Homework Sheet #13

MasterMath: Set Theory
2021/22: 1st Semester
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Deadline for Homework Set #13: Monday, 13 December 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(43) Let $\mathbb{P}$ be a partial order. Define, by recursion on α:
- $N_0 = \emptyset$,
- $N_{\alpha+1} = \mathcal{P}(N_\alpha \times \mathbb{P})$, and
- $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$ if α is a limit ordinal.

Prove that $\bigcup_\alpha N_\alpha$ is equal to the class $V^{\mathbb{P}}$ of all $\mathbb{P}$-names.

(44) Let M be a countable model of ZFC and $\mathbb{P} \in M$ a partial order. Generalize Problem (41) from last week and prove the following general statement: a filter G on $\mathbb{P}$ is M-generic if and only if it interescts every maximal antichain in $\mathbb{P}$ that is an element of M.

(45) The results in class imply that $p \Vdash (\exists x)(\varphi(x,\tau))$ is equivalent to the set

$$E = \bigl\{q \leqslant p : (\exists\sigma)\bigl(q \Vdash \varphi(\sigma,\tau)\bigr)\bigr\}$$

being dense below p.
This problem proves that it is in fact equivalent to

$$(\exists\sigma)\bigl(p \Vdash \varphi(\sigma,\tau)\bigr).$$

(as one would probably expect).
a. Prove that there is a maximal antichain A in E.
b. Prove that there is a function that chooses for every $q \in A$ a name σ_q such that $q \Vdash \varphi(\sigma_q,\tau)$.

Let $D = \bigcup\{\operatorname{dom}\sigma_q : q \in A\}$. Define

$$\sigma = \{\langle\pi,r\rangle : (\exists q \in A)(\exists t \in \mathbb{P})(r \leqslant q \wedge r \leqslant t \wedge \langle\pi,t\rangle \in \sigma_q)\}$$

c. Prove that $p \Vdash \varphi(\sigma,\tau)$. *Hint*: If G is M-generic then $G \cap A$ consists of exactly one point q; prove that $\operatorname{val}(\sigma,G) = \operatorname{val}(\sigma_q,G)$.