Homework Sheet #9

MasterMath: Set Theory
2020/21: 1st Semester
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Deadline for Homework Set #9: Monday, 9 November 2020, 2pm.

(29) Let κ be infinite. This exercise establishes that there are at least 2^{2^κ} many ultrafilters on κ. Let X be the following set:

$$\bigl\{\langle x, F\rangle : x \in [\kappa]^{<\aleph_0}, F \subseteq \mathcal{P}(x)\bigr\}$$

a. Prove that X has cardinality κ.

From now on we work on X. For every subset P of κ define

$$A_P = \bigl\{\langle x, F\rangle \in X : P \cap x \in F\bigr\}$$

b. Prove: if P_1, P_2, …, P_k, Q_1, Q_2, …, Q_l are distinct subsets of κ then

$$A_{P_1} \cap A_{P_2} \cap \cdots \cap A_{P_k} \cap (X \setminus A_{Q_1}) \cap (X \setminus A_{Q_2}) \cap \cdots \cap (X \setminus A_{Q_l}) \neq \emptyset$$

We write $A_P(1) = A_P$ and $A_P(0) = X \setminus A_P$.

c. Prove: for every function $f : \mathcal{P}(\kappa) \to \{0, 1\}$ the family $G_f = \bigl\{A_P\bigl(f(P)\bigr) : P \in \mathcal{P}(\kappa)\bigr\}$ has the finite intersection property.

d. Deduce that there are at least 2^{2^κ} many ultrafilters on X (and hence on κ).

(30) Prove: if U is a σ-complete ultrafilter on $\mathbb{R}$ then U is principal. *Hint*: Consider, for every $q \in \mathbb{Q}$, the sets $(-\infty, q]$ and (q, ∞).

(31) For every countable ordinal $\alpha \geqslant \omega$ let $f_\alpha : \alpha \to \omega$ be a bijection. For α and n define

$$U(\alpha, n) = \{\beta \in \omega_1 : \beta > \alpha \text{ and } f_\beta(\alpha) = n\}$$

Prove:

a. For every $n \in \omega$ the family $\{U(\alpha, n) : \alpha \geqslant \omega\}$ is pairwise disjoint.

b. For every $\alpha \geqslant \omega$ there is an n such that $U(\alpha, n)$ is stationary in ω_1.

c. There is an n such that $\{\alpha \geqslant \omega : U(\alpha, n) \text{ is stationary}\}$ is uncountable.

d. Every stationary subset of ω_1 can be decomposed into $\aleph_1$ many pairwise disjoint stationary sets.

(32) Let $\{F_\alpha : \alpha < \omega_1\}$ be a family of finite subsets of ω_1. Prove that there are a finite set R and a stationary set S such that $F_\alpha \cap F_\beta = R$ whenever $\alpha, \beta \in S$ and $\alpha \neq \beta$. *Hint*: Use Fodor's Pressing-Down Lemma to find R and a stationary set T such that $F_\alpha \cap \alpha = R$ for $\alpha \in T$.
(A family like $\{F_\alpha : \alpha \in S\}$ is called a Δ-system with root R.)