SET THEORY

2020-12-14

1. THE WEAK COMPACTNESS THM
2. INDESCRIBABILITY.

THE INFINITARY LANGUAGES ALLOW YOU TO CHEAT.

"THE GROUP G IS CYCLIC"

$\exists x \forall y \exists n \in \mathbb{Z} : y = x^n$

NOT IN G

$\exists x \forall y (y = x \vee y = x * x \vee y = x * x^{-1}$
$\vee y = xxx \vee y = x^{-1}x^{-1}x^{-1}$
$\vee y = x^{-1}x^{-1} \vee - - - - - - -)$

$\mathcal{L}_{\omega_1, \omega}$

$\exists_{n<\omega} x_n$ $\qquad$ $\forall_{n<\omega} x_n$

C_p $\qquad$ C_π

IF $\mathcal{L}_{\kappa\omega}$ SATISFIES WEAK COMPACTNESS AND κ INACCESSIBLE THEN κ IS WEAKLY COMPACT

WE PROVED THE TREE PROPERTY

IF κ IS WEAKLY COMPACT THEN $\mathcal{L}_{\kappa,\kappa}$ SATISFIES WEAK COMPACTNESS.

WE HAVE A LANGUAGE $\mathcal{L}$ OF TYPE $\mathcal{L}_{\kappa,\kappa}$

$\exists_{\xi<\alpha} \quad \forall_{\xi<\alpha}$

$\bigvee_{\xi<\alpha} \bigwedge_{\xi<\alpha} \varphi_\xi$

- Σ A SET OF ~~FORMULAS~~ SENTENCES $|\Sigma| = \kappa$
  EVERY $S \in [\Sigma]^{<\kappa}$ HAS A MODEL
- WE MUST CONSTRUCT A MODEL FOR Σ

- CONSIDER ONLY THE SYMBOLS OF $\mathcal{L}$ THAT OCCUR IN Σ AND DISREGARD THE REST NOW $|\mathcal{L}| = \kappa$

$\mathcal{L}_0 = \mathcal{L}$

$\gamma < \kappa$: FOR EVERY $\mathcal{L}_{\gamma}$-FORMULA φ WITH FREE VARIABLES $\langle v_\xi : \xi < \alpha \rangle$ INTRODUCE NEW CONSTANTS

$c(\varphi, \xi, \gamma) \quad \xi < \alpha$

WE GET $\mathcal{L}_{\gamma+1}$

$\mathcal{L}_0 \longrightarrow \mathcal{L}_1 \quad : \quad |\mathcal{L}_1| = \kappa$

$\mathcal{L}_\gamma \longrightarrow \mathcal{L}_{\gamma+1} \quad : \quad |\mathcal{L}_{\gamma+1}| = \kappa$ IF $|\mathcal{L}_\gamma| = \kappa$

γ LIMIT $\quad \mathcal{L}_\gamma = \bigcup_{\delta<\gamma} \mathcal{L}_\delta$

$\mathcal{L}^* = \bigcup_{\gamma<\kappa} \mathcal{L}_\gamma$

IF φ IS AN $\mathcal{L}^*$-FORMULA
WITH FREE VARIABLES $\langle v_\xi : \xi < \alpha \rangle$
THEN $\varphi \in \mathcal{L}_\gamma$ FOR SOME (MANY) γ
BECAUSE φ CONTAINS
FEWER THAN κ MANY SYMBOLS
$\mathcal{L}^*$ HAS CONSTANTS
$c(\varphi, \xi, \gamma)$ $\xi < \alpha$
THAT DO NOT OCCUR IN φ ITSELF.

WE MAKE Σ LARGER
FOR EACH $\mathcal{L}^*$-FORMULA φ
TAKE γ_φ SUCH THAT φ IS $\mathcal{L}_{\gamma_\varphi}$
DEFINE
$\sigma_\varphi : (\exists_{\xi<\alpha} v_\xi \; \varphi(v_\xi : \xi < \alpha)) \rightarrow (\varphi(c(\varphi, \xi, \gamma_\varphi) : \xi < \alpha))$
$\Sigma^* = \Sigma \cup \{ \sigma_\varphi : \varphi \text{ IS AN } \mathcal{L}^*\text{-FORMULA} \}$

Σ^* IS LIKE Σ:
IF $S \in [\Sigma^*]^{<\kappa}$
THEN S HAS A MODEL
TAKE MODEL FOR $S \cap \Sigma$: $\mathbb{A}$
$\mathbb{A}$ IS JUST AN $\mathcal{L}$-STRUCTURE.
MAKE THAT A MODEL FOR S
BY INTERPRETING
$(c(\varphi, \xi, \gamma_\varphi) : \xi < \alpha)$
FOR EVERY φ FOR WHICH
$\sigma_\varphi \in S \setminus \Sigma$
IF $\exists_{\xi<\alpha} v_\xi \; \varphi(v_\xi : \xi < \alpha)$ IS TRUE IN $\mathbb{A}$
MAKE SURE $\varphi(c(\varphi, \xi, \gamma_\varphi) : \xi < \alpha)$ IS TRUE
OTHERWISE ARBITRARY

NO CONFLICT

IF $\varphi \neq \psi$ THEN

$\{c(\varphi, \xi, \gamma_\varphi) : \xi < \alpha\} \cap \{c(\psi, \xi, \gamma_\psi) : \xi < \beta\} = \emptyset$

. ENUMERATE ALL SENTENCES IN $\mathcal{L}^*$ AS $\langle \sigma_\alpha : \alpha < \kappa \rangle$

T WILL BE A SUBTREE OF $2^{<\kappa}$.

$t \in T$ IF $\delta = \text{dom}\, t \in \kappa$

AND $t : \delta \to 2$ IS SUCH THAT

THERE IS A MODEL $\mathbb{A}$ FOR

$\Sigma^* \cap \{\sigma_\alpha : \alpha < \delta\}$

SUCH THAT

$\to$ $\mathbb{A} \models \sigma_\alpha$ IFF $t(\alpha) = 1$

T HAS CARDINALITY κ

SO IT HAS A BRANCH $\mathbb{B}$!

$\mathbb{B}$ DETERMINES $x : \kappa \to 2$

$x = \bigcup \mathbb{B}$

$\Delta = \{\sigma_\alpha : x(\alpha) = 1\}$

$\Sigma^* \subseteq \Delta$

– FOR EVERY σ; $\sigma \in \Delta$ OR $\neg\sigma \in \Delta$

AS SOON AS $\{\sigma, \neg\sigma\} \subseteq \{\sigma_\alpha : \alpha < \delta\}$

JUST ONE OF σ AND $\neg\sigma$ GETS VALUE 1

SO Δ IS COMPLETE

WE BUILD A MODEL $\mathbb{A}$

DOMAIN A CONSISTS OF ALL CLOSED TERMS OF $\mathcal{L}^*$.

CLOSED TERMS:
CONSTANTS
CLOSED UNDER APPLICATION OF FUNCTION SYMBOLS

|| $e = ee = eee = eeee$
$e^{-1} = e^{-1}e = ee^{-1} = \ldots$

INTERPRETATION
- CONSTANTS ⟶ THEMSELVES
- FUNCTION SYMBOLS ⟶ THE NATURAL THING.
- P PREDICATE:
||| $\mathbb{A} \models P(\tau_1, \ldots, \tau_k)$ IFF $P(\tau_1, \ldots, \tau_k) \in \Delta$ |||||

THE REST IS AS IN THE STANDARD PROOF.
FOR ALL SENTENCES σ
$\mathbb{A} \models \sigma$ IFF $\sigma \in \Delta$

σ, τ ⟶ $\sigma \vee \tau$ $\sigma \wedge \tau$
$\neg\sigma$, $\sigma \to \tau$

$\sigma = \exists_{\xi<\alpha} v_\xi\, \varphi(v_\xi : \xi < \alpha)$ |||

! || σ_φ IS IN Δ
IF σ IS IN Δ THEN
SO IS $\varphi(c(\varphi, \xi, \sigma_\varphi) : \xi < \alpha) \in \Delta$
$\mathbb{A} \models$ — " —

IF $\mathcal{A} \models \varphi(\tau_\xi : \xi < \alpha)$

THEN $\varphi(\tau_\xi : \xi < \alpha) \in \Delta^{\Sigma}$

$$\boxed{\varphi(\tau_\xi : \xi < \alpha) \longrightarrow \exists_{\xi<\alpha} v_\xi\, \varphi(v_\xi : \xi < \alpha)}$$

SO $\mathcal{A}$ MODELS Σ^*
AND HENCE Σ.

THEOREM [HANF-SCOTT]
κ IS WEAKLY COMPACT IFF
IT IS Π^1_1-INDESCRIBABLE

Π^0_n $\quad \forall x_0\, \exists x_1 \cdots \overset{\forall}{\exists} x_{n-1} (\varphi \cdots)$ WITHOUT QUANTIFIERS

Σ^0_n $\quad \exists x_0\, \forall x_1\, \overset{\exists}{\forall} x_{n-1}$

Π^1_1 $\quad \forall X\, \varphi(X)$ ← REST FIRST-ORDER WITH X AS PREDICATE "$x_i \in X$"

$\uparrow_n$ SECOND ORDER / SUBSET OF STRUCTURE

Σ^1_1 $\quad \exists X\, \varphi(X)$

$\uparrow_n$

Π^2_1 $\quad \forall \mathcal{X}\, \varphi(\mathcal{X})$

$\uparrow$ THIRD ORDER FAMILY OF SUBSETS

FIRST- AND SECOND ORDER

κ IS Π^1_1 - INDESCRIBABLE MEANS WHENEVER σ IS A Π^1_1-SENTENCE

$\forall X\ \varphi(X)$ ← NO FREE VAR'S.

AND $U \subseteq V_\kappa$

ARE SUCH THAT

$(V_\kappa, \in, U) \models \sigma$

↑ CAN BE USED AS A PREDICATE IN σ

THEN THERE IS $\alpha < \kappa$ SUCH THAT

$(V_\alpha, \in, U \cap V_\alpha) \models \sigma$

κ WEAKLY COMPACT

- LET $C \subseteq \kappa$ BE CUB (ACT AS U)
- $\varphi(F)$ SAYS "F IS A FUNCTION" $\wedge$ "IF DOM F IS AN ORDINAL THEN $\exists \alpha\ \forall \beta\ F(\beta) < \alpha$"

THIS SAYS "κ IS REGULAR"

($\forall \alpha\ \exists \beta\ \beta > \alpha\ \beta \in C$) sup $C \cap \alpha \in C$

$(V_\kappa, \in, C) \models \forall F(\varphi(F) \wedge$ "C IS CUB")

THEN $\exists \alpha < \kappa$

$(V_\alpha, \in, C \cap \alpha) \models \forall F(\varphi(F) \wedge$ "C IS CUB")

α IS REGULAR

$C \cap \alpha$ IS CUB IN α

HENCE

$\{\alpha < \kappa : \alpha \text{ REGULAR}\}$ IS STATIONARY

"κ IS MAHLO"

MAHLO IS Π^1_1

$\longrightarrow \left[(\forall X)(X \text{ IS CUB} \to \exists \alpha \in X \; cf\,\alpha = \alpha) \right]$

"C IS CUB"

$\{\alpha < \kappa : \alpha$ IS MAHLO$\}$ IS STATIONARY

→ IF κ IS WEAKLY COMPACT AND $S \subseteq \kappa$ IS STATIONARY IN κ THEN THERE $\lambda < \kappa$ SUCH THAT $S \cap \lambda$ IS STATIONARY IN λ

REG. UNCOUNTABLE

— κ IS REGULAR: Π^1_1

— κ IS UNCOUNTABLE IS Π^1_1 IN $(V_\kappa, \in)$

— S IS STATIONARY IS Π^1_1 IN $(V_{\kappa}, \in, S)$

$(\forall C)(C \text{ IS CUB} \to \exists \alpha \in C \cap S)$

BY THE MAGIC OF INDESCRIBABILITY

$\exists \lambda < \kappa$

$(V_\lambda, \in, S \cap \lambda) \models$ "ALL OF THE ABOVE"

DEFINITION OF WEAK COMPACTNESS IS

Π^1_2 $\quad (\forall F)(\exists H)\, \varphi(F, H)$

YOU FILL THIS IN

κ SINGULAR $\longrightarrow$ κ IS Π^0_m (Σ^0_m)-DESCRIBABLE FOR SOME m.

$\underbrace{\exists f}_{\text{PRED}}\ \underbrace{\exists \lambda}_{\text{PRED}}$ ["f IS A FUNCTION" "DOM $f = \lambda$" "RAN f IS COFINAL"]

$\varphi(f, \lambda)$

$(V_\kappa, \in, f, \lambda) \models \varphi(f, \lambda)$

THERE IS NO $\alpha < \kappa$ SUCH THAT

$(V_\alpha, \in, f \cap V_\alpha, \lambda \cap V_\alpha) \models \varphi(f, \lambda)$

(1) $\alpha < \lambda$ $\lambda \cap V_\alpha$ TOO SMALL

(2) $\alpha > \lambda$ SOME $\beta < \lambda$ HAVE NO VALUES.

IF $\lambda < \kappa \leq 2^\lambda$ Π^0_m—DESCRIBILE λ AND $f : \mathcal{P}(\lambda) \twoheadrightarrow \kappa$

$\notin V_\kappa$

IF κ IS Π^1_1—INDESCRIBABLE THEN κ IS INACCESSIBLE

WE SHOW κ HAS THE TREE PROPERTY.

TAKE A TREE T OF CARDINALITY κ WITH SMALL LEVELS

PROOF OF 9.26: WLOG $T \subseteq 2^{<\kappa}$

ASSUME T HAS NO BRANCH

→ $(\forall B)(B : \kappa \to 2 \Rightarrow (\exists \alpha)(B \restriction \alpha \notin T))$

HOLDS IN $(V_\kappa, \in, T)$

IT HOLDS IN $(V_\lambda, \in, T \cap 2^{<\lambda})$ FOR SOME $\lambda < \kappa$

THAT SAYS: $T_\lambda = \emptyset$ CONTRADICTION

17.17 IF κ IS WEAKLY COMPACT AND $U \subseteq V_\kappa$

THEN THERE IS A TRANSITIVE ELEMENTARY EXTENSION

$(M, \in, U')$ OF $(V_\kappa, \in, U)$

WITH $\kappa \in M$

$\mathbb{V} = (V_\kappa, \in, U, \{x\}_{x \in V_\kappa})$

↑ EACH $x \in V_\kappa$ IS A CONSTANT

Σ: ALL $\mathcal{L}_{\kappa,\kappa}$ — SENTENCES IN OUR LANGUAGE

$\in, U, \{x\}_{x \in V_\kappa} \cup \{c\}$

THAT ARE TRUE IN $\mathbb{V}$

PLUS ALL THE SENTENCES

→ "c is AN ORDINAL" "$\alpha \in c$" $(\alpha < \kappa)$

$|\Sigma| = \kappa$

IF $S \in [\Sigma]^{<\kappa}$

THEN IT HAS A MODEL

THE MODEL IS JUST $\mathbb{V}$

WITH $\in, U, x \; (x \in V_\kappa)$

INTERPRETED AS THEMSELVES

TAKE c LARGE ENOUGH

SO Σ HAS A MODEL!

$\mathbb{A} = (A, E, U_A, \{x_A\}_{x \in V_\kappa}, c_A)$

REPLACE x_A BY x

SO $V_\kappa \subseteq A$

$E \cap V_\kappa^2 = \in$

$U_A \cap V_\kappa = U$

$\mathbb{V} \prec \mathbb{A}$ TRUTH OF SENTENCES

↑ ELEM SUBSET

(A, E) IS WELL-FOUNDED

$\neg (\exists_{n<\omega} v_n)(\bigwedge_{n<\omega} v_{n+1} \in v_n)$

HOLDS IN $\mathbb{V}$ ($\in$ ↦ E)

$\mathcal{L}_{\omega_1,\omega_1}$

LET M BE THE TRANSITIVE COLLAPSE

THE INTERPRETATION OF C IN M IS AN ORDINAL ABOVE ALL $\alpha < \kappa$

SO $\kappa \in M$!!

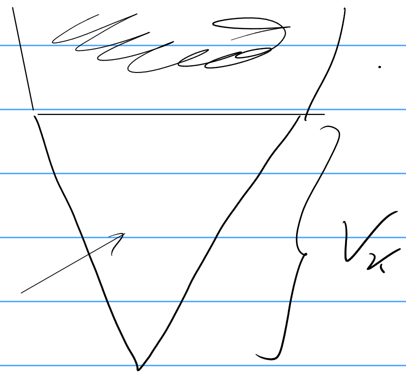
ASSUME κ IS WEAKLY COMPACT $U \subseteq V_\kappa$ AND

$$(*) \quad (V_\kappa, \in, U) \models (\forall X)\, \varphi(X)$$

TAKE $(M, \in, U')$ AS ABOVE

BY ELEMENTARITY AND THE USE OF $\mathcal{L}_{\kappa,\kappa}$ YOU GET

$$V_\kappa^M = V_\kappa$$



$(*)$ MEANS

$$(\forall X \subseteq V_\kappa)\big((V_\kappa, \in, \overset{U' \cap V_\kappa}{U}) \models \varphi(X)\big)$$

SO THIS ALSO HAPPENS IN M

$$(M, \in, U') \models \text{"}(\forall X \subseteq V_\kappa)(\langle \overset{V_\kappa^M}{V_\kappa}, \in, \overset{U}{U' \cap V_\kappa}\rangle \models \varphi(X))$$

$\kappa \in M$

$$(M, \in, U') \models (\exists \alpha)(\forall X \subseteq V_\alpha)(\langle V_\alpha, \in, U' \cap V_\alpha\rangle \models \varphi(X))$$

ELEMENTARITY

$$(V_\kappa, \in, U) \models \text{———————— " ————————}$$

THIS MEANS $(\exists \alpha < \kappa)(\langle V_\alpha, \in, U \cap V_\alpha\rangle \models \sigma)$.

ULTRAFILTERS ⟶ MEASURABLE

PARTITIONS / KÖNIG'S LEMMA ⟶ WEAKLY COMPACT

REGULAR STRONG LIMIT ⟶ INACCESSIBLE

WEAKLY COMPACT: TONS OF INACCESSIBLES BELOW IT

MEASURABLE: TONS OF WEAKLY COMPACTS BELOW IT

$j: V \longrightarrow M \subset V$

KUNEN 17.7

THIS IS NECESSARY

THERE IS NO NON-TRIVIAL ELEM EMBEDDING $j: V \to V$.