Homework Sheet #11

MasterMath: Set Theory
2021/22: 1st Semester
K. P. Hart, Steef Hegeman, Benedikt Löwe & Robert Paßmann

Deadline for Homework Set #11: Monday, 29 November 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(37) Upward absolute and downward absolute. Let M and N be sets with $M \subseteq N$ and $\varphi(y_1, \ldots, y_k)$ a formula (with its free variables shown). We say φ is *upward absolute* for M, N if

$$\forall m_1, \ldots, m_k \in M \big(\varphi^M(m_1, \ldots, m_k) \to \varphi^N(m_1, \ldots, m_k)\big)$$

We say φ is *downward absolute* for M, N if

$$\forall m_1, \ldots, m_k \in M \big(\varphi^N(m_1, \ldots, m_k) \to \varphi^M(m_1, \ldots, m_k)\big)$$

a. Verify that φ is absolute for M, N iff it is both upward and downward absolute for M, N.

Now assume that $\varphi(x, y_1, \ldots, y_k)$ is absolute for M, N.

b. Prove that $(\exists x)\varphi(x, y_1, \ldots, y_k)$ is upward absolute for M, N.

c. Prove that $(\forall x)\varphi(x, y_1, \ldots, y_k)$ is downward absolute for M, N.

(38) Absoluteness of well-orders. Let X be a set and R a binary relation on X.

a. Write down a Δ_0-formula $\varphi(x, y, z)$ such that "R is a wellorder of X" can be expressed as

$$(\forall A)\varphi(A, X, R)$$

b. Write down a Δ_0-formula $\psi(x, y, z)$ such that "R is a wellorder of X" can be expressed as

$$(\exists f)\psi(f, X, R)$$

c. Let M and N be transitive sets/classes that satisfy the (finitely many) axioms used in the proof of the Representation Theorem for Wellorders (Lecture 5). Prove that "R is a wellorder of X" is absolute for M, N.

(39) Consider the following order of $X = \{0, 1\} \times \omega$:

$$\langle i, m\rangle \; R \; \langle j, n\rangle \text{ iff } \begin{cases} i = 0 \text{ and } j = 1 \\ i = j = 0 \text{ and } m < n \\ i = j = 1 \text{ and } n < m \end{cases}$$

a. Verify that R is a linear order, but not a well-order. (It is actually the ordered sum $(\omega, <) \oplus (\omega, >)$.)

Let $M = V_\omega \cup \mathcal{P}(\{0\} \times \omega) \cup \{X, R\}$

b. Verify that $V_\omega \subseteq M \subseteq V_{\omega+1}$ and that M is a transitive set.

c. Prove that every subset of X that is in M has a minimum with respect to R.

d. Deduce that the definition of "R is a wellorder of X" is not absolute for transitive sets.