SET THEORY 2020-10-26

COFINALITY

A CARDINAL κ IS SINGULAR
IFF
THE EXIST A CARDINAL $\lambda < \kappa$
AND SETS $\{S_\alpha : \alpha < \lambda\}$
SUCH THAT

- $|S_\alpha| < \kappa$ ALL α
- $\bigcup_{\alpha<\lambda} S_\alpha = \kappa$.

THE LEAST SUCH λ IS $cf\,\kappa$.

[PROOF AND VARIATIONS
GROUP INTERACTION]

CARDINAL ARITHMETIC

HOW DOES γ DEPEND
ON α AND β
IN $\aleph_\alpha^{\aleph_\beta} = \aleph_\gamma$

WHICH γ GOES WITH
0 AND 0?

$\aleph_0^{\aleph_0} = 2^{\aleph_0} = |\mathbb{R}| = \aleph_\gamma$?
ξ ?

$$\kappa + \lambda = |\kappa \times \{0\} \cup \lambda \times \{1\}|$$

$$\kappa \cdot \lambda = |\kappa \times \lambda|$$

FOR INFINITE κ, λ : $\max\{\kappa, \lambda\}$

$$\Rightarrow \kappa^\lambda = |\,{}^\lambda\kappa\,| \quad (|{}^\lambda \kappa|)$$

CARDINAL   SET OF FUNCTIONS

NOTE: WE USE $\cancel{AC}$ TO GET $|{}^\lambda\kappa|$

$\mathbb{R}$ AND SO ALSO $\mathcal{P}(\omega)$ NEED NOT BE WELL-ORDERABLE.

$2^{\aleph_0} = \ldots \qquad 2^{\aleph_1} = \ldots$

$\aleph_{2020}^{\aleph_{2021}} = \ldots$!

$2^{\aleph_0} = \aleph_1$ IS CANTOR'S CONTINUUM HYPOTHESIS

IF $X \subseteq \mathbb{R}$ IS INFINITE THEN $|X| = |\mathbb{N}|$ OR $|X| = |\mathbb{R}|$

CONSISTENT WITH & INDEP. OF ZFC

$2^{\aleph_0} = \aleph_2, \ldots, \aleph_{2020}, \ldots$

$\aleph_{\omega_1}, \ldots$

$2^{\aleph_0} = \aleph_\omega$ IS FALSE

WHAT CAN WE PROVE?

- λ INFINITE AND $2 \leq \kappa \leq 2^\lambda$
  THEN $\underline{\kappa^\lambda = 2^\lambda}$

$2^\lambda \leq \kappa^\lambda$: EASY

$\kappa^\lambda \leq (2^\lambda)^\lambda = 2^{\lambda \cdot \lambda} = 2^\lambda$

CANTOR - SCHRÖDER - BERNSTEIN - DEDEKIND

$\aleph_0^{\aleph_0} = \aleph_1^{\aleph_0} = \underline{(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}}$

APPARENTLY THINGS ARE INTERESTING IF

$\lambda < 2^\lambda < \kappa$.

CERTAINLY;
$\kappa \leq \kappa^\lambda \leq 2^\kappa$

NON-TRIVIAL: $\kappa < \kappa^{\mathrm{cf}\,\kappa}$

κ REG: $\kappa^\kappa = 2^\kappa$

LET $\langle \alpha_\eta : \eta < \mathrm{cf}\,\kappa \rangle$ BE INCREASING AND COFINAL IN κ.
LET $\langle f_\alpha : \alpha < \kappa \rangle$ BE A SEQUENCE OF FUNCTIONS FROM $\mathrm{cf}\,\kappa$ TO κ.

IF $\eta < \mathrm{cf}\,\kappa$ THEN

$$|\{f_\alpha(\eta) : \alpha < \alpha_\eta\}| \leq |\alpha_\eta| < \kappa.$$

DIAGONALISE DEFINE

$$f(\eta) = \min \kappa \setminus \{f_\alpha(\eta) : \alpha < \alpha_\eta\}$$

$f \neq f_\alpha$ FOR ALL α.

$\aleph_\omega^{\aleph_0} > \aleph_\omega$

EVEN IF $2^{\aleph_0} < \aleph_\omega$

[SO IF $\mathrm{cf}\,\kappa \leq \lambda < \kappa$
THEN $\kappa^\lambda > \kappa$.]

USEFUL: $[A]^\lambda = \{X \subseteq A : |X| = \lambda\}$

$(\lambda \leq |A|)$

$[\kappa]^\lambda = \kappa^\lambda$

$\geq$ κ^λ IS A SUBSET OF $[\lambda \times \kappa]^\lambda$

$\uparrow$ FUNCTIONS $\qquad \lambda \leq \kappa \qquad \sim [\kappa]^\lambda$

$\leq$ $I = \{f \in \kappa^\lambda : f \text{ is } 1\text{-}1\}$

$f \longmapsto \mathrm{ran}\, f$ IS SURJECTIVE
FROM I TO $[\kappa]^\lambda$

AC SAYS: THERE IS AN
INJECTION $[\kappa]^\lambda \longrightarrow I$.

$|[\mathbb{R}]^{\aleph_0}| = 2^{\aleph_0}$

Sums/Products of sets of cardinals

$\langle \kappa_i : i < \lambda \rangle$ sequence of cardinals.

All κ equal: $\sum_{i<\lambda} \kappa_i = \kappa \cdot \lambda$

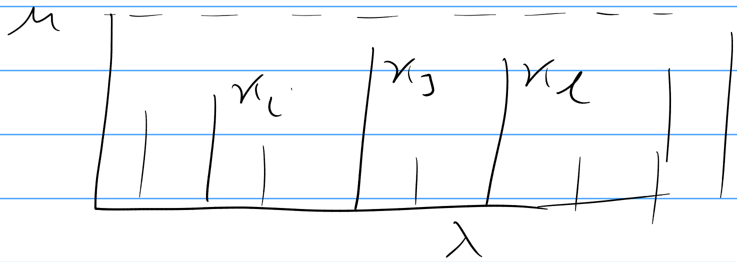
$$\sum_{i<\lambda} \kappa_i = \left| \bigcup_{i<\lambda} \{i\} \times \kappa_i \right|$$

If λ is infinite and $\kappa_i > 0$ for all i then

$$\sum_{i<\lambda} \kappa_i = \lambda \cdot \sup_{i<\lambda} \kappa_i$$

$\mu = \sup_{i<\lambda} \kappa_i$

$\leq$: $\bigcup_{i<\lambda} \{i\} \times \kappa_i \subseteq \lambda \times \mu$



$\geq$: $\cdot$ $\lambda = \sum_{i<\lambda} 1 \leq \sum_{i<\lambda} \kappa_i$

$\cdot$ $\kappa_j \leq \sum_{i<\lambda} \kappa_i$ for all j

$\mu \leq \sum_{i<\lambda} \kappa_i$

$\lambda \cdot \mu = \max\{\lambda, \mu\} \leq \sum_{i<\lambda} \kappa_i$

- $\prod_{i<\lambda} \kappa_i = \left| \underbrace{\prod_{i<\lambda} \kappa_i}_{\text{CART. PRODUCT}} \right|$

  $\uparrow$ CARDINAL

WHY CARDINAL AS INDEX SET

IF $f : J \longrightarrow I$ IS A BIJECTION

THEN $\prod_{i\in I} \kappa_i = \prod_{j\in J} \kappa_{f(j)}$

$\uparrow$ $\underline{\omega+25}$ $\qquad$ $\uparrow$ $\underline{\omega}$

IF $\kappa_i \geq 2$ FOR ALL $i<\lambda$

THEN

$$\sum_{i<\lambda} \kappa_i \leq \prod_{i<\lambda} \kappa_i$$

WHY $\kappa_i \geq 2$: $1+1 > 1\cdot 1$

ALSO THIS IS SHARP $2+2 = 2\cdot 2$

λ FINITE : INDUCTION

OTHERWISE

GIVEN $i<\lambda$ AND $\alpha \in \kappa_i$

PICK $f \in \prod_{i<\lambda}$ WITH $f(i)=\alpha$

$\langle i, \alpha \rangle \longmapsto \langle i, f \rangle$ INJECTIVE (CHECK)

FROM $\bigcup_{i<\lambda} \{i\}\times\kappa_i$ TO $\lambda \times \prod_{i<\lambda} \kappa_i$

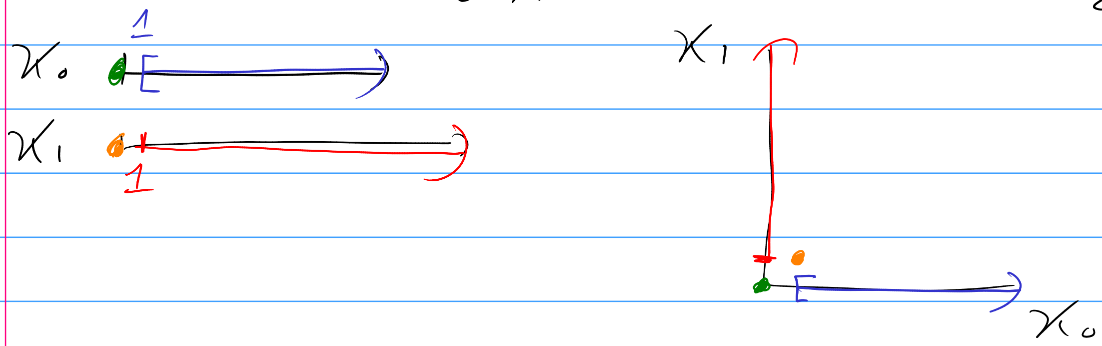
$\lambda < 2^\lambda \leq \prod_{i<\lambda} 2 \leq \prod_{i<\lambda} \kappa_i$

SO $\lambda \cdot \prod_{i<\lambda} \kappa_i = \prod_{i<\lambda} \kappa_i$

PROJECT

DEFINE AN INJECTION FROM

$$\bigcup_{i<\lambda} \{i\} \times \kappa_i \text{ TO } \prod_{i<\lambda} \kappa_i$$



- KÖNIG.

IF $\kappa_i < \lambda_i$ FOR $i < \mu$

THEN

$$\sum_{i<\mu} \kappa_i < \prod_{i<\mu} \lambda_i$$

WLOG $\kappa_i > 0$

SO $\lambda_i \geq 2$

① $\leq$ $\quad \sum_{i<\mu} \kappa_i \leq \sum_{i<\mu} \lambda_i \leq \prod_{i<\mu} \lambda_i$ $\quad \lambda_i \geq 2$

[EXPLICIT INJECTION POSSIBLE]

② $\neq$ LET $f : \bigcup_{i<\mu} \{i\} \times \kappa_i \to \prod_{i<\mu} \lambda_i$

BE A MAP; WE SHOW f IS NOT SURJECTIVE.

FIX j

$$\{j\} \times \kappa_j \xrightarrow{f} \prod_{i<\mu} \lambda_i \xrightarrow{\pi_j} \lambda_j$$

$$(j,\alpha) \longmapsto f(j,\alpha) \longmapsto f(j,\alpha)(j)$$

THIS IS NOT ONTO

DEFINE $x \in \prod_{j<\mu} \lambda_j$

BY $x(j) = \min \lambda_j \setminus \{ f(j,\alpha)(j) : \alpha \in \kappa_j \}$

THEN $x \notin \operatorname{RAN} f$

CONSEQUENCES

— $\kappa < 2^\kappa$: $\kappa_i := 1,\ \lambda_i := 2$

— $\kappa < \mathrm{cf}\, 2^\kappa$:

TAKE $\kappa_i < 2^\kappa$ FOR $i < \kappa$

LET $\lambda_i = 2^\kappa$

$$\sum_{i<\kappa} \kappa_i < \prod_{i<\kappa} \lambda_i = (2^\kappa)^\kappa = 2^\kappa$$

$\mathrm{cf}\, 2^{\aleph_0} > \aleph_0$ THAT IS WHY

$$2^{\aleph_0} \neq \aleph_\omega$$

— $\kappa < \kappa^{\mathrm{cf}\,\kappa}$

$\kappa = \sum_{i<\mathrm{cf}\,\kappa} \kappa_i \qquad \kappa_i < \kappa$

TAKE $\lambda_i = \kappa \qquad i < \mathrm{cf}\,\kappa$

$$\kappa = \sum_{i<\mathrm{cf}\,\kappa} \kappa_i$$

$$< \prod_{i<\mathrm{cf}\,\kappa} \lambda_i = \kappa^{\mathrm{cf}\,\kappa}$$

- THE CONTINUUM FUNCTION

$$\kappa \longmapsto 2^\kappa$$

$$\aleph_\alpha \longmapsto 2^{\aleph_\alpha} = \aleph_{???}$$

GENERALIZED CONTINUUM HYPOTHESIS

$$2^{\aleph_\alpha} = \aleph_{\alpha+1} \quad \text{FOR } \underline{\text{ALL}}\ \alpha.$$

- NOT <u>FALSE</u> [GÖDEL, 1940]

GCH IS CONSISTENT WITH ZFC

- NOT PROVABLE: [COHEN, 1963]

$2^{\aleph_0} = \aleph_2$ IS CONSISTENT.

GCH MAKES LIFE EASY

- $2 \le \kappa \le 2^\lambda$: $\kappa^\lambda = 2^\lambda = \lambda^+$

  $\alpha \le \beta$: $\aleph_\alpha^{\aleph_\beta} = \aleph_{\beta+1}$
- $\mathrm{cf}\,\kappa \le \lambda \le \kappa$: $\kappa^\lambda = \kappa^+$

  $[\kappa < \kappa^\lambda \le \kappa^\kappa = 2^\kappa = \kappa^+]$
- $\lambda < \mathrm{cf}\,\kappa$: $\kappa^\lambda = \kappa$

  AS SETS $\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda$

  $|\alpha^\lambda| = |\alpha|^\lambda \le 2^{|\alpha|\cdot\lambda} = (|\alpha|\cdot\lambda)^+ \le \kappa.$

  $\kappa^\lambda \le \sum_{\alpha<\kappa} |\alpha^\lambda| = \kappa \cdot \sup_{\alpha<\kappa} |\alpha|^\lambda = \kappa$

BUT WHAT IN GENERAL?

(*)
- $\kappa < \lambda \longrightarrow 2^\kappa \le 2^\lambda$

  $[2^{\aleph_0} = 2^{\aleph_1}$ IS POSSIBLE, I.E., CONSISTENT WITH ZFC$]$
- $\kappa < \mathrm{cf}\, 2^\kappa$

(*) THESE ARE THE ONLY RESTRICTIONS FOR REGULAR κ.

EASTON: IF F: CARD $\to$ CARD SATISFIES (*)

THEN "FOR ALL REGULAR κ

$$F(\kappa) = 2^\kappa"$$

IS CONSISTENT

- κ LIMIT SO $\kappa = \sum_{i<\mathrm{cf}\,\kappa} \kappa_i$ $(\kappa_i < \kappa)$

$\boxed{2^\kappa = (2^{<\kappa})^{\mathrm{cf}\,\kappa}}$ $\quad 2^{<\kappa} = \sup\{2^\lambda : \lambda < \kappa,\ \lambda \text{ CARD}\}$

κ REGULAR: $2^\kappa = 2^\kappa$

$$2^\kappa = 2^{\sum \kappa_i} = \prod_i 2^{\kappa_i} \le \prod_i 2^{<\kappa} = (2^{<\kappa})^{\mathrm{cf}\,\kappa} \le 2^\kappa$$

- IF κ IS SINGULAR AND THERE IS A $\mu < \kappa$ SUCH THAT
$$2^\nu = 2^\mu$$
FOR ALL ν WITH $\mu \le \nu < \kappa$
THEN $2^\kappa = 2^\mu$.
WLOG $\mu \ge cf\kappa$.
  - $2^{<\kappa} = 2^\mu$
  - $2^\kappa = (2^\mu)^{cf\kappa} \le 2^{\mu\cdot\mu} = 2^\mu$

GIMEL $\gimel(\kappa) = \kappa^{cf\kappa}$.
(κ REGULAR $\gimel(\kappa) = 2^\kappa$)

- IF FOR EVERY $\mu < \kappa$ THERE IS ν BETWEEN μ AND κ WITH $2^\nu > 2^\mu$
THEN $cf(2^{<\kappa}) = cf\kappa$.
THEN
$$2^\kappa = (2^{<\kappa})^{cf\kappa}$$
$$= \gimel(2^{<\kappa})$$

SUMMARY FOR 2^κ:
- κ SUCCESSOR: $2^\kappa = \kappa^\kappa = \gimel(\kappa)$
- κ LIMIT 2^ν CONSTANT ON A TAIL BELOW κ:
$$2^\kappa = 2^{<\kappa} \cdot \gimel(\kappa).$$
  - κ REGULAR: $\gimel(\kappa) = 2^\kappa$
  - κ SINGULAR $2^{<\kappa} = 2^\kappa = \kappa^\kappa \ge \gimel(\kappa)$
- κ LIMIT 2^ν NOT CONSTANT ON A TAIL:
$$2^\kappa = \gimel(2^{<\kappa})$$

GIMEL IS IMPORTANT.

$\beth_\omega$ STRONG LIMIT $\to 2^{\beth_\omega} = \beth_\omega^{\beth_0} < \beth_{\omega_1}$

$2^{<\kappa} = \kappa$

BACK TO EXPONENTIATION.

$\kappa^{<\lambda} = \sup\{\kappa^\mu : \mu < \lambda, \mu \text{ CARD}\}$

(IF $\lambda = \mu^+$ THIS IS JUST κ^μ)

USEFUL FACT IF λ IS AN INFINITE CARDINAL AND $\langle \kappa_i : i < \lambda \rangle$ IS NON-DECREASING $(i < j \to \kappa_i \leq \kappa_j)$ $\kappa_i > 0$

THEN $\prod_{i<\lambda} \kappa_i = (\sup_{i<\lambda} \kappa_i)^\lambda$

[HOMEWORK]

TOWARDS A SUMMARY FOR κ^λ.

- IF κ IS REGULAR AND $\lambda < \kappa$ THEN $\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda$ (SETS) AND SO

$$\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\lambda \quad \text{(CARDS)}$$

- HAUSDORFF: $\beth_{\alpha+1}^{\beth_\beta^\lambda} = \beth_\alpha^{\beth_\beta^\lambda} \cdot \underline{\beth_{\alpha+1}^\lambda}$

MAXIMUM TERM

NUMBER OF TERMS

$\beta \leq \alpha$

$\beta > \alpha$ $\quad \beth_{\alpha+1}^\lambda \leq 2^{\beth_\beta^\lambda}$

$\lim_{\alpha\to\kappa} \alpha^\lambda = \sup\{\mu^\lambda : \mu < \kappa, \mu \text{ CARD}\}$

IF κ IS A LIMIT AND $\lambda \geq cf\kappa$
THEN

$$\kappa^\lambda = (\underbrace{\lim_{\alpha\to\kappa} \alpha^\lambda}_{})^{cf\kappa}$$

$$\kappa = \sum_{i<cf\kappa} \kappa_i \quad \text{WITH } 2 \leq \kappa_i < \kappa$$

$$\kappa^\lambda \leq \left(\prod_{i<cf\kappa} \kappa_i\right)^\lambda = \prod_{i<cf\kappa} \kappa_i^\lambda$$

$$\leq \prod_{i<cf\kappa} (\lim_{\alpha\to\kappa} \alpha^\lambda)$$

$$= (\lim_{\alpha\to\kappa} \alpha^\lambda)^{cf\kappa}$$

$$\leq (\kappa^\lambda)^{cf\kappa}$$

$$= \kappa^\lambda$$

SUMMARY: κ^λ

- $2 \leq \kappa \leq 2^\lambda$: $\kappa^\lambda = 2^\lambda$
- IF THERE IS $\mu < \kappa$ WITH $\kappa \leq \mu^\lambda$ THEN $\kappa^\lambda = \mu^\lambda$
- IF $\kappa > \lambda$ AND $\mu^\lambda < \kappa$ FOR ALL $\mu < \kappa$ THEN:
  - $\lambda \geq cf\kappa$: $\kappa^\lambda = \cancel{\kappa^{cf\kappa}}$ $\Leftarrow$ BECAUSE $\lim_{\alpha\to\kappa} \alpha^\lambda = \kappa$
  - $\lambda < cf\kappa$: $\kappa^\lambda = \cancel{\kappa}$
    - $\kappa = \mu^+$: HAUSDORFF $\kappa^\lambda = \mu^\lambda \cdot \kappa$
    - κ LIMIT $\kappa^\lambda = \lim_{\alpha\to\kappa} \alpha^\lambda = \kappa$
    - SETS: $\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda$
- κ^λ CAN BE
  - 2^λ
  - κ
  - $\exists(\mu)$ FOR SOME μ WITH $cf\mu \leq \lambda < \mu$.

THAT $\mu = \min\{\nu < \kappa : \nu^{\lambda} \geq \kappa\}$
(IN THE CASE THAT $2^{\lambda} < \kappa$)

SINGULAR CARDINALS HYPOTHESIS
FOR EVERY SINGULAR κ
IF $2^{\mathrm{cf}\kappa} < \kappa$ THEN $\kappa^{\mathrm{cf}\kappa} = \kappa^{+}$

SCH SAYS
IF $2^{\aleph_0} < \aleph_\omega$
THEN $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1}$