SOME REALCOMPACT SPACES

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Abstract. We present examples of realcompact spaces with closed subsets that are C^*-embedded but not C-embedded, including one where the closed set is a copy of $\mathbb{N}$.

Introduction

The purpose of this note is to provide some examples of realcompact (but not compact) spaces that have closed subspaces that are C^*-embedded but not C-embedded, and in particular an example where the closed subspace is a copy of the discrete space $\mathbb{N}$ of natural numbers — what we henceforth call *a closed copy of* $\mathbb{N}$.

The reason for our interest is that we are not aware of any such examples. The examples in [5], for instance, of C^*- but not C-embedded subsets are not all closed and when they are closed the pseudocompactness of the ambient space makes C-embedding impossible.

The only explicit question of this nature that we could find is in the paper [7, Question 1], which asks whether C^*-embedded subsets (not necessarily closed) of first-countable spaces are C-embedded. In that case there is an independence result: there is a counterexample if $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$, and in the model obtained by adding supercompact many random reals the implication holds, see [1].

The more specific question of having a closed copy of $\mathbb{N}$ that is C^*-embedded but not C-embedded arises from an analysis of their position in powers of the real line; see Section 2 for an explanation.

It is clear that our examples should be non-normal Tychonoff spaces. After some preliminaries we briefly discuss two classical examples, the Tychonoff and Dieudonné planks, and introduce a further plank $\mathbb{J}$.

The latter is pseudocompact but we modify it in two steps. The first step yields a plank that is neither pseudocompact nor realcompact, and the second step gives us our first example.

Our second example is constructed in Section 5 and it contains a closed copy of $\mathbb{N}$ that is C^*- but not C-embedded.

1. Preliminaries

We follow [4] and [5] as regards general topology and rings of continuous functions. As is common $C(X)$ and $C^*(X)$ denote the rings of real-valued continuous and bounded continuous functions respectively.

A subset A of a space X is *C-embedded* if every continuous function $f : A \to \mathbb{R}$ admits a continuous extension $\bar{f} : X \to \mathbb{R}$. It is *$C^*$-embedded* if every bounded continuous function $f : A \to \mathbb{R}$ admits a bounded continuous extension $\bar{f} : X \to \mathbb{R}$.

We define a space X to be *realcompact* if it can be embedded into a power of the real line as a closed subset. The most useful characterization for this paper is that

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every zero-set ultrafilter with the countable intersection property has a non-empty intersection, see [4, Theorem 3.11.11].

Planks. As noted above our examples will be non-normal Tychonoff spaces. Non-normal because we need a closed subset that is not C-embedded and Tychonoff because that is part of the definition of realcompactness.

There are various examples of such spaces, such as the Tychonoff plank $\mathbb{T}$ ([9] or [8, Example 87]), and the Dieudonné plank $\mathbb{D}$ ([2] or [8, Example 89]).

Both start with the product set $X = (\omega_1 + 1) \times (\omega_0 + 1)$ and take the subset $P = X \setminus \{\langle \omega_1, \omega_0 \rangle\}$ as the underlying set of the space.

In each case P has the subspace topology where X has a product topology induced by topologies on the factors.

For $\mathbb{T}$ one takes the order topologies on both ordinals. For $\mathbb{D}$ one enlarges the order topology of $\omega_1 + 1$ by making all points of ω_1 isolated.

We shall consider a third variation in Section 3 below.

2. Context

To begin we have the following proposition, which may be well-known, but bears repeating here because it shows that if one has a non-C-embedded copy of $\mathbb{N}$ in a realcompact space then that copy contains many infinite subsets that *are* C-embedded.

Proposition 2.1. *Let X be realcompact and A a subset whose closure is not compact, then A contains a countably infinite subset that is closed, discrete and C-embedded in X.*

Proof. Take a point x_0 in $\beta X \setminus X$ that is in the closure of A. Apply [4, Theorem 3.11.10] to find a continuous function $f : \beta X \to [0, 1]$ such that $f(x_0) = 0$ and $f(x) > 0$ if $x \in X$. Because x_0 is in the closure of A we can find a sequence $\langle a_n : n \in \mathbb{N} \rangle$ in A such that $\langle f(a_n) : n \in \mathbb{N} \rangle$ is strictly decreasing with limit 0.

The set $N = \{a_n : n \in \mathbb{N}\}$ is closed and C-embedded in X. It is closed as a locally finite set of points. If $g : N \to \mathbb{R}$ is given then we can take a continuous function $h : (0, 1] \to \mathbb{R}$ such that $h(f(a_n)) = g(a_n)$ for all n. Then $h \circ f$ is a continuous extension of g. □

The space in Section 5 illustrates this proposition quite well: one can point out very many infinite C-embedded subsets of the non-C-embedded copy of $\mathbb{N}$ explicitly.

This proposition also shows why the initial planks in Section 3 are not realcompact: there are not enough C-embedded copies of $\mathbb{N}$.

Closed copies of $\mathbb{N}$ in other spaces. Here we collect a few natural questions that arise when one considers C^*- and C-embedding of closed copies of $\mathbb{N}$.

Suppose one has two closed copies, $\mathbb{N}_1$ and $\mathbb{N}_2$ say, of the space of natural numbers in a Tychonoff space X.

(1) If $\mathbb{N}_1$ and $\mathbb{N}_2$ are C-embedded is their union C-embedded?
(2) If $\mathbb{N}_1$ and $\mathbb{N}_2$ are C^*-embedded is their union C^*-embedded?
(3) If $\mathbb{N}_1$ is C-embedded and $\mathbb{N}_2$ is C^*-embedded is their union C^*-embedded?

Questions (1) and (3) have positive answers.

For question (3) one uses a continuous extension $f : X \to \mathbb{R}$ of a bijection between $\mathbb{N}_1$ and $\mathbb{N}$ to obtain a discrete family $\{O_x : x \in \mathbb{N}_1\}$ of open sets with $x \in O_x$ for all $x \in \mathbb{N}_1$. Then, given a bounded function $g : \mathbb{N}_1 \cup \mathbb{N}_2 \to \mathbb{R}$ one first takes a bounded extension $\bar{g} : X \to \mathbb{R}$ of $g \restriction \mathbb{N}_2$ and then modifies $\bar{g}$ on each O_x to obtain an extension of g.

The argument for question (1) is similar but easier because one can find a single discrete family of open sets that separates the points of $\mathbb{N}_1 \cup \mathbb{N}_2$.

A counterexample to question (2) can be obtained by taking Katětov's example of a pseudocompact space with a closed C^*-embedded copy of $\mathbb{N}$, see [6] or [4, Example 3.10.29]. The example is $\mathbb{K} = \beta\mathbb{R} \setminus \mathbb{N}^*$, and the copy of $\mathbb{N}$ is just $\mathbb{N}$ itself. Take the sum of two copies of this space, $\mathbb{K} \times \{0, 1\}$, and for every $x \in \mathbb{K} \setminus \mathbb{R}$ identify the points $\langle x, 0\rangle$ and $\langle x, 1\rangle$. The copies $\mathbb{N} \times \{0\}$ and $\mathbb{N} \times \{1\}$ are both C^*-embedded in the resulting quotient, but their union is not.

Below we shall show that question (2) also has a negative answer in the class of realcompact spaces.

Closed copies of $\mathbb{N}$ in powers of $\mathbb{R}$. The discrete space $\mathbb{N}$ is realcompact, hence it admits many embeddings into powers of $\mathbb{R}$ as a closed and C-embedded set.

The specific question from the introduction is equivalent to the question whether there is a closed copy of $\mathbb{N}$ in some power of $\mathbb{R}$ that is C^*-embedded but not C-embedded. Indeed the latter is a special case of the former and a positive answer to the former answers the latter by embedding the example as a closed C-embedded copy into some power of $\mathbb{R}$; the copy of $\mathbb{N}$ is then not C-embedded in that power.

The difference between C- and C^*-embedding manifests itself also in the way certain maps can be factored through partial products.

Assume first that $\mathbb{N}$ is C-embedded in a power of $\mathbb{R}$, say $\mathbb{R}^\kappa$. Then there is a continuous function $f : \mathbb{R}^\kappa \to \mathbb{R}$ such that $f(n) = n$ for all $n \in \mathbb{N}$. It is well known that f factors through a countable subset of κ: there are a countable subset I of κ and a continuous function $g : \mathbb{R}^I \to \mathbb{R}$ such that $f = g \circ \pi$, where π is the projection onto $\mathbb{R}^I$, see [4, Problem 2.7.12]. Then the projection $\pi[\mathbb{N}]$ of $\mathbb{N}$ in $\mathbb{R}^I$ is C-embedded and we see that every function from $\mathbb{N}$ to $\mathbb{R}$ has an extension that factors through the partial power $\mathbb{R}^I$.

Now assume $\mathbb{N}$ is C^*-embedded but not C-embedded in $\mathbb{R}^\kappa$. Then every bounded function from $\mathbb{N}$ to $[0, 1]$ has a continuous extension to $\mathbb{R}^\kappa$. Such a continuous extension will then factor through a partial product with countably many factors but the set of factors will vary with the function.

Indeed, assume that there is a single countable set I such that every bounded function $f : \mathbb{N} \to [0, 1]$ has a continuous extension that factors through $\mathbb{R}^I$. Apply this with the function $f(n) = 2^{-n}$; using a factorization $\bar{f} = g \circ \pi$, as above, of an extension $\bar{f}$ of f that the projection π onto $\mathbb{R}^I$ is injective on $\mathbb{N}$ and that $\pi[\mathbb{N}]$ is relatively discrete in $\mathbb{R}^I$.

We also find that $\pi[\mathbb{N}]$ is C^*-embedded in the metric space $\mathbb{R}^J$, and hence closed. But then $\pi[\mathbb{N}]$ is C-embedded in $\mathbb{R}^J$ and $\mathbb{N}$ is C-embedded in $\mathbb{R}^\kappa$.

Using the plank $\mathbb{A}$ from Section 5 we obtain such a copy of $\mathbb{N}$ in a power of $\mathbb{R}$. The standard embedding of $\mathbb{A}$ in the power $\mathbb{R}^{C(\mathbb{A})}$ yields a closed C-embedded copy of $\mathbb{A}$. The right-hand side R is a closed copy of $\mathbb{N}$ that is C^*-embedded in $\mathbb{A}$ and hence in $\mathbb{R}^{C(\mathbb{A})}$, but not C-embedded in $\mathbb{R}^{C(\mathbb{A})}$.

This then suggests the following question.

Question 1. What is the minimum cardinal κ such that $\mathbb{R}^\kappa$ contains a closed copy of $\mathbb{N}$ that is C^*-embedded but not C-embedded?

Since $\mathbb{R}^{\omega_0}$ is metrizable and, as we shall see, $|C(\mathbb{A})| = \mathfrak{c}$ we know that $\aleph_0 < \kappa \leq \mathfrak{c}$. This means that the Continuum Hypothesis settles this question, but there may be some variation under other assumptions.

Our answer to question (2) from the list on page 2 produces, in the same way, a closed copy of $\mathbb{N}$ in $\mathbb{R}^\mathfrak{c}$ that is not C^*-embedded. After we submitted this paper we were able to answer the analogue of Question 1: the smallest cardinal κ such that $\mathbb{R}^\kappa$ contains a closed copy of $\mathbb{N}$ that is not C^*-embedded is $\aleph_1$. See [3] for a surprising (to us) variety of closed copies of $\mathbb{N}$ in $\mathbb{R}^{\omega_1}$ that are not C^*-embedded.

3. The plank $\mathbb{J}$ and a variation

In our third variation of the idea of the plank the topology on $\omega_0 + 1$ remains as it is and we let, from now on, $\omega_1 + 1$ carry the topology of the one-point compactification of the discrete space ω_1, with ω_1 the point at infinity.

In this case we denote the resulting space by $\mathbb{J}$. It is a minor variation of [4, Example 2.3.36]: in the terminology of that book $\mathbb{J} = A(\aleph_1) \times A(\aleph_0) \setminus \{\langle x_0, y_0\rangle\}$, where we have specified the underlying sets of the factors explicitly.

As in the case of $\mathbb{T}$ and $\mathbb{D}$ the top line $T = \omega_1 \times \{\omega_0\}$ and the right-hand side $R = \{\omega_1\} \times \omega_0$ cannot be separated by open sets in $\mathbb{J}$. Hence their union is not C^*-embedded in the space $\mathbb{J}$.

A more careful analysis of the continuous functions on $\mathbb{J}$ will reveal that neither T nor R is C^*-embedded.

Indeed: let $f : \mathbb{J} \to \mathbb{R}$ be continuous. For each $n \in \omega_0$ the set $\{\alpha \in \omega_1 : f(\alpha, n) \neq f(\omega_1, n)\}$ is countable. It follows that there is an α in ω_1 such that $f(\beta, n) = f(\omega_1, n)$ for all n and all $\beta \geq \alpha$. By continuity this implies that $f(\beta, \omega_0) = f(\alpha, \omega_0)$ for all $\beta \geq \alpha$. This shows that the function $\langle\alpha, \omega_0\rangle \mapsto \alpha \bmod 2$ (the characteristic function of the odd ordinals), which is continuous on T, has no continuous extension to $\mathbb{J}$.

If we let $r = f(\alpha, \omega_0)$ then it follows that $\lim_{n\to\infty} f(\omega_1, n) = r$. We see that the function $\langle\omega_1, n\rangle \mapsto n \bmod 2$, which is continuous on R, has no continuous extension to $\mathbb{J}$ either.

This argument also shows that $\mathbb{J}$ is not realcompact: the co-countable sets on the top line form a zero-set ultrafilter with the countable intersection property that has an empty intersection. Alternatively use Proposition 2.1: no infinite subset of R is C^*-embedded.

The space $\mathbb{J}$ is not pseudocompact either: the diagonal $\{\langle n, n\rangle : n \in \omega_0\}$ is a clopen discrete subset.

Ensuring C^*-embeddedness. To ensure that R is C^*-embedded we change the second factor in our product.

We let $X = (\omega_1 + 1) \times \beta\omega_0$ and $P = X \setminus (\{\omega_1\} \times \omega_0^*)$. The right-hand side R remains unchanged but the top line T now becomes $\omega_1 \times \omega_0^*$.

To see why this makes the right-hand side C^*-embedded let $f : R \to [0, 1]$ be continuous. Take the unique continuous extension of $n \mapsto f(\omega_1, n)$ to $\beta\omega_0$ and it on every vertical line $\{\alpha\} \times \beta\omega_0$ to get an extension of f to the plank P.

This does not make the right-hand side C-embedded: the analysis of the continuous functions on $\mathbb{J}$ shows that for any extendable function f the function $n \mapsto f(\omega_1, n)$ should be extendable from ω_0 to $\beta\omega_0$ and hence should be bounded.

When we adapt the analysis of continuous functions on $\mathbb{J}$ to continuous functions on P we obtain that the intersection of a zero-set with the top line T contains a set of the form $A(\alpha, Z) = [\alpha, \omega_1) \times Z$, where $\alpha \in \omega_1$ and Z is a zero-set of ω_0^* (and Z could be empty of course).

Now take any point u in ω_0^* and let $\mathcal{Z}_u$ be the family of zero-sets of ω_0^* that contain u. Then $\{A(\alpha, Z) : \alpha \in \omega_1, Z \in \mathcal{Z}_u\}$ generates a zero-set ultrafilter with the countable intersection property that has an empty intersection. Thus, the present plank is not realcompact. Again, Proposition 2.1 applies as well: no closed copy of $\mathbb{N}$ (and there are many) in R is C-embedded.

4. The plank $\mathbb{V}$

It should be clear that the fact that continuous functions on ω_1+1 are constant on co-countable sets is the main cause that the two previous examples are not realcompact.

To alleviate that we replace ω_1+1 by $\beta\omega_1$, where ω_1 still has the discrete topology. We take the product $\Pi = \beta\omega_1 \times \beta\omega_0$; our example is $\mathbb{V} = \Pi \setminus (\omega_1^* \times \omega_0^*)$.

The top line and the right-hand side now become $T = \omega_1 \times \omega_0^*$ and $R = \omega_1^* \times \omega_0$.

The right-hand side R is C^*-embedded in $\mathbb{V}$. This is proved almost as in the case of the plank P.

Let $f : R \to [0,1]$ be continuous. Apply the Tietze-Urysohn extension theorem to each horizontal line H_n to obtain a continuous extension $f_n : H_n \to [0,1]$ of the restriction of f to $\omega_1^* \times \{n\}$.

Next take, for each $\alpha \in \omega_1$, the unique extension g_α of the map $\langle\alpha, n\rangle \mapsto f_n(\alpha, n)$ to $\{\alpha\} \times \beta\omega_0$. The union of the maps g_α and f_n is an extension of f to $\mathbb{V}$.

The right-hand side R is not C-embedded in $\mathbb{V}$. Define $f : R \to \mathbb{R}$ by $f(x,n) = n$. Assume $g : \mathbb{V} \to \mathbb{R}$ is a continuous extension of f. For each n and k the set

$$\{\alpha \in \omega_1 : \bigl|g(\alpha,n) - n\bigr| \geq 2^{-k}\}$$

is finite, hence for each n the set $\{\alpha : g(\alpha,n) \neq n\}$ is countable. It follows that there are co-countably many $\alpha \in \omega_1$ such that $g(\alpha,n) = n$ for all n. For each such α the restriction of g to the compact set $\{\alpha\} \times \beta\omega_0$ would be unbounded, which is a contradiction.

The space $\mathbb{V}$ is realcompact. Let $\mathcal{Z}$ be a zero-set ultrafilter with the countable intersection property. We show that its intersection is nonempty.

To begin: if for some n the clopen 'horizontal line' $H_n = \beta\omega_1 \times \{n\}$ belongs to $\mathcal{Z}$ then the compactness of this line implies that $\bigcap \mathcal{Z}$ is nonempty.

In the opposite case the complements of the H_n belong to $\mathcal{Z}$; the intersection of these complements is equal to the top line T. By the countable intersection property we find that every member of $\mathcal{Z}$ intersects T, hence $T \in \mathcal{Z}$.

For every subset A of ω_1 the partial top line $T_A = A \times \omega_0^*$ is a zero-set as it is the intersection of T with the clopen subset $\operatorname{cl} A \times \beta\omega_0$ of Π.

Consider $u = \{A : T_A \in \mathcal{Z}\}$. This is an ultrafilter on ω_1 and it has the countable intersection property and therefore, because ω_1 is not a measurable cardinal, it is a principal ultrafilter. Take $\alpha \in \omega_1$ such that $u = \{A \subseteq \omega_1 : \alpha \in A\}$.

It follows that the compact set $\{\alpha\} \times \omega_0^*$ belongs to $\mathcal{Z}$ so that $\bigcap \mathcal{Z} \neq \emptyset$.

Comments. The natural maps from $\beta\omega_1$ onto ω_1+1 and and from $\beta\omega_0$ onto ω_0+1-as-the-one-point-compactification are perfect and irreducible. Hence so is the product map from Π onto $(\omega_1+1) \times (\omega_0+1)$. It follows that the restriction of this map to $\mathbb{V}$ is perfect as well, because $\mathbb{V}$ is the preimage of $\mathbb{J}$.

We have seen that $\mathbb{J}$ is not realcompact, so we have here a very simple perfect map that does not preserve realcompactness.

We also note that $\mathbb{V}$ is extremally disconnected and it is in fact the absolute of $\mathbb{J}$.

5. Another plank

In this section we construct a realcompact space with a closed copy of $\mathbb{N}$ that is C^*-embedded but not C-embedded.

We let D be the tree $2^{<\omega}$ with the discrete topology and we topologize $D \cup 2^\omega$ so as to obtain a natural compactification cD of D. If $x \in 2^\omega$ then its nth neighbourhood $U(x,n)$ will be the 'wedge' above $x \restriction n$:

$$U(x,n) = \{s \in cD : x \restriction n \subseteq s\}$$

Let $e : \beta D \to cD$ be extension of the identity map.

This yields a partition of D^* into closed sets, indexed by 2^ω: simply let $K_x = \{u \in D^* : e(u) = x\}$.

To construct our plank we take a point ∞ not in 2^ω and topologize $\mathfrak{C} = 2^\omega \cup \{\infty\}$ by making every point of 2^ω isolated and letting

$$\{U : \infty \in U \wedge |\mathfrak{C} \setminus U| \le \aleph_0\}$$

be a local base at ∞.

Let us note that $\mathfrak{C}$ has a property in common with the horizontal lines in our planks above: for every continuous function $f : \mathfrak{C} \to \mathbb{R}$ there is a neighbourhood of ∞ (a co-countable set) on which f is constant.

We let $\mathbb{A}$ be the following subspace of $\mathfrak{C} \times \beta D$:

$$\mathbb{A} = (\mathfrak{C} \times D) \cup \bigcup_{x \in 2^\omega} \{x\} \times K_x$$

We let $R = \{\infty\} \times D$ denote the right-hand side of the plank. The top line $T = \bigcup_{x \in 2^\omega} \{x\} \times K_x$ is not as smooth as in the other examples; every point u of D^* occurs just once in the top line, when $e(u) = x$.

R is C^*-embedded. This is as in the previous examples: R is even C^*-embedded in $R \cup (2^\omega \times \beta D)$. Given $f : R \to [0,1]$ let $g : \beta D \to [0,1]$ be the Čech-Stone extension of $s \mapsto f(\infty, s)$ and then define $\bar{f} : \mathbb{A} \setminus R \to [0,1]$ by $\bar{f}(x,u) = g(u)$ (replicate g on each vertical line but restrict it to $\{x\} \times (\omega_0 \cup K_x)$ each time). Then $f \cup \bar{f}$ is a continuous extension of f.

R is not C-embedded. Below we show that $\mathbb{A}$ is realcompact, so Proposition 2.1 implies that R has many infinite C-embedded subsets. Therefore the unbounded function without continuous extension must be chosen with some care.

Define $f(\infty, s) = |s|$ (the length of s). Assume $g : \mathbb{A} \to \mathbb{R}$ is a continuous extension. By the remark above there is a neighbourhood U of ∞ such that g is constant on $U \times \{s\}$ for every $s \in D$. But then for every $x \in U \setminus \{\infty\}$ and $n \in \omega_0$ we have $g(x, x \restriction n) = g(\infty, x \restriction n) = f(\infty, x \restriction n) = n$. Since $K_x = \bigcap_n \operatorname{cl}_{\beta D}\{x \restriction i : i \ge n\}$ this would imply that $g(x,u) \ge n$ for all n when $u \in K_x$.

$\mathbb{A}$ is realcompact. In the plank P in Section 3 we used ω_0^* everywhere in the top line. Combined with the fact that continuous functions were constant on a tail on each horizontal line this implied that P is not realcompact, mainly because unbounded (to the right) zero-sets in the top line contain sets of the form $[\alpha, \omega_1) \times Z$, where Z is a zero-set of ω_0^*. In the present example the disjointness of the K_x will provide us with a richer supply of zero-sets; these will help ensure realcompactness of $\mathbb{A}$.

Let $\mathcal{Z}$ be a zero-set ultrafilter on $\mathbb{A}$ with the countable intersection property.

For each $s \in D$ the horizontal $\mathfrak{C} \times \{s\}$ is clopen, hence a zero-set.

The continuous function $f : \mathbb{A} \to [0,1]$ determined by setting $f(x,s) = 2^{-|s|}$ for all $\langle x, s\rangle \in \mathfrak{C} \times D$ has the top line T as its zero-set.

This means that we have a partition of $\mathbb{A}$ into countably many zero-sets. It follows that one of these sets must belong to $\mathcal{Z}$.

If $\mathfrak{C} \times \{s\} \in \mathcal{Z}$ then either $\langle \infty, s\rangle \in \bigcap \mathcal{Z}$ or there is a $Z \in \mathcal{Z}$ is such that $\infty \notin Z$. But then Z is discrete and countable because $\{x \in \mathfrak{C} : \langle x, s\rangle \notin Z\}$ is open in $\mathfrak{C}$ and contains ∞. Then $\mathcal{Z}$ determines a countably complete ultrafilter on Z, which is fixed because $|Z|$ is countable.

We are left with the case that $T \in \mathcal{Z}$. Here is where we use the partition $\{K_x : x \in 2^\omega\}$ of D^* to show that T may be split into zero-sets in many ways.

We show that whenever A is clopen in the Cantor set 2^ω the union $Z(A) = \bigcup_{x \in A}\{x\} \times K_x$ is a zero-set in $\mathbb{A}$.

By compactness and zero-dimensionality of cD we know there is a continuous function $f : cD \to \{0,1\}$ such that $f[A] = \{0\}$ and $f[2^\omega \setminus A] = \{1\}$ (we assume both A and its complement are non-empty).

We use f to define $F : \mathbb{A} \to \{0,1\}$ by $F(x,s) = f(s)$ if $\langle x,s\rangle \in \mathfrak{C} \times D$ and $F(x,u) = f(x)$ if $u \in K_x$.

The function F is continuous on $\mathbb{A}$ and we have $Z(A) = T \cap Z_F$, so $Z(A)$ is a zero-set of $\mathbb{A}$.

Using this we build countably many pairs of complementary zero sets in T. For every $n \in \omega$ we let $A_n = \{x \in 2^\omega : x(n) = 0\}$ and $B_n = \{x \in 2^\omega : x(n) = 1\}$; these clopen sets determine the zero-sets $Z(n,0) = \bigcup_{x \in A_n}\{x\} \times K_x$ and $Z(n,1) = \bigcup_{x \in B_n}\{x\} \times K_x$ respectively.

Since $\mathcal{Z}$ is a zero-set ultrafilter and $T \in \mathcal{Z}$ we deduce that for every n there is an element $x(n)$ of $\{0,1\}$ such that $Z(n,x(n)) \in \mathcal{Z}$. Thus we get an $x \in 2^\omega$ such that $\{Z(n,x(n)) : n \in \omega\}$ is a subfamily of $\mathcal{Z}$.

Its intersection is equal to $\{x\} \times K_x$ and because $\mathcal{Z}$ has the countable intersection property this compact set belongs to $\mathcal{Z}$. It follows that $\bigcap \mathcal{Z} \neq \emptyset$.

As mentioned before, Proposition 2.1 implies that R has many infinite C-embedded subsets. A lot of these can be pointed out explicitly.

For every $x \in 2^\omega$ the set $N_x = \{\langle\infty, x\restriction n\rangle : n \in \omega\}$ is C-embedded in $\mathbb{A}$. Given a function $f : N_x \to \mathbb{R}$ we extend it to R first by setting $\bar{f}(\infty,s) = 0$ for all other s. Then we extend $\bar{f}$ horizontally: $\bar{f}(y,s) = \bar{f}(\infty,s)$ for all y and s, *except* for $y = x$, we set $\bar{f}(x,s) = 0$ for all s. Now we can set $\bar{f}(t) = 0$ for all t in the top line to get our continuous extension to all of $\mathbb{A}$.

In a similar fashion every infinite antichain in $2^{<\omega}$ yields an infinite C-embedded subset as well.

More answers. We can use $\mathbb{A}$ and some variations to answer some of the questions raised earlier in this paper.

The smallest power of $\mathbb{R}$. The set $\mathfrak{C} \times D$ is dense in $\mathbb{A}$, so every member of $C(\mathbb{A})$ is determined by its restriction to this set. Using the fact that continuous functions on $\mathfrak{C}$ are constant on co-countable sets we see that there are $\mathfrak{c}$ many such restrictions. We conclude that $C(\mathbb{A})$ has cardinality $\mathfrak{c}$, as claimed in the discussion of Question 1.

The union of two closed C^*-embedded copies of $\mathbb{N}$. We can use $\mathbb{A}$ much like we used $\mathbb{K}$ to create a realcompact space with two closed C^*-embedded copies of $\mathbb{N}$ whose union is not C^*-embedded. Take $\mathbb{A} \times \{0,1\}$ and identify the points $\langle t,0\rangle$ and $\langle t,1\rangle$ for all t in the top line T. Then $R \times \{0\}$ and $R \times \{1\}$ are still C^*-embedded in the resulting quotient space, but their union is not: mapping $\langle r,i\rangle$ to i results in a bounded function without a continuous extension. The proof that the quotient space is realcompact is almost verbatim that of the realcompactness of $\mathbb{A}$. Note that the $R \times \{0\}$ and $R \times \{1\}$ are separated (neither intersects the closure of the other), so their union is a closed copy of $\mathbb{N}$ that is not C^*-embedded. The quotient space also has $\mathfrak{c}$ many real-valued continuous functions, hence also we obtain a closed copy of $\mathbb{N}$ in $\mathbb{R}^{\mathfrak{c}}$ that is not C^*-embedded. This copy is quite unlike the closed copies of $\mathbb{N}$ in $\mathbb{R}^{\omega_1}$ that are constructed in [3].

Another closed copy of $\mathbb{N}$ *that is not C^*-embedded.* If we replace βD by cD in $\mathbb{A}$ then we obtain a realcompact plank where the right-hand side is a closed copy of $\mathbb{N}$ that is not C^*-embedded.

The analogue of $\mathbb{A}$ is the following subspace of $\mathfrak{C} \times cD$:

$$(\mathfrak{C} \times D) \cup \{\langle x, x\rangle : x \in 2^\omega\}$$

That this space is realcompact is shown exactly as for $\mathbb{A}$. However in this space the right-hand side R is not C^*-embedded.

Since 2^ω is homeomorphic to its own square it is relatively easy to produce two disjoint open sets U and V in 2^ω with a dense union and whose common boundary F is homeomorphic to 2^ω itself.

Via the map $e : \beta D \to cD$ we can find a subset C of D such that $\operatorname{cl} U \subseteq \operatorname{cl} C$ and $\operatorname{cl} V \subseteq \operatorname{cl}(D \setminus C)$.

Define $f : R \to [0, 1]$ by $f(\infty, s) = \chi(s)$, where χ is the characteristic function of C. As before, given a continuous extension $\bar{f}$ of f, we would have a countable set B such that $\bar{f}(x, s) = f(\infty, s)$ for all $x \in 2^\omega \setminus B$ and all $s \in D$. But then $\bar{f}$ would not be continuous at $\langle x, x\rangle$ whenever $x \in F \setminus B$.

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