bar{X}_i \backslash X_i$, so let $H_i \subseteq \omega$ be infinite such that $G_i \in x_{ij}$ for $j \in H_i$ and

$$\bar{X}_{i+1} \cap \overline{\{x_{ij} : j \in H_i\}} = \varnothing.$$

Next set $G_i^0 = G_i \cap \bigcup\{P_{ij} : j \in H_i\}$ and inductively

$$G_i^{l+1} = G_i^l \cap \bigcup\{P_{i-(l+1),j} : G_i^l \in x_{i-(l+1),i}\}.$$

Then set $W_i = G_i^i$. In the end let $V_\alpha = \bigcup_{i \in \omega} W_i$. Clearly $H_i \subseteq \{j : V_\alpha \cap G_i \in x_{ij}\}$, so (i) is fulfilled. For (iii), note that if $k > i$ then $W_i^* \cap X_k = \varnothing$ and if $k \le i$ then $W_i^* \cap X_k \subseteq G_i^* \cap X_k \subseteq V_{s_\alpha(k)} \cap X_k$. Hence $V_\alpha^* \cap X_k \subseteq V_{s_\alpha(k)}^* \cap X_k$ for every $k$.

**§8. A summary.** Using the results from §§3–7 we can now state our consistency result.

**8.0.** THEOREM. *It is relatively consistent with ZFC that $2^\omega$ is arbitrarily large and there are ultrafilters of character $\omega_1$ of the following types*:

a) *selective,*

b) *P-point but not selective,*

c) *$\omega_1$-OK but not P-point,*

d) *having $0, 1, 2, \ldots \le_{\mathrm{RF}}$ predecessors,*

e) *having $2^\omega$ $\le_{\mathrm{RF}}$-predecessors, and*

f) *having exactly $\omega$ $\le_{\mathrm{RF}}$-predecessors. Moreover*

g) *There is an unbounded $\le_{\mathrm{RF}}$-chain of cofinality $\omega_1$ consisting wholly of ultrafilters of character $\omega_1$.*

□ Start with a model of CH, take $\kappa$ as desired and force with $\mathbf{P}(\kappa)$. Then Theorem 4.0 gives a); Theorem 4.1 gives b); Theorem 5.1 gives c); d) was noted in §7, as was e); f) was established in §7; and for g) take $\{u_\alpha : \alpha \in \omega_1\}$ from §6. The chain is $X = \{x : \exists \alpha \in \omega_1, x \le_{\mathrm{RF}} u_\alpha\}$. Then $X$ is unbounded and, since by [BuBu] if $u \le_{\mathrm{RF}} v$ then $\chi(u) \le \chi(v)$, every element of $X$ has character $\omega_1$. □

As noted in the Introduction, a) and b) were obtained before in various ways. Kunen [Ku3] showed that if there is a selective ultrafilter of character $\omega_1$ then there is an ultrafilter as in e).

**§9. Some questions and additional remarks.** We have constructed a variety of $\mathbf{P}(\omega)$-indestructible ultrafilters. A question that comes to mind is "Is every ultrafilter $\mathbf{P}(\omega)$-indestructible?" The answer is negative. In fact Kunen [Ku3] pointed out to me that there are always ultrafilters such that, no matter how a real is added to the world, they do not remain ultrafilters after that. So already one Sacks real destroys some ultrafilters.

As noted in the Introduction, many of the known ultrafilters of character less than $2^\omega$ are selective, or at least $P$-points; the others are usually constructed using something selective as a starting point. The same is true for the ultrafilters constructed in this paper. In §5 it is practically unavoidable that a selective ultrafilter or a $P$-point is constructed along the way: $\{I_\alpha : \alpha \in \omega_1\}$ generates a $P$-point. This leads to the following question posed first by Bukovský:

**9.0.** QUESTION. a) Can there be an ultrafilter of character less than $2^\omega$ which has nothing to do with $P$-points? or even:

b) Is it consistent that there are no $P$-points, yet there is an ultrafilter of character less than $2^\omega$?

The answer to this question with "selective" instead of "$P$-points" is positive: First recall that $\mathfrak{d} = \min\{|D| : D \subseteq {}^\omega\omega \wedge \forall g \in {}^\omega\omega\, \exists f \in D : g \leq f\}$ [vD]. Now in [BlSh] it is shown that in the model obtained by iterating rational perfect set forcing $\omega_2$ times (starting with CH) one has $\mathfrak{d} = 2^\omega = \omega_2$, and for every $x \in \omega^*$ there is a finite-to-one $f \in {}^\omega\omega$ such that $\chi(f(x)) < \mathfrak{d}$ (so $\chi(f(x)) = \omega_1$).

So there are many ultrafilters of character less than $2^\omega$. Now in this model there are no selective ultrafilters: if $x$ is selective then $\chi(x) \geq \mathfrak{d}$ and if $f \in {}^\omega\omega$ then either $f(x) \in \omega$ or $f(x) \sim x$. This model definitely does not answer Question 9.0: for $x \in \omega^*$, $\chi(x) = \omega_1$ iff $x$ is a $P$-point.

A consequence of Lemma 6.3 is that if $Q$ is either $\mathbf{P}(\omega)$ or $\mathbf{P}_\alpha$ (an $\alpha$-stage iteration of **PF**), then $I = \{u \in \omega^* : u \text{ is } Q\text{-indestructible}\}$ is a (by CH nonempty) countably compact subspace of $\omega^*$. What, if anything, can be said about this space? By [BaLa], $I$ contains *all* selective ultrafilters in case $Q = \mathbf{P}_\alpha$. A final question is

**9.1.** QUESTION. Can we do the same things with $\omega_2, \omega_3, \ldots$? The approach of this paper will not work immediately: if $u$ were a $\mathbf{P}(\kappa)$-indestructible ultrafilter of character $\omega_2$, then after forcing with $\mathbf{P}(\kappa)$, $2^\omega$ would be collapsed to $\omega_1$ (because $\mathbf{P}(\kappa)$ contains $\mathrm{Fn}(\kappa, 2, \omega_1)$), so that in the extension $u$ would have character $\omega_1$ anyway.

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