SET THEORY 2020-11-02

(ULTRA)FILTERS

STATIONARY SETS

An ultrafilter is a truth-value assignment to the family of subsets of a set, and a method of convergence to infinity. From the first (logical) property arises its connection with two-valued logic and model theory; from the second (convergence) property arises its connection with topology and set theory. Both these descriptions of an ultrafilter are connected with compactness. The model-theoretic property finds its expression in the construction of the ultraproduct and the compactness type of theorem of Łoś (implying the compactness theorem of first-order logic); and the convergence property leads to the process of completion by the *adjunction* of an ideal element for every ultrafilter—*i.e.*, to the Stone-Čech compactification process (implying the Tychonoff theorem on the compactness of products).

CARTAN 1937

A FILTER F ON A SET X IS A FAMILY OF SUBSETS WITH $X \neq \emptyset$

- $X \in F$, $\emptyset \notin F$
- $A, B \in F \rightarrow A \cap B \in F$
- $A \in F$, $B \supseteq A \rightarrow B \in F$.

DUAL: IDEAL I

- $\emptyset \in I$, $X \notin I$
- $A, B \in I \rightarrow A \cup B \in I$
- $A \in I$, $B \subseteq A \rightarrow B \in I$

IDEAL IN THE RING $(\mathcal{P}(X), \Delta, \cap)$

F IS A FILTER THEN

$$I = \{A : X \setminus A \in F\}$$

IS AN IDEAL AND VICE VERSA.

PRINCIPAL

TRIVIAL: $\{X\}$

LESS TRIVIAL: FIX $A \subseteq X$, $A \neq \emptyset$

$F_A = \{B : A \subseteq B\}$

$F_x = F_{\{x\}}$ IF $x \in X$.

Topology: THE NEIGHBOURHOODS OF A POINT.

X INFINITE:

$[X]^{<\aleph_0} = \{A \subseteq X : A \text{ FINITE}\}$

IS AN IDEAL.

$Fr = \{X \setminus A : A \text{ FINITE}\}$

COFINITE / FRÉCHET - FILTER

Fr IS NOT PRINCIPAL

IT IS <u>FREE</u>: $\bigcap Fr = \emptyset$

GENERALLY

$\{A \subseteq X : |X \setminus A| < |X|\}$

GENERALIZED FRÉCHET

$I = \{A \subseteq \mathbb{N} : \sum_{n \in A} \frac{1}{n+1} < \infty\}$

IS AN IDEAL

$F = \{A : \mathbb{N} \setminus A \in I\}$ IS A FILTER

$\mathcal{P}(\mathbb{N}) \setminus I$ <u>IS NOT</u>

GENERATING FILTERS.

SUPPOSE $G \subseteq \mathcal{P}(X)$ HAS THE
FINITE INTERSECTION PROPERTY

FIP [IF $A_1, A_2, \ldots, A_n \in G$ THEN
$A_1 \cap A_2 \cap \cdots \cap A_n \neq \emptyset$]

THEN THERE IS A FILTER F
SUCH THAT $G \subseteq F$.

START WITH $H = \{A_1 \cap \cdots \cap A_n : A_1, \ldots, A_n \in G\}$
DEFINE
$$F = \{A : (\exists B \in H)(B \subseteq A)\}$$
CHECK THAT THIS WORKS.
AND F IS THE SMALLEST
FILTER THAT CONTAINS G.
IT IS GENERATED BY G

IF F AND G ARE FILTERS
AND $F \subseteq G$
WE SAY G IS FINER THAN F
F IS COARSER THAN G.

- IF $\mathcal{F}$ IS A FAMILY OF FILTERS
THEN $\bigcap \mathcal{F}$ IS A FILTER.
- $X \in F$ FOR ALL F SO THERE
- $A, B \in \bigcap \mathcal{F}$ THEN $A, B \in F$ FOR ALL F
SO $A \cap B \in F$ FOR ALL F
- IF $A \in \bigcap \mathcal{F}$ AND $B \supseteq A$ THEN
$B \in \bigcap \mathcal{F}$

— IF $\mathcal{F}$ IS A NON-EMPTY FAMILY OF FILTERS AND IT IS LINEARLY ORDERED BY $\subseteq$ THEN

$\bigcup\mathcal{F}$ IS A FILTER.

EXAMPLE IF $x \neq y$ THEN

$F_{\{x\}} \cup F_{\{y\}}$ $\quad \{x\}\cap\{y\}=\emptyset$

IS NOT A FILTER

- $\emptyset \notin \bigcup\mathcal{F}$ ✓ $\quad X \in \bigcup\mathcal{F}$ ✓
- $A, B \in \bigcup\mathcal{F}$ --- $A \in F_1$
  --- $B \in F_2$

(*) $F_1 \subseteq F_2$ OR $F_2 \subseteq F_1$

- $A, B \in F_2$ SO $A \cap B \in F_2 \subseteq \bigcup\mathcal{F}$
- $A \in \bigcup\mathcal{F}$, SAY $A \in F$
  THEN $B \in F$ IF $B \supseteq A$.

A FILTER $\mathcal{U}$ IS AN <u>ULTRAFILTER</u> IF IT IS <u>MAXIMAL</u> IN THE CONGERIES OF FILTERS ORDERED BY $\subseteq$

congeries /kon-jē'ri-ēz or kon'jə-rēz/
noun (*pl* **conger'ies**)
A collection, mass or heap
ORIGIN: L *congeriēs*, from *con-* together, and *gerere*, *gestum* to bring

$\mathcal{U}$ IS ULTRA IF IT IS A FILTER AND IF F IS A FILTER WITH $\mathcal{U} \subseteq F$ THEN $F = \mathcal{U}$

THERE IS NO STRICTLY FINER FILTER.

ARE THERE ANY?

YES: $F_x = \{A \subseteq X : x \in A\}$ $(x \in X)$

SUPPOSE $G \supseteq F_x$ IS A FILTER
LET $A \in G$ BECAUSE $\{x\} \in G$
WE GET $A \cap \{x\} \neq \emptyset$
AND SO $x \in A$ AND $A \in F_x$.

ANY OTHERS?
FREE/NON-PRINCIPAL ULTRAFILTER??

ULAM 1929, TARSKI 1930:
THERE ARE FREE ULTRAFILTERS ON $\mathbb{N}$
CARTAN 1937 IN GENERAL:
EVERY FILTER CAN BE EXTENDED TO A ULTRAFILTER.

PROOF: ZORN'S LEMMA.

THE CONGERIES Φ OF ALL FILTERS ON X SATISFIES THE CONDITIONS OF ZORN'S LEMMA.
EVERY CHAIN HAS AN UPPER BOUND
WE PROVED THAT!!
IF F IS GIVEN APPLY THIS TO

$$\{ G \in \Phi : F \subseteq G \}$$

WHAT DID ULAM AND TARSKI DO?

CONSTRUCT $\mu : \mathcal{P}(\mathbb{N}) \to \{0,1\}$
SUCH THAT

- $\mu(\{n\}) = 0 \qquad n \in \mathbb{N}$
- $\mu(\mathbb{N}) = 1$
- $A \cap B = \emptyset \to \mu(A \cup B) = \mu(A) + \mu(B)$

- $U_\mu = \{ A \subseteq \mathbb{N} : \mu(A) = 1 \}$ IS AN ULTRAFILTER
- CONVERSELY IF $\mathcal{U}$ IS AN U.F. THEN $\mu(A) = \begin{cases} 1 & A \in \mathcal{U} \\ 0 & A \notin \mathcal{U} \end{cases}$

DEFINES SUCH A MEASURE

CHARACTERIZATION OF U.FS.

FOR A FILTER $\mathcal{U}$ ON X TFAE

1. $\mathcal{U}$ IS AN ULTRAFILTER
2. IF A SATISFIES $(\forall B \in \mathcal{U})(A \cap B \neq \emptyset)$ THEN $A \in \mathcal{U}$.
3. IF $A, B \subseteq X$ AND $A \cup B \in \mathcal{U}$ THEN $A \in \mathcal{U}$ OR $B \in \mathcal{U}$
4. $\Rightarrow$ (4) IF $A \subseteq X$ THEN $A \in \mathcal{U}$ OR $X \setminus A \in \mathcal{U}$.

(3) $\to$ (4) SPECIAL CASE

$$A \cup (X \setminus A) = X \in \mathcal{U}$$

(2) $\to$ (3) IF $A \cup B \in \mathcal{U}$ THEN

$$(\forall C \in \mathcal{U})(C \cap (A \cup B) \neq \emptyset) \Longleftarrow$$

$$\longrightarrow (\forall C \in \mathcal{U})(C \cap A \neq \emptyset \vee C \cap B \neq \emptyset)$$

WE GET $(\forall C \in \mathcal{U})(C \cap A \neq \emptyset)$

$\vee$ $(\forall C \in \mathcal{U})(C \cap B \neq \emptyset)$

NOT AUTOMATIC ∇

SUPPOSE NOT: $C_A \in \mathcal{U}$ WITH $C_A \cap A = \emptyset$

$C_B \in \mathcal{U}$ —— $C_B \cap B = \emptyset$

NOW – $C_A \cap C_B \in \mathcal{U}$

– $(C_A \cap C_B) \cap (A \cup B) = \emptyset$

CONTRADICTION

(1) $\to$ (2) LOOK AT

$$G = \{C : (\exists B \in \mathcal{U})(A \cap B \subseteq C)\}$$

– $X \in G$, $\emptyset \notin G$

– $C \in G$, $D \supseteq C \to D \in G$

– $C, D \in G \to C \cap D \in G$

$$\left.\begin{array}{l} A \cap B_1 \subseteq C \\ A \cap B_2 \subseteq D \end{array}\right\} A \cap (B_1 \cap B_2) \subseteq C \cap D$$

G IS A FILTER

- $A \in G$: take $B = X$.
- $U \subseteq G$ if $B \in U$ then $B \cap A \subseteq B$

U is ultra: $U = G$ so $A \in U$.

(4) $\to$ (1) Let G be a filter with $U \subseteq G$.

Take $A \in G$

well either $A \in U$ or $X \setminus A \in U$

But $X \setminus A \in U$ leads to

$X \setminus A \in G$ and $(X \setminus A) \cap A = \emptyset \in G$

A contradiction

so $A \in U$.

Homework:

If $|X| = \kappa$ then there are 2^{2^κ} many ultrafilters on X.

Ultrafilters on ω

Sierpiński:

If there is a free ultrafilter on ω then there is a non-measurable set in $\mathbb{R}$.

A free ultrafilter U on ω is selective or Ramsey if for every partition $\{A_n : n \in \omega\}$ either $A_n \in U$ for some n or $(\exists A \in U)(\forall n)(|A \cap A_n| \leq 1)$

EQUIVALENT:

FOR EVERY $f:\omega \to \omega$
THERE IS $A \in \mathcal{U}$ SUCH THAT
$f \upharpoonright A$ IS CONSTANT OR $f \upharpoonright A$ IS INJECTIVE

DO THESE ULTRAFILTERS EXIST?

- CH SAYS YES

- THERE ARE MODELS WITHOUT
RAMSEY ULTRAFILTERS.
[RANDOM REAL MODEL]
- CONSTRUCTION FROM CH
$\langle f_\alpha : \alpha \in \omega_1 \rangle$ COUNTS THE
FUNCTIONS FROM ω TO ω.
BUILD A SEQUENCE OF INFINITE SETS
$\langle X_\alpha : \alpha < \omega_1 \rangle$ AS FOLLOWS
- $X_0 = \omega$
- GIVEN X_α TAKE $X_{\alpha+1} \subseteq X_\alpha$
INFINITE ON WHICH
f_α IS CONSTANT OR 1-1.
[CHECK THIS]
- AT LIMITS? GIVEN $\{X_\beta : \beta < \alpha\}$
WHAT IS X_α?
ENUMERATE α AS $\{\beta_n : n < \omega\}$
$\{X_\beta : \beta < \alpha\}$ HAS THE SFIP.
MAKE A NEW SEQUENCE

$$A_n = \left(\bigcap_{i \le n} X_{\beta_i}\right) \setminus n.$$

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots$$

$x_0 = \min A_0$

$x_{n+1} = \min A_{n+1} \setminus \{x_i : i \leq n\}$

$X = \{x_n : n \in \omega\}$

NOTE $X \setminus A_n \subseteq \{x_i : i < n\}$

CERTAINLY

$X \setminus X_{\beta_n} \subseteq X \setminus A_n \subseteq \{x_i : i < n\}$

$\rightarrow X \subseteq^* X_{\beta_n}$ — ALMOST FINITELY MANY EXCEPTIONS.

$X_\alpha = X$.

$X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots X_n \supseteq \cdots \quad \supseteq^* X_\omega$

WE GET $\alpha < \beta \rightarrow X_\beta \subseteq^* X_\alpha$

THIS GIVES YOU THE STRONG FIP

$X_{\beta_1} \cap \cdots \cap X_{\beta_n}$ IS ALWAYS INFINITE.

$=$

A FILTER F ON X IS κ-COMPLETE (CARDINAL)

IF FOR EVERY SUBFAMILY F' OF F WITH $|F'| < \kappa$

THE INTERSECTION $\bigcap F'$ IS IN F.

(κ SINGULAR :

κ-COMPLETE $\rightarrow \kappa^+$ COMPLETE)

IF κ IS REGULAR THEN

$F_\kappa = \{ A \subseteq \kappa : |\kappa \setminus A| < \kappa \}$

IS κ-COMPLETE.

F_κ IS VEERY COMPLETE

σ-COMPLETE $=$ $\aleph_1$-COMPLETE

ARE THERE σ-COMPLETE UF'S

NOT ON ω $\quad \bigcap_{n \in \omega} \omega \setminus \{n\} = \emptyset$

NOT ON $\mathbb{R}$ [HOMEWORK]

ANYWHERE?

THIS LEADS TO VEERY LARGE CARDINAL: MEASURABLE ONES.

THE μ FROM ULAM AND TARSKI WOULD BE σ-ADDITIVE;

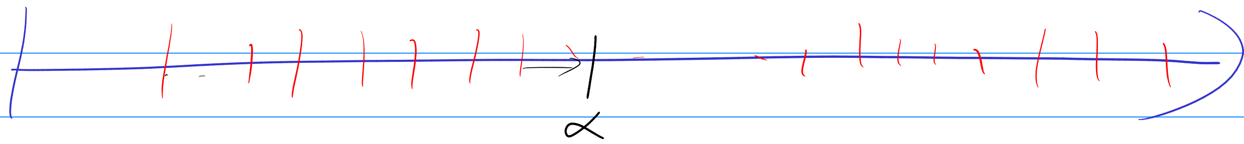
A (REAL) MEASURE

κ IS REGULAR UNCOUNTABLE

$C \subseteq \kappa$ IS CLOSED AND UNBOUNDED (CUB)

- IT IS UNBOUNDED = COFINAL
- IT IS CLOSED: EVERY LIMIT POINT OF C IS IN C

IF $\alpha = \sup(C \cap \alpha)$
THEN $\alpha \in C$



[EXERCISE THE INTERSECTION OF CLOSED SET IS CLOSED.]

- C AND D CUB $\to$ $C \cap D$ CUB

TAKE α ARBITRARY

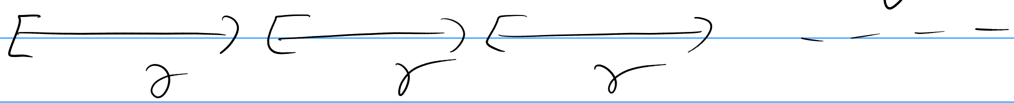
$$\alpha < \underset{\in C}{\alpha_0} < \underset{\in D}{\alpha_1} < \underset{\in C}{\alpha_2} < \underset{\in D}{\alpha_3} < \cdots$$

LET $\beta = \sup_n \alpha_n$: $\underline{\beta < \kappa}$

$$\beta = \sup_n \alpha_{2n} \quad \text{so} \quad \beta \in C$$
$$= \sup_n \alpha_{2n+1} \quad \text{so} \quad \beta \in D$$

- LET $\langle C_\alpha : \alpha < \gamma \rangle$ BE A SEQUENCE OF CUB SETS WITH $\gamma < \kappa$.

TAKE THE ORDINAL $\gamma \cdot \omega$



MAKE, GIVEN $\beta < \kappa$
A SEQUENCE $\langle \delta_\eta : \eta < \gamma \cdot \omega \rangle$

$$\beta < \underset{\in C_0}{\delta_0} < \underset{\in C_1}{\delta_1} < \cdots < \underset{\in C_\alpha}{\delta_\alpha} < \cdots$$

$$< \underset{\in C_0}{\delta_\gamma} < \underset{\in C_1}{\delta_{\gamma+1}} < \cdots < \underset{\in C_\alpha}{\delta_{\gamma+\alpha}} < \cdots$$

$$< \delta_{\gamma \cdot i} < \delta_{\gamma \cdot i+1} < \cdots < \delta_{\gamma \cdot i+\alpha} < \cdots$$
$$\in C_0 \qquad \in C_1 \qquad \in C_\alpha$$

$$\delta = \sup\{\delta_\eta : \eta < \gamma \cdot \omega\} < \kappa$$
$$= \sup_n \delta_{\gamma \cdot n} \quad \text{so } \delta \in C_0$$
$$= \sup_n \delta_{\gamma \cdot n+1} \quad \text{so } \delta \in C_1$$
$$\vdots$$
$$= \sup_n \delta_{\gamma \cdot n+\alpha} \quad \text{so } \delta \in C_\alpha$$

$$\therefore \quad \delta \in \bigcap_{\alpha<\gamma} C_\alpha$$

CLOSED

IF $\delta = \sup(\delta \cap \bigcap_{\alpha<\gamma} C_\alpha)$

THEN $\delta = \sup(\delta \cap C_\alpha)$ ALL α

$\mathcal{C}_\kappa = \{A \subseteq \kappa :$ THERE IS A CUB C WITH $C \subseteq A\}$

$\mathcal{C}_\kappa$ IS A κ-COMPLETE FILTER

IN MODEL THEORY THE (ELEMENTARY) SUBMODELS OF SMALLER CARDINALITY FORM A CUB SET

THERE IS MORE!

SUPPOSE $\langle C_\alpha : \alpha < \kappa \rangle$ IS A SEQUENCE OF CUB SETS

MOST LIKELY $\bigcap_\alpha C_\alpha = \emptyset$

$$\bigcap_{\alpha<\kappa} [\alpha, \kappa) = \emptyset$$

$$\Delta_{\alpha<\kappa} C_\alpha = \{ \delta : (\forall \alpha < \delta)(\delta \in C_\alpha) \}$$

DIAGONAL INTERSECTION

IS CUB !

- CLOSED

$$\delta = \sup(\delta \cap \Delta_\alpha C_\alpha)$$

TAKE $\gamma < \delta$

THERE IS $\beta > \gamma$ $\beta < \delta$

IN $\Delta_\alpha C_\alpha$

SO $\beta \in \bigcap_{\alpha<\beta} C_\alpha \subseteq C_\gamma$

SO IF $\gamma < \beta < \delta$ AND

$\beta \in \Delta_\alpha C_\alpha$ THEN $\beta \in C_\gamma$

SO $\delta = \sup(\delta \cap C_\gamma) \therefore \delta \in C_\gamma$

UNBOUNDED:

$\beta < \kappa$: $\delta_0 > \beta$ IN $\bigcap_{\alpha \leq \beta} C_\alpha$

$\delta_1 > \delta_0$ IN $\bigcap_{\alpha \leq \delta_0} C_\alpha$

$\delta_{n+1} > \delta_n$ IN $\bigcap_{\alpha \leq \delta_n} C_\alpha$

IN THE END $\delta = \sup_n \delta_n$
IS IN $\Delta_\alpha C_\alpha$

IF $\alpha < \delta_n$ THEN
$\{\delta_i : i \geq n\} \subseteq C_\alpha$
SO $\delta \in C_\alpha$.

$\mathcal{C}_\kappa$ IS NOT AN ULTRAFILTER.

YOU CAN SPLIT κ INTO κ MANY STATIONARY SETS

S IS STATIONARY IF $S \cap C \neq \emptyset$ FOR ALL CUB C.

FODOR'S PRESSING DOWN LEMMA

IF $S \subseteq \kappa$ IS STATIONARY AND $f : S \to \kappa$ IS SUCH THAT $f(\alpha) < \alpha$ $(\alpha \in S \setminus \{0\})$

THEN f IS CONSTANT ON A STATIONARY SET.

IF NOT THEN FOR EVERY α THERE IS A CUB SET C_α SUCH THAT: $\beta \in C_\alpha \cap S \to f(\beta) \neq \alpha$.

WE $\langle C_\alpha : \alpha < \kappa \rangle$

LOOK AT $D = \Delta_{\alpha<\kappa} C_\alpha$

$D \cap S \neq \emptyset$

TAKE $\alpha \in D \cap S$

NOW $f(\alpha) < \alpha$

SO $\alpha \in C_{f(\alpha)}$

SO $f(\alpha) \neq f(\alpha)$

CONTRADICTION!

Une fonction $f(x)$, définie sur un sous-ensemble A de $\mathcal{O}$ et à valeurs dans $\mathcal{O}$, sera dite *régressive* si $f(x) \leq x$ pour tout $x \in$ A, l'égalité étant exclue sauf pour $x = 1$, si $1 \in$ A. Un ensemble A de points de $\mathcal{O}$ est *stationnaire* s'il est non dénombrable et si pour toute fonction régressive $f(x)$ définie sur A, on a $\lim_{x \succ \Omega,\, x \in A} f(x) < \Omega$; autrement dit s'il existe au moins un point a tel que $\overset{-1}{f}(a)$ soit non dénombrable. Dans tous les autres cas, A est *non stationnaire*.

THÉORÈME I. — *Une condition nécessaire et suffisante pour qu'un ensemble soit stationnaire est que tout sous-ensemble fermé de son complément soit au plus dénombrable.*

THE κ-COMPLETENESS OF C_κ TELLS US:

IF $\gamma < \kappa$ AND $\{S_\alpha : \alpha < \gamma\}$ IS A FAMILY OF SUBSETS OF κ SUCH THAT $\bigcup_{\alpha<\gamma} S_\alpha$ IS STATIONARY THEN SOME S_α IS STATIONARY