(1)

SET THEORY 2020-11-18

WHY PARTITION THEOREMS?

RAMSEY "A PROBLEM OF FORMAL LOGIC"

ERDŐS-SZEKERES-KLEIN
- FIVE POINTS IN THE PLANE NO THREE ON ONE LINE, FOUR POINTS WILL FORM A CONVEX 4-GON.

- FOR EVERY n THERE IS AN N SUCH THAT AMONG N POINT IN $\mathbb{R}^2$ NO THREE ON A LINE YOU CAN FIND A CONVEX n-GON

$$\underline{\underline{N}} \longrightarrow (n, 5)^4$$

$$\underline{\underline{N}} \longrightarrow (n)^3_2$$

HAJNAL-JUHÁSZ

IF X IS HAUSDORFF
FIRST-COUNTABLE
CCC "EVERY PWD FAMILY OF NONEMPTY OPEN SETS IS CTBLE"

THEN $|X| \leq 2^{\aleph_0}$

HEDRLIN - KUREPA:
IF $X_1, X_2, \ldots, X_n$ ARE CCC SPACES
WHAT ABOUT
$$X_1 \times X_2 \times \cdots \times X_n$$
ALL PWD FAMILIES OF NONEMPTY OPEN SETS HAVE CARDINALITY $\underline{\leq 2^{\aleph_0}}$

$f : \mathbb{Q} \longrightarrow \mathbb{Z}$ BIJECTION

$$F(\{x,y\}) = \begin{cases} 1 & \text{IF } x < y \text{ IFF } f(x) < f(y) \\ 0 & \text{OPPOSITE} \end{cases}$$

HOMOGENEOUS. f IS ORDER PR. OR f IS ORDER-REV.

ERDŐS - DUSHNIK - MILLER THM
IF κ IS INFINITE THEN
$$\kappa \longrightarrow (\kappa, \aleph_0)^2$$
IF $F : [\kappa]^2 \longrightarrow 2 \ (=\{0,1\})$
THEN THERE IS $H \subseteq \kappa$
H IS 0-HOMOG. AND $|H| = \kappa$
OR H IS 1-HOMOG AND $|H| = \aleph_0$

$R = F^{\leftarrow}(0) \qquad B = F^{\leftarrow}(1)$
FOR $\alpha \in \kappa$ WRITE
$R(\alpha) = \{\beta : \{\alpha, \beta\} \in R\}$
$B(\alpha) = \{\beta : \{\alpha, \beta\} \in B\}$

CASE 1: κ REGULAR

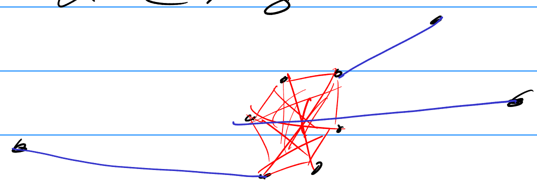
ASSUME: $|H| < \kappa$ WHENEVER $[H]^2 \subseteq R$.

WE FIND $x_0, x_1, x_2, \ldots, x_n \ldots$ SUCH THAT $[\{x_n : n \in \omega\}]^2 \subseteq B$

LET A_0 BE MAXIMAL SUCH THAT $[A_0]^2 \subseteq R$.

GI [WHY IS ZORN'S LEMMA APPLICABLE]

- IF $\alpha \in A_0$ THEN $B(\alpha) \subseteq \kappa \setminus A_0$
- IF $\beta \notin A_0$ THEN $\beta \in B(\alpha)$ FOR SOME $\alpha \in A_0$



- $\kappa \setminus A_0 = \bigcup_{\alpha \in A_0} B(\alpha)$
- REGULARITY: TAKE x_0 WITH $|B(x_0)| = \kappa$
- TAKE $A_1 \subseteq B(x_0)$ MAXIMAL WITH $[A_1]^2 \subseteq R$

  SAME STORY: THERE IS $x_1 \in A_1$ SUCH THAT $|B(x_1) \cap B(x_0)| = \kappa$
- $n \to n+1$: TAKE $A_{n+1} \subseteq \bigcap_{i \le n} B(x_i)$ MAXIMAL WITH $[A_{n+1}]^2 \subseteq R$ AND x_{n+1} AS BEFORE: $|\bigcap_{i \le n+1} B(x_i)| = \kappa$

IN THE END $[\{x_n : n\in\omega\}]^2 \subseteq B$.

NO GUARANTEE THAT

$$\bigcap_{n\in\omega} B(x_n) \neq \emptyset.$$

CASE 2 κ SINGULAR.

CASE 2a

EVERY SUBSET, X, OF κ OF CARDINALITY κ HAS AN ELEMENT x WITH $|X \cap B(x)_0| = \kappa$.

$\cdot$ $x_0 \in X$ WITH $|X \cap B(x_0)| = \kappa$

$\vdots$

$x_{n+1} \in X \cap \bigcap_{i\le n} B(x_i)$ AND $|\bigcap_{i\le n+1} B(x_i) \cap X| = \kappa$

CASE 2b

THERE IS AN X SUCH THAT $|X| = \kappa$, $\underline{|B(x) \cap X| < \kappa}$ ALL $x\in X$.

ASSUME NO INFINITE B-HOMOG. SET.

WLOG: $X = \kappa$

GI [ASSUME κ IS REGULAR AND MAKE AN R-HOMOG. SET OF CARD. κ]

$\langle \kappa_\eta : \eta < \mathrm{cf}\,\kappa \rangle$ INCREASING COFINAL IN κ, EACH κ_η REGULAR AND $\mathrm{cf}\,\kappa < \kappa_0$.

IN PREPARATION OF THE RECURSIVE STEPS:

LET λ BE REGULAR WITH
$\mathrm{cf}\,\kappa < \lambda < \kappa$. (ONE OF THE κ_η)
LET $X \subseteq \kappa$ WITH $|X| = \lambda$

$$X_\eta = \{\alpha \in X : |B(\alpha)| \le \kappa_\eta\}$$

So $X = \bigcup_{\eta < \mathrm{cf}\,\kappa} X_\eta$

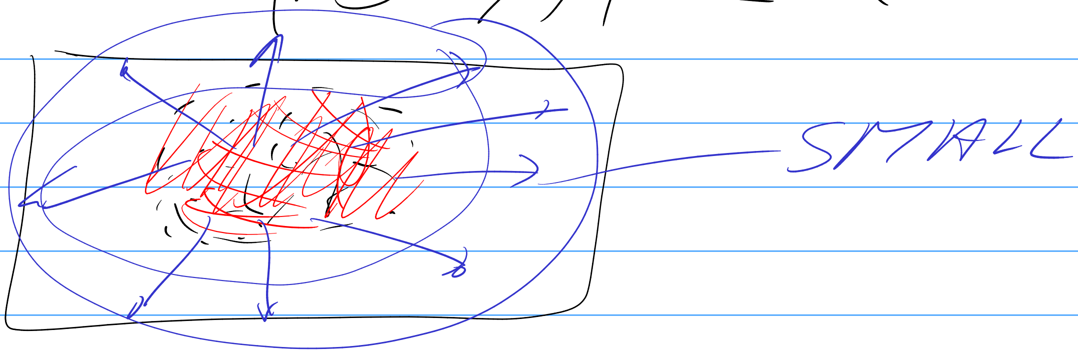
So THERE IS AN η WITH $|X_\eta| = \lambda$
$B(X_\eta) = \bigcup\{B(x) : x \in X_\eta\}$
HAS CARDINALITY AT MOST

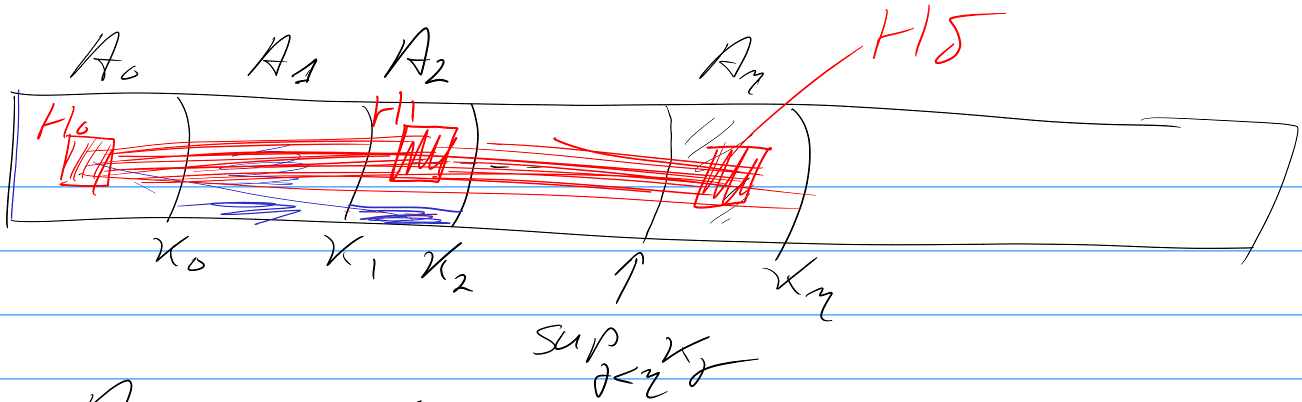
$$\lambda \cdot \kappa_\eta < \kappa$$

REGULAR CASE:

X_η HAS AN R-HOMOGENEOUS
SUBSET Y_η
OF CARDINALITY λ

IF $\lambda < \kappa$ IS REGULAR $\lambda > \mathrm{cf}\,\kappa$
THEN EVERY SET OF SIZE λ
HAS AN R-HOMOGENEOUS
SUBSET Y OF SIZE λ
AND $|B(Y)| < \kappa$.





$$A_\eta = \kappa_\eta \setminus \bigcup_{\gamma<\eta} \kappa_\gamma$$

$|A_\eta| = \kappa_\eta$ (REGULARITY)

TAKE $H_0 \subseteq A_0$: CARDINALITY κ_0
R-HOMOGENEOUS
$|B(H_0)| < \kappa$.

TAKE $\eta_1 = \min\{\eta > \eta_0 : |B(H_0)| < \kappa_\eta\}$, $\eta_0 = 0$
($\eta_1 = 2$)

TAKE $H_1 \subseteq A_{\eta_1} \setminus B(H_0)$

- CARDINALITY κ_{η_1}
- R-HOMOGENEOUS
- $|B(H_1)| < \kappa$

IN GENERAL IF $\delta < cf\kappa$
TAKE $X = \bigcup_{\gamma<\delta} B(H_\gamma)$ $H_\gamma \subseteq A_{\eta_\gamma}$
BECAUSE $\delta < cf \kappa$
$|X| < \kappa$

FIRST η_δ ABOVE THE η_γ's
SUCH THAT $|X| < \kappa_{\eta_\delta}$

TAKE $H_\delta \subseteq A_{\eta_\delta} \setminus X$

R-HOMOGENEOUS
CARD. κ_{η_δ}
$|B(H_\delta)| < \kappa$

THE CONSTRUCTION LASTS $\mathrm{cf}\,\kappa$ MANY STEPS AND GIVES US $H = \bigcup_{\gamma < \mathrm{cf}\,\kappa} H_{\eta_\gamma}$

- $|H| = \kappa$
- H IS R-HOMOGENEOUS.

APPLICATION:
IF $(P, <)$ IS A PARTIAL ORDER THEN EITHER THERE IS A CHAIN OF CARDINALITY $|P|$

THEN THERE IS AN INFINITE SET I SUCH THAT

$$p \not< q, \; q \not< p$$

WHENEVER $p, q \in I$.

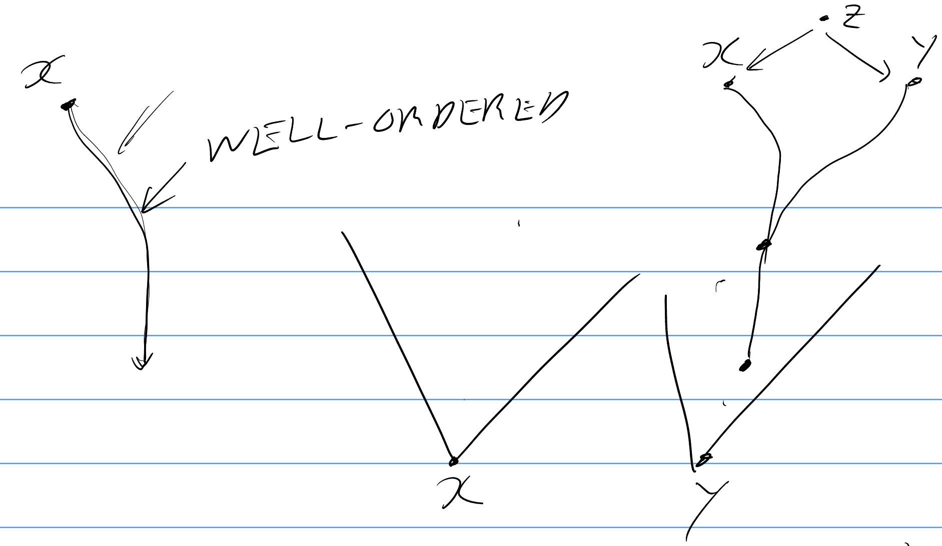
$$? \; \kappa \longrightarrow (\kappa)^2_2 \; ?$$

TREES

A TREE IS A PARTIALLY ORDERED SET $(T, <)$ WITH THE PROPERTY THAT

$$\{ y : y < x \}$$

IS WELL-ORDERED FOR ALL $x \in T$.



$O(x)$ = ORDER TYPE OF $\{y : y < x\}$

α TH LEVEL: $\{x : O(x) = \alpha\}$ (T_α)

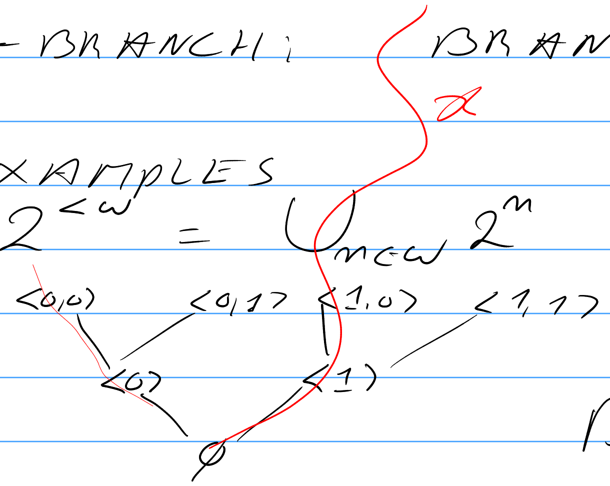
HEIGHT(T) = $\sup\{O(x)+1 : x \in T\}$

BRANCH: MAXIMAL CHAIN

α-BRANCH: BRANCH OF ORDER TYPE α.

EXAMPLES

$2^{<\omega} = \bigcup_{n\in\omega} 2^n$ THE BINARY TREE



BRANCH $\longleftrightarrow$ $x \in 2^\omega$

$x \rightsquigarrow \{x\restriction n : n \in \omega\}$

$A^{<\omega_1}$ ALL FUNCTIONS FROM CTBLE ORDINALS TO A

$\bigcup_{\alpha<\omega_1} A^\alpha$

SUBTREE OF $\mathbb{R}^{<\omega_1}$

$T = \{s : s$ IS STRICTLY INCREASING$\}$

$O(s) = \operatorname{dom} s$

HEIGHT $T = \omega_1$

NO ω_1-BRANCHES

Über eine Schlussweise aus dem Endlichen ins Unendliche.

(Punktmengen. — Kartenfärben. — Verwandtschaftsbeziehungen. — Schachspiel.)

Von Dénes König in Budapest.

§ 1.

Beim Analysieren eines Beweises, den Stephan Valkó und ich für einen Satz der allgemeinen Mengenlehre gegeben haben,[1]) wurde ich auf folgendes Lemma geführt,[2]) welches sich als eigentlicher Kern des erwähnten Beweises erwies.

A) *Es sei $E_1, E_2, E_3 \ldots$ eine abzählbarunendliche Folge endlicher, nicht leerer Mengen und R eine binäre Relation, die so beschaffen ist, dass zu jedem Element x_{n+1} von E_{n+1} mindestens ein solches Element x_n von E_n gehört, welches zu x_{n+1} in der Relation R steht, was wir durch $x_n R x_{n+1}$ ausdrücken wollen. Dann kann man in jeder der Mengen E_n je ein Element a_n derart bestimmen, dass für die unendliche Folge $a_1, a_2, a_3, \ldots$ stets $a_n R a_{n+1}$ bestehe* $(n = 1, 2, 3, \ldots)$

KÖNIG'S

- T AN INFINITE TREE
- FINITE LEVELS: $|T_\alpha| < \aleph_0$

THEN T HAS AN INFINITE BRANCH.

PROOF:

- T_0 IS FINITE, SO
  $\bigcup\{\{y : y > x\} : x \in T_0\}$ $(= T \setminus T_0)$
  IS INFINITE
  PICK $x_0 \in T_0$ WITH $\{y : y > x_0\}$ INFINITE.
- T_1 IS FINITE
  $\bigcup\{\{y : y > x\} : x \in T_1, x > x_0\}$
  IS INFINITE IT IS
  $\{y : y > x_0\} \setminus T_1$
  PICK $x_1 \in T_1$ WITH $\{y : y > x_1\}$ INFINITE
- GIVEN $x_n \in T_n$ WITH
  $\{y : y > x_n\}$ INFINITE
  TAKE $x_{n+1} \in T_{n+1}$ WITH
  $x_{n+1} > x_n$, $\{y : y > x_{n+1}\}$ INFINITE

$\{x_n : n \in \omega\}$ IS AN INFINITE CHAIN
SITS IN AN INFINITE BRANCH.

GI [HOW DOES THE PICK/TAKE WORK??]

APPLICATIONS:
- RAMSEY'S THEOREM [HOMEWORK]
- COMPACTNESS OF $[0,1]$.
- COMPACTNESS IN LOGIC.
- "COMPACTNESS IN COMBINATORICS"

WHAT ABOUT $\aleph_1$?

IF T IS UNCOUNTABLE WITH COUNTABLE LEVELS IS THERE AN UNCOUNTABLE CHAIN / BRANCH?

[21] Ce théorème est dû à M.N. Aronszajn qui a bien voulu me communiquer, vers la fin du mois de juin 1934, un exemple d'une suite ramifiée ambiguë de complexes construite par des considérations suivantes: $r_1, r_2, \dots r_n, \dots$ étant la suite de l'ensemble R de tous les nombres rationnels entre 0 et 1, posons $\varphi(r_n) - 1/n$ pour tout n entre 0 et ω. On considère un système A de complexes $(a_0, \dots a_\xi \dots)_{\xi<\alpha}$, $(\alpha < \omega_1)$, satisfaisant aux conditions suivantes:

a) pour tout complexe $(a_0, \dots a_\xi \dots)_{\xi<\alpha}$ de A, les a_ξ appartiennent à R, sont deux à deux distincts et tels que $\sum_{\xi<\alpha} \varphi(a_\xi) < \infty$;

b) α, α' étant deux ordinaux quelconques tels que $0 < \alpha < \alpha' < \omega_1$, pour chaque élément $(a_0, \dots a_\xi \dots)_{\xi<\alpha}$ de A et si petit que soit le nombre réel $\varepsilon > 0$ il existe un complexe $(a'_0, \dots a'_\xi \dots)_{\xi<\alpha'}$ de A tel que $a'_\xi = a_\xi$ pour tout $\xi < \alpha$ et $\sum_{\alpha \le \eta < \alpha'} \varphi(a'_\eta) < \varepsilon$;

c) pour tout $0 < \alpha < \omega_1$, l'ensemble des complexes $(a_0, \dots a_\xi \dots)$ de rang α de A est dénombrable.

Par le procédé de l'induction complète, on s'assure que la construction de A est possible. Il est clair que A n'admet aucune descente monotone.

ANSWER: NO

THERE A TREE A
- CARDINALITY $\aleph_1$
- COUNTABLE LEVELS (HEIGHT: ω_1)
- NO ω_1-BRANCHES.

CONSIDER $\mathbb{Q}^{<\omega_1}$ (OR $\mathbb{R}^{<\omega_1}$)

$$\bigcup_{\alpha<\omega_1} \mathbb{Q}^\alpha$$

$T = \{ s \in \mathbb{Q}^{<\omega_1} : s$ IS STRICTLY INCREASING $\}$

- HEIGHT ω_1
- NO UNCOUNTABLE BRANCHES
- EVERY LEVEL ω AND UP HAS CARDINALITY $2^{\aleph_0}$

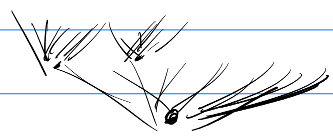
WE BUILD $A \subseteq T$

HEIGHT ω_1 ALL LEVELS CTBLE

$A_0 = \{ \emptyset \}$

$A_1 = \{ \langle q \rangle : q \in \mathbb{Q} \}$ COUNTABLE

$A_2 = \{ \langle q_0, q_1 \rangle : q_0, q_1 \in \mathbb{Q}, q_0 < q_1 \}$



WE WILL CONSTRUCT THE LEVELS A_α SO THAT ALWAYS:

(*) IF $\alpha < \beta$, IF $s \in A_\alpha$ THEN s IS BOUNDED AND FOR EVERY $q >$ SUP RAN s THERE IS A $t \in A_\beta$ SUCH THAT

- $s \subset t$
- SUP RAN $t < q$

SUCCESSOR: $A_\alpha \longrightarrow A_{\alpha+1}$

$$A_{\alpha+1} = \{ s^\frown\langle q\rangle : s\in A_\alpha,\ q > \sup \mathrm{RAN}\, s\}$$

CONCATENATION

$$s = \langle s(\xi) : \xi<\alpha\rangle$$

$$s^\frown\langle q\rangle = \underline{\langle s(\xi) ; \xi<\alpha\rangle} \cup \{\underline{\langle \alpha, q\rangle}\}$$

DOM $= \alpha+1$

- A_α CTBL $\longrightarrow$ $A_{\alpha+1}$ CTBL
- WE HAVE ⊛

$\beta < \alpha+1$ $\qquad$ $\beta = \alpha$

$$\beta < \alpha < \alpha+1$$

s — t $\qquad$ q

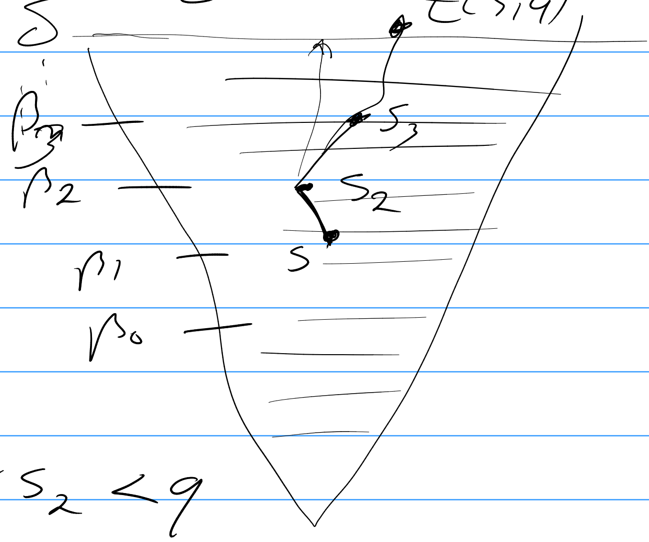
SUP RAN $t < q$

$t^\frown\langle p\rangle$

δ A LIMIT

$\langle A_\alpha : \alpha<\delta\rangle$ FOUND

WE MUST MAKE A_δ



$\langle \beta_n : n\in\omega\rangle$
INCREASING
COFINAL IN δ

$s \in \bigcup_{\alpha<\delta} A_\alpha$

$q > \sup \mathrm{RAN}\, s$

$s_2 \in A_{\beta_2}$ $\quad$ SUP RAN $s_2 < q$

$s_3 \in A_{\beta_3}$ $\quad$ SUP RAN $s_3 < q$

$s_{n+1} \in A_{\beta_{n+1}}$, $s_{n+1} \supset s_n$

$\sup \operatorname{ran} s_{n+1} < q$

IN THE END $t = \bigcup_n s_n$
IS A SEQUENCE
WITH - $\operatorname{dom} t = \delta$
- $s \subset t$
- $\sup \operatorname{ran} t \leq q$

$t = t(s, q)$

$A_\delta = \{ t(s,q) : s \in \bigcup_{\alpha<\delta} A_\alpha,\ q > \sup \operatorname{ran} s \}$

COUNTABLE SET

WE HAVE $\circledast$ BY CONSTRUCTION.

→ κ HAS
THE TREE PROPERTY
IF EVERY TREE OF CARDINALITY κ, WITH LEVELS OF CARD. LESS THAN κ HAS A κ-BRANCH.

$\underline{\kappa \text{ IS REGULAR}}$

→ KÖNIG: $\aleph_0$ HAS TREE PROP.

→ ARONSZAJN: $\aleph_1$ DOES NOT.

$\aleph_2$?? CH IMPLIES NO
MAKE AN $\aleph_2$-ARONSZAJN TREE
WITHOUT CH -- THINGS GET INTERESTING.

VARIATIONS:

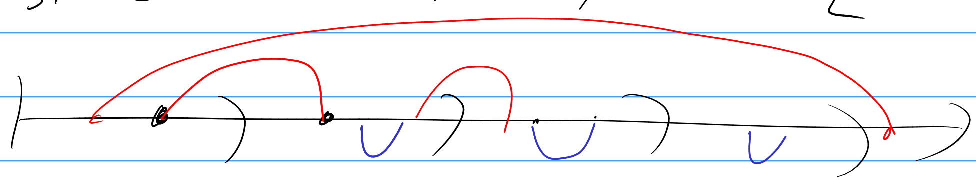
- YOU CAN ENSURE THAT $\sup \operatorname{ran} S \in \mathbb{Q}$ FOR ALL S.
- $\{ A \subseteq \mathbb{Q} : A$ IS WELL-ORDERED$\}$ $A < B$ IF A IS AN INITIAL SEGMENT OF B.

$$\aleph_0 \longrightarrow (\aleph_0)^n_k \qquad (n, k \in \omega)$$

$$\aleph_1 \not\longrightarrow (\aleph_1)^2_2$$

$$\kappa^+ \not\longrightarrow (\kappa^+)^2_2$$

κ SINGULAR $\quad \kappa \not\longrightarrow (\kappa)^2_2$



DEFINITION

κ IS **WEAKLY COMPACT** IF IT IS UNCOUNTABLE AND $\kappa \longrightarrow (\kappa)^2_2$

κ IS (STRONGLY) INACCESSIBLE

WC $\equiv$ STR. INACC.
+ TREE PROPERTY