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Set Theory (Chapter 4)

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CHAPTER IV

The point of gravity of the chapter is the final § 3 devoted to the method of generic extensions of models of set theory. The explanation is based on the theory of Boolean algebras which is explained in the first two paragraphs. After the introductory § 1, containing basic notions, statements and examples, we shall deal in § 2 with structural properties of Boolean algebras. We emphasize complete Boolean algebras and those properties of them that are relevant for investigations of generic extensions, for example dense subsets, partitions of unity, saturatedness and distributivity. There we also give constructions of new algebras using free products and factorization, and the Stone Theorem about duality. In § 3 we motivate the notion of a generic set, we present the Generic Extension Theorem, we show by some examples how to use it and in the end we prove it.

§ 1 Boolean operations

We shall define Boolean algebras and algebras of sets. We shall prove basic statements and become familiar with important types of Boolean algebras, in particular with κ–complete and complete Boolean algebras, algebras of clopen sets, regular open sets and sets having the Baire property.

<u>1.1. Definition</u>. A <u>Boolean algebra</u> is a structure $\langle B, \wedge, \vee, -, \mathbf{0}, \mathbf{1})$ consisting of a set B, two binary operations of meet $\wedge$ and join $\vee$, a unary operation of complement $-$, and two elements $\mathbf{0}, \mathbf{1} \in B$, in which the following axioms hold. For every $x, y, z \in B$

(1) commutativity of meet and join

$$x \wedge y = y \wedge x, \qquad x \vee y = y \vee x;$$

(2) distributive laws

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

(3) neutrality of the null and unit elements

$x \vee \mathbf{0} = x, \qquad x \wedge \mathbf{1} = x,$

(4) axioms of complement

$x \wedge (-x) = \mathbf{0}, \qquad x \vee (-x) = \mathbf{1},$

(5) condition of nondegeneracy

$\mathbf{0} \neq \mathbf{1}.$

In other words, every structure of the language $\{\wedge, \vee, -, \mathbf{0}, \mathbf{1}\}$, in which all formulas (1)–(5) hold, is a Boolean algebra.

The operations of meet, join and complement are called Boolean operations. Let us agree that Boolean algebras will be denoted by the same symbol as their underlying sets.

By the cardinality of a Boolean algebra we mean the cardinality of its underlying set. An algebra B is finite (countable) if its underlying set is finite (countable).

1.2 Examples. (a) Two–element algebras. In any Boolean algebra B let us consider the designated constants **0** and **1**. The axioms (iii) and (iv) guarantee that all Boolean operations applied to them give again the value **0** or **1** by the following table:

x	y	x∧y	x∨y	−x
0	0	0	0	1
0	1	0	1	1
1	0	0	1	0
1	1	1	1	0

This means that the simplest type of Boolean algebras, which we call trivial or two–element, are the algebras consisting of the elements **0** and **1**. It is obvious that all two–element Boolean algebras are pairwise isomorphic. The best known interpretations of such a Boolean algebra represent zero and unit as truth–values; **0** is false, **1** is true. The Boolean operations then correspond to truth–values of conjuction, disjunction and negation depending on truth values of individual components.

(b) For a nonempty set X the power set algebra $\mathcal{P}(X)$ with the set–theoretic operations of intersection and union of two sets, operation of complement in the set X and the designated elements $\emptyset$ and X is a Boolean algebra. In this algebra for any $Y, Z \subseteq X$ we have

$$Y \wedge Z = Y \cap Z, \qquad Y \vee Z = Y \cup Z,$$

$$-Y = X - Y,$$

$$\mathbf{0} = \emptyset, \qquad \mathbf{1} = X.$$

In the sequel when we deal with collections of subsets of a given set by "the usual set – theoretic operations" we shall mean union, intersection and complement.

(c) <u>Interval algebras</u>. Let $< X, \leq >$ be a linear ordering, $X \neq \emptyset$, and consider all halfclosed intervals $[a,b)$, $[a,-)$ and intervals $(-,a)$, $(-,-)$, where $a < b$. The collection of all finite unions of halfclosed intervals with the usual set–theoretic operations is a Boolean algebra, which we call the interval algebra of the linear ordering $< X, \leq >$. The interval algebra of the rational numbers is an example of a countable Boolean algebra.

Let $\mathcal{J}$ be an ideal on a set X and let $\mathcal{J}^*$ be the dual filter. Then $\mathcal{J} \cup \mathcal{J}^*$ together with the usual set–theoretic operations is a Boolean algebra. In particular, if $\mathcal{J} = [X]^{<\omega}$ then $\mathcal{J} \cup \mathcal{J}^*$ is the algebra of finite subsets and complements of finite subsets of the set X. The algebra of finite subsets and complements of finite subsets of natural numbers is also the interval algebra of the ordering $(\omega, <)$.

(e) <u>Algebras of clopen sets. Totally disconnected topological spaces</u>. Let X be a nonempty topological space. A set $A \subseteq X$ is said to be <u>clopen</u> if A is both open and closed. Let CO(X) be the family of all clopen sets of the space X. CO(X) with the usual set–theoretic operations is a Boolean algebra and is called the <u>algebra of clopen sets of the space X.</u>

In nice metric spaces, like the Euclidean n–dimensional space $\mathbb{R}^n$ and the Hilbert cube $^{\omega}\mathbb{I}$, there are only two clopen sets – the empty set and the whole space. Such spaces are called <u>connected</u>. The algebra of clopen sets is then trivial.

From a Boolean algebraic standpoint the disconnected spaces in which any two distinct points can be separated by clopen sets are most interesting. This condition is equivalent to the condition that for different points x and y there is a clopen neighbourhood of the point x not containing the point y. Topological spaces with this property are called <u>totally disconnected.</u>

(f) The simplest example of a totally disconnected space is a discrete topological space X. The space is not very interesting because $CO(X) = \mathscr{P}(X)$. The Cantor discontinuum D, which we introduced in I.6.44, is a nontrivial example of a totally disconnected space. It is a compact metric space without isolated points. Let us recall that the Cantor discontinuum is homeomorphic to the topological product $^{\omega}2$ of countably many two–point discrete spaces.

If κ is an infinite cardinal, then the topological product $^{\kappa}2$ of κ two–element discrete spaces is called the generalized Cantor discontinuum and is denoted by $D(\kappa)$. It is a compact totally disconnected space without isolated points, and if $\kappa > \omega$ then $D(\kappa)$ is not metrizable.

We shall show that $|CO(D(\kappa))| = \kappa$. For a finite set $K \subseteq \kappa$ and a function $f : K \to 2$ let us denote by $V(f)$ the set of all functions from $^{\kappa}2$ which extend f. The set $V(f)$ is clopen and, moreover, the family $S = \{V(f) : K \in [\kappa]^{<\omega} \ \& \ f{:}K{\to}2\}$ is a base for the topology of the space $D(\kappa)$. Any clopen set is open, and hence it is the union of some sets from the base, and also closed, hence compact, which means that it is the union of only finitely many basic sets. Therefore we have

$$\kappa = |S| \leq |\ CO\ (D(\kappa))| \leq |\ [S]^{<\omega}| = \kappa$$

or

$$|CO(D(\kappa))| = \kappa.$$

Thus we showed that there are Boolean algebras of arbitrary infinite cardinality.

<u>1.3</u>. The Boolean algebras from examples (b) – (f) have one property in common, namely that they are subalgebras of some power set algebra $\mathscr{P}(X)$ and the Boolean operations of these algebras are identical with the usual set–theoretic operations.

1.4. Definition. Subalgebras. Any nonempty subset C of a Boolean algebra B, which is closed under the Boolean operations of the algebra B, is called a subalgebra of the algebra B.

Hence the Boolean operations of the subalgebra C are restrictions of the operations of the algebra B to the set C.

It is obvious that every subalgebra of the algebra B contains **0** and **1** and that every algebra contains the trivial algebra and itself as subalgebras.

1.5. Definition. Algebras of sets. Let X be a nonempty set. A subalgebra of the power set algebra $\mathcal{P}(X)$ is called an algebra of subsets of the set X, or more briefly, an algebra of sets.

An algebra of sets is uniquely determined by a nonempty family S of subsets of a nonempty underlying set X, which contains together with any two sets Y, Z also the union $Y \cup Z$ and the complement $X - Y$.

From de Morgan's laws it follows that the condition that the system S be closed under unions can be replaced by the condition that S be closed under intersections.

The algebras of sets are special cases of Boolean algebras (there are Boolean algebras which are not algebras of sets, for instance the algebra RO ($\mathbb{R}^2$) from 1.25), but they are a representative type. We shall later show that every Boolean algebra is isomorphic to some algebra of sets.

1.6. Algebraic duality. If we look at the axioms for Boolean algebras we see that every formula from a pair of axioms arises from the other one by replacing the operation of meet by join, join by meet and by changing the role of constants, i.e. **0** goes to **1** and vice versa. We say that the formulas written in one line are dual to each other. In the case of the last axiom we get $\mathbf{1} \neq \mathbf{0}$ which is equivalent to the original axiom.

In other words, if $< B, \wedge, \vee, -, \mathbf{0}, \mathbf{1} >$ is a Boolean algebra, then also the structure $< B, \vee, \wedge, -, \mathbf{1}, \mathbf{0} >$ is a Boolean algebra.

This means that the following principle of algebraic duality holds: if φ is a formula provable from the axioms (1) – (5) then the dual formula to it is also provable from them. The practical importance consists of the fact that it is sufficient to prove only one half of the theorems: the second half we get for free from the principle of duality.

We shall also meet another and more important notion of duality, namely the so–called Stone duality, which gives a connection between Boolean algebras and compact totally disconnected topological spaces.

1.7. Lemma. For any two elements x, y of a Boolean algebra we have

(i) idempotence laws

$x \wedge x = x,$ $\qquad$ $x \vee x = x.$

(ii) $x \wedge \mathbf{0} = \mathbf{0},$ $\qquad$ $x \vee \mathbf{1} = \mathbf{1}.$

(iii) absorption laws

$x \wedge (x \vee y) = x,$ $\qquad$ $x \vee (x \wedge y) = x.$

Proof. Relations given in the same line are dual, so it is sufficient to prove the first of them.

(i) Using the axioms in the order (3), (4), (2), (4), (3), we get

$x = x\wedge\mathbf{1} = x\wedge(x\vee-x) = (x\wedge x) \vee (x\wedge-x) = (x\wedge x)\vee\mathbf{0} = x\wedge x.$

(ii) If we use the axioms (3), (4), (2), (1), and (3), (1), then we get

$x\wedge\mathbf{0} = (x\wedge\mathbf{0})\vee\mathbf{0} = (x\wedge\mathbf{0})\vee(x\wedge-x) = x\wedge(\mathbf{0}\vee-x) = x\wedge-x = \mathbf{0}.$

(iii) If we use axiom (3), relation (ii) of this lemma and two times axiom (2) then we get

$x = x\wedge\mathbf{1} = x\wedge(\mathbf{1}\vee y) = (x\wedge\mathbf{1})\vee(x\wedge y) = x\vee(x\wedge y).$

1.8. Definition. Difference and symmetric difference. The difference x–y and the symmetric difference $x \,\Delta\, y$ of two elements of a Boolean algebra are defined by the relations

$$x-y = x\wedge(-y),$$

$$x\Delta y = (x-y)\vee(y-x).$$

From axiom (3) and commutativity it is obvious that

$$-x = \mathbf{1}-x.$$

By rewriting the distributive law we get

$$(x\vee y)-z = (x-z)\vee(y-z).$$

In an algebra of sets, the Boolean (symmetric) difference is identical to the (symmetric) difference of sets. That is why we allowed ourselves to use the same notation for both operations.

<u>1.9. Lemma</u>. The following conditions are equivalent for any x,y:

(i) $x\wedge y = x$,

(ii) $x\vee y = y$,

(iii) $x-y = \mathbf{0}$.

<u>Proof</u>. (i) - (ii). By the absorption law we have $y = y\vee(x\wedge y)$. By assumption $x\wedge y = x$, and if we substitute x for $x\wedge y$, we get $y = y\vee x$, and this is (ii).

(ii) - (iii). Suppose that $x\vee y = y$, then $\mathbf{0} = y-y = (x\vee y)-y = (x-y)\vee\mathbf{0} = x-y$.

(iii) - (i). If we assume that

$x-y = \mathbf{0}$ then $x = x\wedge 1 = x\wedge(y\vee -y) = (x\wedge y)\vee(x-y) = (x\wedge y)\vee\mathbf{0} = x\wedge y$, and hence (i) holds.

<u>1.10.</u> For two sets A and B the equality $A\cap B = A$ is equivalent with the inclusion $A \subseteq B$ and this fact motivates the definition of the ordering of a Boolean algebra with similar properties as the set–theoretic inclusion.

<u>1.11. Definition. Canonical ordering</u>. Let B be a Boolean algebra. For any $x, y \in B$ let us set

$$x \leq y \;—\; x\wedge y = x.$$

The relation $\leq$ on B is called the canonical ordering of the algebra B. If it is clear what ordering we consider, we shall omit the word canonical.

In an algebra of sets the canonical ordering is identical with inclusion. In every algebra $0 < 1$.

The canonical ordering of Boolean algebras closely connects their study with the study of ordered sets. Moreover, the ordering allows us to extend the operations of meet and join to some infinite sets of elements of a Boolean algebra.

<u>1.12. Theorem</u>. Let B be a Boolean algebra. Then

(i) $\leq$ is an ordering on B,

(ii) $\mathbf{0} \leq x \leq \mathbf{1}$ for every $x \in B$,

(iii) $x \wedge y = \inf\{x,y\}$, $x \vee y = \sup\{x,y\}$,

(iv) for every $x \in B$ the system of equations $x \wedge u = \mathbf{0}$, $x \vee u = \mathbf{1}$ has exactly one solution: $u = -x$.

The properties (i) – (iii) say that $(B,\leq)$ is a lattice with a smallest and a largest element.

<u>Proof</u>. (i). By 1.9 we know that $x \leq y$ is equivalent with both $x-y = \mathbf{0}$ and $x \vee y = y$. We shall verify that the relation $\leq$ is reflexive, transitive and (weakly) antisymmetric. The reflexivity is nothing else but idempotence $x \wedge x = x$. For the transitivity let us suppose that $x \leq y$ and $y \leq z$. This means that $x \vee y = y$ and $y-z = \mathbf{0}$ and from these relations we get $\mathbf{0} = y-z = (x \vee y)-z = (x-z) \vee \mathbf{0} = x-z$ or $x \leq z$. The (weak) anti–symmetry follows immediately from the commutativity, because from $x \wedge y = x$ and $y \wedge x = y$ follows $x = y$.

(ii) The inequalities $\mathbf{0} \leq x$ and $x \leq \mathbf{1}$ are by definition only rewritings of axioms (3).

(iii) The laws of absorption $x \wedge (x \vee y) = x$ and $y \vee (x \wedge y) = y$ imply $x \wedge y \leq x$ and $x \wedge y = y$, hence the element $x \wedge y$ is a lower bound of the set $\{x,y\}$. Let z be another lower bound, it means that $z \vee x = x$ and $z \vee y = y$. Then $z \vee (x \wedge y) = (z \vee x) \wedge (z \vee y) = x \wedge y$ or $z \leq x \wedge y$. Hence we have proved that $x \wedge y = \inf\{x,y\}$. Going to the dual relations proves $x \vee y = \sup\{x,y\}$.

(iv) The axioms of complements ensure that the system has a solution $u = -x$. We want to show that there is no other solution. Let us join to both sides of the first equation $-x$, we get $-x = -x \vee (x \wedge y) = -x \vee u$ or $u \leq -x$. The second equation we meet with $-x$ and we get $-x = -x \wedge (x \vee u) = -x \wedge u$ or $-x \leq u$. Thus necessarily $u = -x$.

<u>1.13. Corollary</u>. (i) The operations of meet and join are monotone; if

$$x \leq y \text{ and } u \leq v,$$

then

$$x \wedge u \leq y \wedge v \text{ and } x \vee u \leq y \vee v.$$

(ii) $-(-x)=x$.

(iii) $x \leq y$ — $-y \leq -x$. this means that the operation of complement is a one–to–one antimonotone mapping of B onto B.

<u>Proof</u> (i) The monotonicity of the operations follows from the relation between the operations $\wedge, \vee$ and infimum and supremum in the lattice $(B, \leq)$.

(ii) The system $-x \wedge u = \mathbf{0}$, $-x \vee u = \mathbf{1}$ has a unique solution for u. The solutions are both x and $-(-x)$ as follows from commutativity and the axioms of complement, hence $x = -(-x)$.

(iii) From (ii) it follows immediately that for $x \neq y$, $-x \neq -y$, hence the complementation is a one–to–one mapping of B onto B. The relation $x \leq y$ is equivalent with $x - y = \mathbf{0}$ by (ii) and this is the same as $-y-(-x) = 0$ and this relation is equivalent with $-y \leq -x$.

<u>1.14. Infinite operations</u>. A Boolean algebra B together with the canonical ordering is a lattice $< B, \leq >$ and hence for finitely many elements of B we have

$$x_1 \wedge x_2 \wedge \ldots \wedge x_n = \inf \{x_1, \ldots, x_n\}.$$

The infimum of an infinite subset of a Boolean algebra can but need not exist. The same holds for supremum.

<u>1.15. Example.</u> Let C be the algebra of all finite subsets of the natural numbers and their complements. Then

$$A = \{ \{n\} : n \text{ is even} \}$$

is an infinite subset of the algebra C which has no supremum.

<u>1.16. Definition.</u> <u>Infinite meets and joins.</u> If u is the infimum of the set $A \subseteq B$, we write $u = \bigwedge A$ and the element u is called the <u>meet of the set A</u>. In other words, $\bigwedge$ is the operation of meet which is defined for some subsets of the algebra. Similarly, if v is the supremum of the set $A \subseteq B$ we write

$$v = \bigvee A$$

and $\bigvee$ is called the <u>operation of join</u>. It is obvious that $\bigwedge$ and $\bigvee$ are extensions of the operations $\wedge$ and $\vee$ to all finite and some infinite subsets of the algebra B. Let us realize that $\bigvee \emptyset = 0$, $\bigwedge \emptyset = 1$, $\bigvee\{x\} = \bigwedge\{x\} = x$, $\bigwedge\{x,y\} = x \wedge y$.

If $< a_i : i \in I >$ is a collection of elements of the algebra B, then

$$\bigwedge_{i \in I} a_i \qquad \text{and} \qquad \bigvee_{i \in I} a_i$$

mean

$$\bigwedge \{a_i : i \in I\} \qquad \text{and} \qquad \bigvee \{a_i : i \in I\}.$$

In the sequel we shall work with subsets and collections of elements of a Boolean algebra.

By 1.13. (iii) going to complements is an antimonotone one–to–one mapping of the lattice $(B, \leq)$ onto itself. From this and from the definition of supremum and infimum it follows immediately that $\bigwedge A$ exists if and only if $\bigvee\{-x : x \in A\}$ exists, and we have

<u>1.17. De Morgan's laws.</u> For $A \subseteq B$ we have

$$-\bigwedge A = \bigvee \{-x : x \in A\},$$

$$-\bigvee A = \bigwedge \{-x : x \in A\},$$

whenever at least one side of the equation is defined. In particular,

$$x \wedge y = -(-x \vee -y), \qquad x \vee y = -(-x \wedge -y).$$

We see that the operations appearing in the definition of the Boolean algebra are redundant.

The meet (join) can be defined by means of join (meet) and the operation of complement, and this is also true for infinite operations.

1.18. Associative laws for meet and join were not among the axioms (1) – (5). They can be derived immediately from the properties of supremum and infimum, also for infinite operations.

Let $< J_k : k \in K >$ be a family of sets and

$$I = \bigcup_{k \in K} J_k.$$

Then

$$\bigwedge_{i \in I} a_i = \bigwedge \{ \bigwedge_{i \in J_k} a_i : k \in K \},$$

$$\bigvee_{i \in I} a_i = \bigvee \{ \bigvee_{i \in J_k} a_i : k \in K \},$$

with the following interpretation: if the expressions on the right hand side make sense then also the expression on the left hand side makes sense and both expressions are equal. In particular,

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z, \qquad x \vee (y \vee z) = (x \vee y) \vee z,$$

and hence the expressions

$$x_1 \wedge x_2 \wedge \ldots \wedge x_n, \qquad x_1 \vee x_2 \vee \ldots \vee x_n$$

have a unique meaning.

More interesting and important are relations in which both operations of meet and join occur.

1.19. Infinite distributive laws. If

$$\bigvee_{i \in I} a_i \quad \text{and} \quad \bigvee_{j \in J} b_j$$

or

$$\bigwedge_{i \in I} a_i \quad \text{and} \quad \bigwedge_{j \in J} b_j,$$ respectively, exist.

then for arbitrary $c \in B$ the following hold:

(i) $$c \wedge \bigvee a_i = \bigvee \{c \wedge a_i : i \in I\},$$

(ii) $$c \vee \bigwedge a_i = \bigwedge \{c \vee a_i : i \in I\},$$

(iii) $$\bigvee_{i \in I} a_i \wedge \bigvee_{j \in J} b_j = \bigvee \{a_i \wedge b_j : \langle i,j \rangle \in I \times J\},$$

(iv) $$\bigwedge_{i \in I} a_i \vee \bigwedge_{j \in J} b_j = \bigwedge \{a_i \vee b_j : \langle i,j \rangle \in I \times J\}.$$

Proof. We shall prove the relations (i) and (iii), the remaining ones are dual.

(i) Let us set $a = \bigvee a_i$. Because $a \geq a_i$, we also have

$$c \wedge a \geq c \wedge a_i,$$

and so $c \wedge a$ is an upper bound of the set $\{c \wedge a_i : i \in I\}$. To prove equality it is sufficient to show that for every upper bound d we have $d \geq c \wedge a$. So, let $d \geq c \wedge a_i$ for every $i \in I$. Then

$$a_i = (c \wedge a_i) \vee (-c \wedge a_i) \leq d \vee -c$$

for every $i \in I$ and from this it follows that

$$a \leq d \vee -c.$$

If we meet both sides with the element c, we get the required inequality

$$c \wedge a \leq c \wedge d \leq d.$$

(iii) Let us set $a = \bigvee a_i$, $b = \bigvee b_j$. We use the already established relation (i) from which it follows that

$$a \wedge b = \bigvee \{a \wedge b_j : j \in J\},$$

and also

$$a \wedge b_j = b_j \wedge a = \bigvee \{b_j \wedge a_i : i \in I\}$$

for every $j \in J$. Substituting and using the associative law 1.18 we get

$$a \wedge b = \bigvee \{\bigvee_{i \in I} a_i \wedge b_j : j \in J\} = \bigvee \{a_i \wedge b_j : \langle i,j \rangle \in I \times J\}.$$

The infinite operations allow us to classify Boolean algebras by completeness.

1.20. Definition. Complete Boolean algebras. Let κ be an infinite cardinal.

(i) A Boolean algebra B is called κ–complete if for every set $X \subseteq B$ of cardinality less than κ the meet $\bigwedge X$ exists.

(ii) An algebra B is called a complete Boolean algebra if there exists a meet for every subset of the algebra B.

(iii) We say that a subalgebra C of a complete Boolean algebra B is a complete (κ–complete) subalgebra if for every set $X \subseteq C$ ($|X| < \kappa$) the meet $\bigwedge_B X \in C$ exists.

From de Morgan's laws it follows that in a κ–complete algebra both operations $\bigwedge$ and $\bigvee$ are defined for all subsets of cardinality less than κ. In a complete Boolean algebra B the domain of the operations $\bigwedge$ and $\bigvee$ is the whole power set of the set B. Every algebra is ω–complete and hence finite algebras are complete.

In the algebras of sets the Boolean operations coincide with the usual set–theoretic ones. For algebras of sets we define a stronger notion of κ–completeness in such a way that also the infinite Boolean operations coincide with the usual set–theoretic ones.

1.21. Definition. κ–algebras of sets. We say that an algebra of sets A is a κ–algebra if for every family $S \in [A]^{<\kappa}$, $\bigcap S \in A$.

This means that the meet of the family S exists and is equal to the intersection $\bigcap S$.

A κ–algebra is more than an algebra of sets which is a κ–complete Boolean algebra. Later we shall see that there are algebras of sets which are not closed under intersections of countable subfamilies, so they are not ω_1–algebras of sets, but yet they are ω_1–complete Boolean algebras.

1.22. Mathematically most interesting are ω_1–algebras of sets and ω_1– complete Boolean algebras which have a direct relationship to measure theory and to probability theory. They are known under the older names of σ–algebras of sets and σ–complete Boolean algebras. For set theory complete Boolean algebras have great importance.

1.23. Boolean algebras of regular open sets. Let us recall first some basic topological notions and facts. Let X be a topological space. If $A \subseteq X$, then cl(A) will denote the closure and int(A) will denote the interior of the set A in the space X. Recall that cl (A) is the smallest closed superset and int(A) is the largest open subset of the set A. For $A \subseteq X$ we set r(A) = int cl(A) and r(A) will be called the regularization of the set A.

A set $A \subseteq X$ is said to be nowhere dense in X if $r(A) = \emptyset$. If cl(A) = X then also r(A) = X, and we say that A is dense in X.

A set A is dense if and only if it has a nonempty intersection with every nonempty open set. A set A is nowhere dense if and only if for every nonempty open set U there exists a nonempty open $V \subseteq U$ disjoint with A.

It is obvious that $r(\emptyset) = \emptyset$, r(X) = X and for every open set A, $A \cup \operatorname{int}(X-A)$ is a dense subset in X.

1.24. Definition. (Kuratowski). We say that A is a regular open set if r(A) = A.

We shall denote by RO(X) the family of all regular open sets of the space X.

1.25 Loosely speaking, an open set is regular if it does not have any cracks. The plane $\mathbb{R}^2$ without the x–axis is an open set in $\mathbb{R}^2$ but it is not regular. Its closure and regularization is the whole space. On the other hand both open halfplanes are regular open sets. The union of two regular open sets need not be a regular open set. This means that RO(X) need not be an algebra of sets.

A set is regular open if and only if it is the interior of some closed set. In particular, for an open set A we have $r(X-A) = int(X-A)$. If $A \subseteq B \subseteq X$ then from the monotonicity of the operations of closure and interior we get $r(A) \subseteq r(B)$. This means that regularization is also monotone. It is easily seen that for an open set A, $A \subseteq r(A)$, and that for every set A, $r(r(A)) = r(A)$, and hence $r(A) \in RO(X)$.

1.26. We shall show that the regular open sets with suitably chosen operations form a complete Boolean algebra which we shall call the Boolean algebra of regular open sets.

First we shall verify that for open sets A and B we have

(6) $$r(A \cap B) = r(A) \cap r(B).$$

The intersection $A \cap B$ is a subset of both A and B, and from the monotonicity of regularity it follows that in (6) the left hand side is a subset of the right hand side.

Let us realize that for an open set P and any subset Q of a topological space

(7) $$P \cap cl(Q) \subseteq cl(P \cap Q).$$

Therefore in our case we have

$$A \cap cl(B) \subseteq cl(A \cap B),$$

and hence

$$A \cap r(B) \subseteq r(A \cap B) \subseteq cl(A \cap B).$$

$r(B)$ is an open set and by repeating the argument we get $cl(A) \cap r(B) \subseteq cl(A \cap r(B)) \subseteq cl(A \cap B)$ from which it follows that

$$r(A) \cap r(B) \subseteq r(A \cap B).$$

From the just proved equality (6) it follows that the intersection of a finite number of regular open sets is again a regular open set. On the other hand the intersection of an infinite family of regular open sets need not be a regular open set. Every open interval on the real line is a regular open set, but the intersection of all open intervals containing zero does not belong to $RO(\mathbb{R})$.

<u>1.27. Theorem</u>. The family of regular open sets $RO(X)$ of a nonempty topological space X with operations

$$A\wedge B = A\cap B, \qquad A\vee B = r(A\cup B),$$
$$-A = \mathrm{int}(X-A)$$

and constants $\mathbf{0} = \emptyset$, $\mathbf{1} = X$ forms a complete Boolean algebra. Moreover, if $S \subseteq RO(X)$ then

$$\bigwedge S = r(\bigcap S), \qquad \bigvee S = r(\bigcup S).$$

<u>Proof</u>. We know that the chosen operations when applied to regular open sets give again regular open sets. The commutativity of meet and join follows from the commutativity of the usual set–theoretic operations. It is obvious that $\mathbf{0} \neq \mathbf{1}$, and that both $\mathbf{0}$ and $\mathbf{1}$ are neutral. If $A\in RO(X)$ then

$$A\wedge -A = A \cap \mathrm{int}(X-A) = \emptyset$$

and

$$A\vee -A = r(A \cup \mathrm{int}(X-A) = X,$$

because on the right hand side we have the regularization of a dense set.

We shall verify the distributive laws. Let $A, B, C, \in RO(X)$. For the usual set–theoretic operations we have

$$A\cap(B\cup C) = (A\cap B) \cup (A\cap C).$$

We apply the regularization to both sides of the equality and we use (6). For the left hand side we get

$$r(A\cap(B\cup C)) = r(A) \cap r(B\cup C) = A\wedge(B\vee C)$$

and for the right hand side

$$r((A\cap B) \cup (A\cap C)) = (A\wedge B) \vee (A\wedge C),$$

hence

$$A\wedge(B\vee C) = (A\wedge B) \vee (A\wedge C).$$

The second distributive law can be verified in the same way.

It remains to verify completeness. We know that meet is identical with intersection, this means that the canonical ordering is set–theoretic inclusion. Let $S \subseteq RO(X)$ and set

$A = r(\bigcap S)$. The set A is regular open and for every $B \in S$ we have $A \subseteq r(B) = B$. A is hence a lower bound of the family S. Let $C \in RO(X)$ be any lower bound of the family S. Then $C \subseteq \bigcap S$ and from the monotonicity of regularization follows $C \subseteq A$. Hence A is the greatest lower bound of the set S, which means that $A = \bigwedge S$. Similarly one can show that $\bigvee S = r(\bigcup S)$. Hence RO(X) is a complete Boolean algebra.

Let us notice that for every topological space we have $CO(X) \subseteq RO(X)$ and that, moreover, the algebra of clopen sets is a subalgebra of the algebra RO(X).

1.28. Definition. A topological space X is said to be extremally disconnected if $CO(X) = RO(X)$.

A space is extremally disconnected if and only if the closure of any open set is open, hence a clopen set. In extremally disconnected spaces the Boolean algebra of regular open sets is an algebra of sets.

In § 2 we shall see that every Boolean algebra is isomorphic to an algebra of regular open sets of some topological space. The space $\beta\omega$ of all ultrafilters on the natural numbers is the simplest example of an infinite compact extremally disconnected Hausdorff space.

1.29. Examples. In the following examples we assume that X is a nonempty topological space.

(a) Borel sets. The smallest σ–algebra of subsets of the space X containing all open sets is called the σ–algebra of Borel sets and is denoted by Borel (X). Its elements are called Borel sets.

This means that we can obtain Borel sets from the open sets using complementation, countable unions and countable intersections repeatedly.

All closed sets are Borel and Borel(X) is also the smallest σ–algebra containing the closed sets.

<u>Hierarchy of Borel sets.</u> (Lebesgue, 1905). We define subsets Σ_α and Π_α of the power set $\mathscr{P}(X)$ of the space X:

Σ_0 is the set of all open sets.

Π_0 is the set of all closed sets, hence complements of sets from Σ_0.

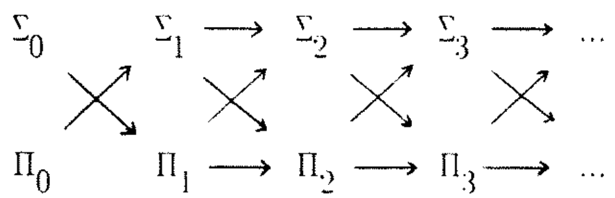
for $\alpha>0$

Σ_α is the set of all unions of at most countable subfamilies of the set $\bigcup_{\beta<\alpha} \Pi_\beta$.

Π_α is the set of all intersections of at most countable subfamilies of the set $\bigcup_{\beta<\alpha} \Sigma_\beta$.

The elements of Σ_α and Π_α are called Σ_α–sets and Π_α–sets, respectively. Instead of Σ_1 and Π_1 also the notation F_σ and G_δ, respectively, is used. Hence a G_δ–set is an intersection of countably many open sets.

The basic inclusions among the Borel classes are represented by the arrows in the following diagram:



If X is a metric space then also $\Pi_0 \subseteq \Pi_1$ and $\Sigma_0 \subseteq \Sigma_1$ hold, because every closed set A can be obtained as the intersection of open $\frac{1}{n}$ – neighbourhoods of the set A.

The first uncountable ordinal ω_1 is a regular cardinal. Therefore for any countable family

$$S \subseteq \bigcup_{\alpha<\omega_1} \Pi_\alpha$$

there exists a $\beta<\omega_1$ such that $S \subseteq \Pi_\beta$. This implies that

$$\text{Borel}(X) = \bigcup_{\beta<\omega_1} \Pi_\beta = \bigcup_{\beta<\omega_1} \Sigma_\beta.$$

The σ–algebra of Borel sets serves as a natural domain for σ–additive measures on a topological space. Such measures are called Borel measures.

A trivial example of a Borel measure is the two–valued measure determined by a point. from Example II.8.22. An important example is the Lebesgue measure on $\mathbb{R}$.

(b) Meager sets, sets with Baire property. Every subset of a nowhere dense set and also the union of two nowhere dense sets are nowhere dense sets. The nowhere dense sets form an ideal on X because the whole space X is not a nowhere dense set in X.

We say that $A \subseteq X$ is a meager set (an older term is set of first category) if it is a union of at most countably many nowhere dense sets.

It is clear that every nowhere dense set is meager and that the union of countably many meager sets is again a meager set. If the whole space is not a union of countably many meager sets then the meager sets form a σ–complete ideal on X.

We say that $A \subseteq X$ has the Baire property if there exists an open set G such that both sets $A-G$ and $G-A$ are meager. Hence a set has the Baire property if it differs from an open set only by a meager set.

We shall show that the sets having the Baire property form a σ–algebra of subsets of the space X. We shall denote it by Baire (X). First we prove that the complement $X-A$ has the Baire property whenever $A \in$ Baire (X). Let G be an open set which differs from A by a meager set. Let us set $G_0 = \text{int}\,(X-G) = X-\text{cl}(G)$. Because

$$(X-A) - G_0 \subseteq \text{cl}(G) - A \subseteq (G-A) \cup (\text{cl}(G)-G).$$

$$G_0 - (X-A) \subseteq A - \text{cl}(G) \subseteq A-G$$

and $\text{cl}(G)-G$ is a nowhere dense set. $X-A$ differs from G_0 by a meager set. Hence $X-A \in$ Baire (X).

It remains to show that the sets having the Baire property are closed under unions of countable families. Suppose that for every natural number n, A_n differs from an open set G_n by a meager set. Let us set $A = \bigcup A_n$, $G = \bigcup G_n$. Because

$$A-G \subseteq \bigcup (A_n - G_n),$$

$$G-A \subseteq \bigcup (G_n - A_n),$$

and on the right hand sides we have countable unions of meager sets and hence also meager sets, the set A has the Baire property.

With this we have proved that the family Baire (X) is a σ–algebra of sets. Moreover, Baire (X) is the smallest σ–algebra of subsets of X containing all open and meager subsets of X. This implies that

(8) Borel (X) $\subseteq$ Baire (X).

(c) Baire spaces. Baire Category Theorem. A space X is said to be Baire if no nonempty open set $A \subseteq X$ is meager.

An equivalent condition: the intersection of countably many open dense sets of the space X is dense in X.

Baire Category Theorem. Complete metric spaces and compact topological spaces are Baire.

The Baire Category Theorem belongs to the treasures of mathematics and the reader can find its proof in every textbook on functional analysis or topology. Let us remark that a proof of the Baire Category Theorem for separable complete metric spaces does not need any form of the axiom of choice.

(d) Ultrafilters on ω and Borel sets. By identifying the subsets of the natural numbers with their characteristic functions, we identify $\mathcal{P}(\omega)$ and ${}^{\omega}2$. In this way the topology of the Cantor discontinuum D is carried over to the set $\mathcal{P}(\omega)$. Every ultrafilter on ω is a subset of the space D and we ask when it has the Baire property.

If $\mathcal{U}$ is a trivial ultrafilter over $n \in \omega$ then $\mathcal{U} = \{f \in {}^{\omega}2 : f(n) = 1\}$ is a clopen set of the base S from Example 1.2(f). We see that the smallest σ–algebra of subsets of the space D that contains all trivial ultrafilters contains all open sets and is therefore equal to the algebra of Borel sets.

We shall use the Baire Category Theorem to prove that no nontrivial ultrafilter on ω has the Baire property. From this it follows by (8) that it is not a Borel set either. Consider any clopen set V(f) from the base S given by the mapping f of a finite set $K \subseteq \omega$ into $\{0,1\}$. Define a mapping φ_f: D → D so that if $h \notin V(f)$ then $\varphi_f(h) = h$, and if $h \in V(f)$ then $\varphi_f(h)(n) = 1 - h(n)$ for $n \in \omega - K$ and $\varphi_f(h)(n) = h(n)$ for $n \in K$.

It is obvious that φ_f is a homeomorphism of the space D, and because $\mathcal{U}$ is uniform, for the dual ideal $\mathcal{U}^*$ we have

$$\varphi_f[V(f) \cap \mathcal{U}^*] = V(f) \cap \mathcal{U}.$$

A homeomorphism maps nowhere dense sets onto nowhere dense sets, and hence also meager sets onto meager sets. By the Baire Category Theorem V(f) is not meager, and because

$$V(f) = (V(f) \cap \mathcal{U}) \cup (V(f) \cap \mathcal{U}^*),$$

neither the set $V(f) \cap \mathcal{U}$, nor the set $V(f) \cap \mathcal{U}^*$ is meager.

Let us suppose that an ultrafilter $\mathcal{U}$ has the Baire property. Let G be an open set which differs from $\mathcal{U}$ by a meager set. We know that $\mathcal{U} = V(\emptyset) \cap \mathcal{U}$ is not a meager set, hence G is nonempty. Let us take any basic set $V(f) \subseteq G$. Because $G - \mathcal{U} \supseteq V(f) - \mathcal{U} = V(f) \cap \mathcal{U}^*$, we get that $V(f) \cap \mathcal{U}^*$ is a meager set and this is a contradiction. We have proved that no uniform ultrafilter on ω is a set having the Baire property in the Cantor discontinuum.

§ 2 Structural properties of Boolean algebras

First we shall get acquainted with special types of subsets of Boolean algebras, such as sets of disjoint elements, partitions of unity, dense sets, sets of atoms and sets of generators. We shall show that any ordered set uniquely determines a complete Boolean algebra. We shall deal with a more general notion of distributivity of Boolean algebras, which is connected with refinings of partitions. Collapsing algebras are examples of complete algebras which have only a countable set of complete generators. Moreover, collapsing algebras are universal in the sense that every Boolean algebra with a dense set of size at most κ can be completely embedded into the algebra $C(\omega,\kappa)$. We shall get acquainted with constructions of new algebras using products, free products and factorization. We shall conclude with the introduction of the notions of ideals and ultrafilters on Boolean algebras and we shall recall the classical results of M.H. Stone about the representation of Boolean algebras by algebras of sets and about the Stone duality between Boolean algebras and Boolean topological spaces.

In the sequel the letters A, B and C will always denote Boolean algebras.

2.1. Factors. For a nonzero element $b \in B$ we shall denote by $B \restriction b$ the set

$$\{x \in B : x \leq b\}$$

of all elements of the algebra B which are smaller than or equal to b. The set $B \restriction b$ together with the restrictions of the operations $\wedge$, $\vee$ and the operation of complement $-_b$ defined by $-_b x = b - x$ and constants $\mathbf{0}$ and b is also a Boolean algebra. It is called a factor or a restriction of the algebra B and is denoted, like its universe, by $B \restriction b$.

2.2. Disjointness.

(i) We say that two elements $x, y \in B$ are disjoint, denoted by $x \perp y$, if $x \wedge y = \mathbf{0}$. If the elements x and y are not disjoint, then they are compatible.

(ii) We say that $X \subseteq B$ is a set of disjoint elements if all elements of X are nonzero and pairwise disjoint; such a set is also called an antichain.

(iii) A set P of disjoint elements is a partition of an element $b \in B$ provided $b = \bigvee P$.

By Lemma 1.9 we have

(1) $x \nleq y$ — $x - y \neq \mathbf{0}$

This implies an important relation (2) between the canonical ordering and disjointness:

(2) $x \nleq y$ iff there is a nonzero element $z \leq x$ disjoint with y.

The family of all sets of disjoint elements of an algebra B is ordered by inclusion and satisfies the assumptions of the Principle of Maximality. It follows that every set of disjoint elements can be extended to a maximal one. Moreover we have

2.3. Lemma. A set $P \subseteq B$ is a maximal set of disjoint elements iff P is a partition of the unit element 1_B.

Proof. Let P be a maximal set of disjoint elements. We shall show that $\sup P = \bigvee P = 1$. Let $u \in B$ be an arbitrary upper bound of the set P. If $u \neq 1$, then the complement $-u$ is nonzero and is disjoint with all elements of P, but this contradicts the maximality of the set P. So the unit element is the only upper bound of the set P and $\bigvee P = 1$.

Conversely, suppose that $\bigvee P = 1$. It is sufficient to show that every nonzero u is compatible with some element of P. Let $u \neq \mathbf{0}$. Then $-u \neq 1$, and therefore the element $-u$ is not an upper bound of the set P. Hence for some $x \in P$ we have $x \nleq -u$ and by (1) we get $x \wedge u \neq \mathbf{0}$ and u is compatible with x.

In accordance with a similar definition for ordered sets (III.1.22), the Suslin number $c(B)$ of a Boolean algebra B is the supremum of the set of cardinalities of sets of disjoint elements of the algebra B; the Suslin number of B is also called the cellularity of B.

2.4. Definition. The saturatedness of an algebra B is the smallest cardinal κ such that in the algebra B there is no set of disjoint elements of size κ. Hence sat (B) = min { κ : for every set X $\subseteq$ B of disjoint elements we have $|X| < \kappa$}.

Obviously c (B) $\leq$ sat (B). There can be two possibilities. Either there is a set of disjoint elements of size the Suslin number c(B), then sat (B) = $(c(B))^+$, or no set of disjoint elements has size c(B), then sat (B) = c(B). For infinite algebras we have the following statement which we shall present without proof.

2.5. Theorem. (Erdös, Tarski 1943). The saturatedness of every infinite Boolean algebra is an uncountable regular cardinal.

From this theorem it follows that the equality sat (B) = c(B) can occur only if sat(B) is a weakly inaccessible cardinal. In Example 2.25 (h) we shall show that for every weakly inaccessible cardinal κ there is a Boolean algebra B such that sat(B) = κ.

2.6. Definition. Atoms.

(i) An element a$\in$B is said to be an atom if a $\neq$ **0** and there is no b $\in$ B with **0** < b < a. The set of all atoms of an algebra B is denoted by At(B)

(ii) If At(B) is empty then we say that B is atomless.

(iii) An algebra B is said to be atomic if below every nonzero element there is an atom.

It is obvious that At(B) is a set of disjoint elements. Let us realize that an algebra which is not atomic need not be atomless. A Boolean algebra B is atomic if and only if the set At(B) is a partition of the unit element. One can easily prove:

<u>2.7. Lemma.</u> For every $a \in B$, the following conditions are equivalent:

(i) a is an atom.

(ii) $B|a$ is a two–element algebra

(iii) for every $b \in B$, $a \leq b$ or $a \wedge b = 0$.

<u>2.8.</u> Finite Boolean algebras are atomic. For suppose that B is not atomic, then there exists a nonzero $b \in B$ below which there is no atom. We can choose an infinite decreasing sequence $b = b_0 > b_1 > \ldots$ of elements smaller than b. From this it follows that a non–atomic algebra is always infinite and that every finite algebra is atomic.

Suppose that B is an atomic algebra. Since for distinct $x, y \in B$ the symmetric difference $x \Delta y$ is nonzero, every atom which lies below it must be below exactly one of the elements x and y and must be disjoint with the other one. This means that the mapping f which assigns to every element $b \in B$ the set

(3) $f(b) = \{a \in At(B) : a \leq b\}$.

is a one–to–one mapping of the atomic Boolean algebra B into the power set $\mathcal{P}(At(B))$. Moreover, for every $b \in B$ we have $b = \bigvee f(b)$.

<u>2.9. Theorem</u>. If B is an infinite Boolean algebra then, in B, there are

(i) an infinite decreasing sequence, and

(ii) an infinite set of disjoint elements, and hence $sat(B) \geq \omega_1$.

<u>Proof</u>. Suppose first that B is atomic. Then $At(B)$ is an infinite set of disjoint elements, hence we have (ii). If we choose a one–to–one sequence of atoms $< a_n : n < \omega >$, it is sufficient to set, for every natural number i, $b_i = -\bigvee \{a_n : n \leq i\}$ and we get an infinite decreasing sequence $< b_i : i < \omega >$. Hence we have (i).

Now suppose that B is not atomic. From 2.8. we know that then there exists an infinite decreasing sequence $< b_n : n < \omega >$. If we set $a_n = b_n - b_{n+1}$, then $\{a_n : n < \omega\}$ is an infinite set of disjoint elements.

2.10. Dense sets. A set $H \subseteq B$ of nonzero elements is said to be dense in B if for every nonzero $b \in B$ there is an $x \in H$ such that $x \leq b$.

It is obvious that for every Boolen algebra B the set $B - \{\mathbf{0}\}$ is dense in B. Every dense set must contain all atoms. An algebra B is atomic if and only if the set of its atoms is dense in B.

2.11. Lemma. If H is a dense set in B then for any $b \in B$

(i) $\qquad b = \bigvee \{ x \in H : x \leq b\}$,

(ii) $\qquad$ there is a partition of the element b that consists only of elements of the set H.

Proof. Given $b \in B$ let us set $X = \{x \in H : x \leq b\}$. If $b = \mathbf{0}$ then X is an empty set. hence sup X is the smallest element $\mathbf{0}$ in B. Let $b \neq \mathbf{0}$. Then $X \neq \emptyset$ and it is obvious that b is an upper bound of the set X. We need to verify that for any upper bound c of the set X we have $b \leq c$. If not then $b \nleq c$ and by (2) there is a $u \in H$ such that $u \leq b$. hence $u \in X$. and also u is disjoint with c, a contradiction. In the same way we can prove that every maximal set of disjoint elements of X is a partition of the element b.

The above theorem indicates that every dense set together with the canonical ordering reflects the structure of the whole algebra.

An important property of the ordering of a dense set is separativity. If H is a dense subset of an algebra B. then from (2) it follows that for any $x, y \in H$ we have

(4) $\qquad x \nleq y \quad - \quad (\exists z \in H)\,(z \leq x \;\&\; z \perp y)$.

where $z \perp y$ means disjointness in the ordering $< H, \leq >$. Let us realize that (4) is stated only in terms of the ordering of the set H.

2.12. Definition. Separative orderings. An ordering $\leq$ on a nonempty set H is said to be separative if for every $x, y \in H$ the relation (4) holds.

We know that the canonical ordering is separative on every dense subset of a Boolean algebra. We shall show in 2.17 that every separatively ordered set is a dense set in some uniquely determined complete Boolean algebra. First let us recall the important notions of embeddings and isomorphisms.

2.13. Definition. Complete embeddings, isomorphisms.

(i) A mapping $f : A \to B$ is said to be an embedding of an algebra A into an algebra B if f is a one–to–one mapping and if it preserves finite Boolean operations: for any $x, y \in A$ we have $f(x \wedge y) = f(x) \wedge f(y)$ and $f(-x) = -f(x)$.

(ii) If, moreover,

$$f(\bigwedge X) = \bigwedge f[X]$$

for every set $X \subseteq A$ which has an infimum $\bigwedge X$ in A, then we say that f is a complete embedding of A into B.

(iii) If $f: A \to B$ is an embedding and $f[A] = B$, then we say that f is an isomorphism of A onto B. Two algebras A and B are isomorphic, denoted by $A \cong B$, if there is an isomorphism of A onto B.

It is obvious that an embedding preserves the canonical ordering and the smallest and the largest elements. An isomorphism of an algebra A onto B is also a complete embedding. If two algebras A and B are isomorphic and if one of them is complete or κ–complete, then so is the other one.

2.14. Atomic algebras and algebras of sets. If B is an atomic Boolean algebra, one can easily verify that the mapping (3) from 2.8 is a complete embedding of the algebra B into the power set algebra $\mathcal{P}(At(B))$. Hence every atomic algebra is isomorphic with some algebra of sets. If, moreover, B is finite, then B is isomorphic with the power set algebra $\mathcal{P}(n)$ for some natural number $n > 0$. The cardinalities of finite algebras are exactly all powers 2^n for $n > 0$.

We shall show that a complete Boolean algebra is uniquely determined by any dense subset and its canonical ordering.

<u>2.15.Theorem.</u> If B_1 and B_2 are complete Boolean algebras such that some dense subset $H_1 \subseteq B_1$ is isomorphic to some dense subset $H_2 \subseteq B_2$ with respect to the canonical orderings, then the algebras B_1 and B_2 are isomorphic.

<u>Proof.</u> Suppose that $j: H_1 \to H_2$ is an isomorphism of dense subsets with respect to the canonical orderings $\leq_1$ and $\leq_2$. We shall show that there is in fact a unique isomorphism J of the algebra B_1 onto B_2 which extends the mapping j. For arbitrary $x \in B_1$ let us define

(5) $\quad J(x) = \bigvee_2 \{j(y) : y \in H_1 \ \& \ y \leq_1 x\}.$

The completeness of the algebras implies that J is a mapping from B_1 onto B_2. First we verify that J extends the mapping j. For an arbitrary $x \in H_1$ we have

$J(x) = \bigvee_2 \{z \in H_2 : z \leq_2 j(x)\}.$

By 2.11 (i) the join is equal to $j(x)$, hence $J(x) = j(x)$. We shall show that J is a mapping onto the whole algebra B_2. We shall use the fact that the isomorphism j preserves disjointness. Given an arbitrary $z \in B_2$ let us set $x = \bigvee_1 \{y \in H_1 : j(y) \leq_2 z\}$. Obviously $z \leq_2 J(x)$. If we had $J(x) \nleq_2 z$ then we could find a $z_0 \in H_2$ with $z_0 \leq_2 J(x)$ and $z_0 \wedge_2 z = 0$. If we set $x_0 = j^{-1}(z_0)$ then $x_0 \in H_1$ and x_0 is disjoint with x. From (5) it follows that $J(x)$ is disjoint with $j(x_0) = z_0$, and this is a contradiction, because $\mathbf{0} \neq z_0 \leq_2 J(x)$. We verified that $J(x) = z$. In a similar way we can show that J is a one–to–one mapping. Next, from (5) it follows that the mapping J preserves the canonical orderings. This means that J is an isomorphism of the algebras B_1 and B_2. Moreover, every isomorphism $J : B_1 \to B_2$ which extends the mapping j, must satisfy (5). From this the uniqueness of the isomorphism J follows.

<u>2.14. Ordered sets and complete Boolean algebras.</u> We shall show that every nonempty ordered set completely determines a complete Boolean algebra. This leads to an easier way of describing various Boolean algebras. It is sufficient to construct a suitable ordered set which determines then the whole structure of the corresponding Boolean algebra. Moreover, if the initial set is seperatively ordered, then it is isomorphic with some dense subset of the corresponding Boolean algebra.

Suppose that $< Q, \leq >$ is a nonempty ordered set. We are interested in lower subsets of the set Q, i.e. sets X satisfying : if $y \leq x \in X$ then $y \in X$. It is obvious that the union and intersection of any family of lower subsets is again a lower subset. This means that the family of all lower subsets is a topology on the set Q. We call it the <u>topology of lower subsets.</u>

The set of regular open sets of the space Q with the topology of lower subsets forms, by Theorem 1.27, a complete Boolean algebra B = RO (Q). We say that the complete Boolean algebra B is determined by the ordered set $< Q, \leq >$.

Let us realize that a set X in a topological space is regular open if and only if it is open and moreover satisfies: if some neighbourhood of a point p is contained in the closure cl(X) of the set X, then $p \in X$. For every point p of the space Q with the topology of lower subsets the set $(-, p]$ is the smallest neighbourhood of the point p . From this it follows that $X \subseteq Q$ is a regular open set in Q if and only if X is a lower subset and

$$(\forall p \in Q)\ [p \in X \leftrightarrow (\forall q \leq p)\ (X \cap (-, q] \neq \emptyset)].$$

The following theorem characterizes the complete Boolean algebra determined by an ordered set.

<u>2.17. Theorem.</u> Let Q be a nonempty ordered set. Then there is a complete Boolean algebra B and a mapping $j: Q \to B$ such that

(i) $j[Q]$ is a dense set in B.

(ii) j preserves the ordering, if $p \leq q$ in Q then $j(p) \leq j(q)$ in B.

(iii) j preserves disjointness, if $p \perp q$ in Q then $j(p) \perp j(q)$ in B.

The conditions (i) – (iii) determine the complete algebra B uniquely up to isomorphism. Moreover, if Q is a separative ordered set then the mapping j is one–to–one and then it is an isomorphic embedding of Q onto a dense subset of B.

<u>Proof</u>. First we prove the theorem for the case that Q is a separative ordering. Consider the complete Boolean algebra $B = RO(Q)$ determined by the ordered set Q. We shall prove that the algebra satisfies the conditions of the theorem. It is easily seen that the separativity of the ordering Q is equivalent with the fact that for every $p \in Q$, $(\leftarrow,p]$ is a regular open set. From this it follows that if we set $j(p) = (\leftarrow, p]$ for every $p \in Q$, then j is a one–to–one mapping of the set Q into B. Each $j(p)$ is a nonempty set, and hence a nonzero element of the algebra B. If $X \in B$ and $X \neq \emptyset$, then for every $p \in X$ we have $j(p) \subseteq X$. This means that $j[Q]$ is a dense set in B. It is obvious that the mapping j is an isomorphic embedding of Q into B, therefore (ii) and (iii) are satisfied. Next we show the uniqueness of the algebra B. Consider an arbitrary complete Boolean algebra C and a mapping $k : Q \to C$ such that it satisfies the conditions (i) – (iii). We shall show that k is a one–to–one mapping. Take two distinct elements $p, q \in Q$. We may suppose that $p \nleq q$. From the separativity we know that there is an $r \leq p$ disjoint with q. Then $k(r) \leq k(p)$ and $k(r)$ is disjoint with $k(q)$ in the algebra C. It follows that $k(p) \nleq k(q)$. Hence k is even an isomorphic embedding of Q onto a dense subset of the algebra C. Because j is an isomorphic embedding of Q onto a dense set of the algebra B and k is an isomorphic embedding of Q onto a dense set of the algebra C, both algebras have isomorphic dense sets and by Theorem 2.15 they are isomorphic. There is even an isomorphism $f : B \to C$ such that the composed mapping fj is equal to k.

In the case that Q is not a separative ordered set, we proceed in a similar way. Instead of the set $(-,p]$ in the definition of the mapping j we have to take its regularization (1.23).

2.18. Definition . Completions. A complete Boolean algebra B is said to be the completion of the algebra A, denoted by B = cm (A), if A is a subalgebra of B and the set $A - \{0\}$ is dense in B.

If B is the completion of an algebra A, one can easily prove that the identity $i : A - B$ is a complete embedding of A into B.

The following statement is a consequence of Theorem 2.17.

2.19. Theorem. Every Boolean algebra A has a completion cm (A) and this completion is (up to isomorphism) uniquely determined.

If A is a complete Boolean algebra then it is obvious that cm (A) = A. Let us remark that for any Boolean algebra A, the construction of Mac Neille from I.5.24 also gives the completion cm (A).

2.20. Generators. The intersection of an arbitrary nonempty family of subagebras of a Boolean algebra B is again a subalgebra of the algebra B. From this it follows that for any set $X \subseteq B$

$$A = \bigcap \{C : X \subseteq C \text{ and } C \text{ is a subalgebra of } B\}$$

is the smallest subalgebra which contains all elements of X. We say that the algebra A is generated by the set X or that X is a set of generators of the algebra A.

Similarly, if B is a complete Boolean algebra, we can restrict ourselves to its complete subalgebras. For an arbitrary set $X \subseteq B$,

$$A = \bigcap \{C : X \subseteq C \text{ and } C \text{ is a complete subalgebra of } B\}$$

is the smallest complete subalgebra containing X. We say that A is completely generated by the set X or that X is a set of complete generators of the complete algebra A.

By 2.8, every dense subset $H \subseteq B$ is a set of complete generators of the complete algebra B. If we consider the subalgebra $C \subseteq B$, which is merely generated by the set H, then C need not be complete. We do have, however, cm (C) = B. By proving that any ordered set Q uniquely determines a complete Boolean algebra B, we also proved that Q uniquely determines a (not necessarily complete) Boolean algebra C generated by the set j[Q] (see 2.17).

2.21. Cardinalities of algebras and numbers of generators. If we know that a set $X \subseteq B$ generates an algebra B then we can estimate the cardinality of the algebra B from the size of the set X. If X is infinite then by III.2.37 we have $|B| = |X|$.

A subalgebra $A \subseteq B$ generated by a set $X \subseteq B$ consists of exactly those elements $x \in B$ which can be expressed in the form

(6) $$x = \bigvee_{f \in F} \bigwedge_{y \in Y} (f(y))y,$$

where Y is a finite subset of X, $F \subseteq {}^{Y}\{-1,1\}$ and $(-1)x = -x$, $(1)x = x$. It is obvious that every element $x \in B$ of the form (6) belongs to A. To show that also every element $a \in A$ can be expressed in the form (6) it is sufficient to verify that the set of all elements of the form (6) forms a subalgebra, i.e. that it is closed under the operations of meet and complement.

If X is a finite set of generators of the algebra B, then B is finite, has at most $2^{|X|}$ atoms and $|B| \leq 2^{2^{|X|}}$.

A similar estimate of the size of a complete Boolean algebra from the size of a set of complete generators is not possible. There are arbitrary large complete algebras, for example $C(\omega,\kappa)$ from 2.25(e), which have only a countable set of complete generators.

2.22. Products. The Cartesian product $B_1 \times B_2$ of two Boolean algebras B_1 and B_2 with

Boolean operations defined componentwise is again a Boolean algebra. We call it the product of the algebras B_1 and B_2 and denote it by $B_1 \times B_2$.

If $B = B_1 \times B_2$ then the elements $u_1 = \langle 1_1, 0_2 \rangle$ and $u_2 = \langle 0_1, 1_2 \rangle$ form a partition of the unit element $\langle 1_1, 1_2 \rangle$ in B. For every i = 1,2 the factor $B|u_i$ is isomorphic with the algebra B_i. The isomorphism $j_1 : B_1 \to B|u_1$ is defined by $j_1(x) = \langle x, 0_2 \rangle$ for every $x \in B_1$ and the isomorphism j_2 is defined similarly. Moreover, an arbitrary element $\langle x,y \rangle \in B$ is equal to the join $j_1(x) \vee j_2(y)$. Thus, the product of the algebras B_1 and B_2 is formed by setting the algebras B_1 and B_2 next to each other.

Consider now a complete Boolean algebra B which has some atoms but is not atomic. Then the element $u = \bigvee At(B)$ is different from **0** and **1** and $B \cong B|u \times B|(-u)$. This means that B is a product of an atomic algebra and an atomless algebra.

<u>2.23. Free products.</u> If B is a Boolean algebra, let us denote the set of its nonzero elements by B^+. Let B_1 and B_2 be Boolean algebras. An algebra C is said to be a free product of the algebras B_1 and B_2, denoted by

$$C = B_1 \odot B_2,$$

if for every i = 1, 2 there is an embedding j_i of the algebra B_i into C so that

(i) for every $x \in B_1^+$, $y \in B_2^+$, we have $j_1(x) \wedge j_2(y) \neq \mathbf{0}$ in C.

(ii) the set $j_1[B_1] \cup j_2[B_2]$ generates the algebra C.

Conditions (i) and (ii) and expression (6) imply that the set

$$\{ j_1(x) \wedge j_2(y) : x \in B_1^+ \;\&\; y \in B_2^+ \}$$

is dense in C, and hence the free product of the algebras B_1 and B_2 is determined uniquely up to isomorphism. Moreover, the mapping j_i for i = 1,2 is a complete embedding into C. Let us note that the free product C of complete Boolean algebras B_1 and B_2 need not be a complete algebra. Therefore the free product in the class of complete Boolean algebras is defined as the completion of the algebra C.

2.24. Construction of the free product. So far we didn't prove that the free product of two arbitrary algebras exists. Given two algebras B_1 and B_2, consider the sets B_1^+ and B_2^+ and their canonical orderings $\leq_1$ and $\leq_2$. It is easily seen that the Cartesian product $B_1^+ \times B_2^+$ is separatively ordered by the relation $\leq$, where

$$< x_1, y_1 > \leq < x_2, y_2 > \;\leftrightarrow\; x_1 \leq_1 x_2 \;\&\; y_1 \leq_2 y_2.$$

Let B be the complete Boolean algebra determined by the ordered set $< B_1^+ \times B_2^+ , \leq >$ and let $C \subseteq B$ be the subalgebra generated by the set $B_1^+ \times B_2^+$. We know that B is the completion of the algebra C. Let us define a mapping $j_1 : B_1 \to C$ by $j_1(0_1) = \mathbf{0}$ and $j_1(x) = < x, 1_2 >$ for $x \in B_1^+$. Similarly, a mapping $j_2 : B_2 \to C$ is defined by $j_2(0_2) = \mathbf{0}$ and $j_2(y) = < 1_1, y >$ for $y \in B_2^+$. It is easy to verify that for arbitrary $x \in B_1^+$ and $y \in B_2^+$ in the algebra C we have $< x,y > = j_1(x) \wedge j_2(y)$ and that j_i is an embedding of B_i into C. From this it follows that C is the free product of the algebras B_1 and B_2 together with mappings j_1 and j_2.

2.25. Examples. (a) For arbitrary sets I and J, by F(I,J) we denote the set of all finite functions f such that Dom (f) $\subseteq$ I and Rng (f) $\subseteq$ J, ordered by reverse inclusion.
Hence $f \leq g \leftrightarrow f \supseteq g$.

The empty set always lies in F(I,J). If the set J has at least two elements, then F(I,J) is a separative ordering. The complete Boolean algebra determined by the set F(I,J) is denoted by C(I,J). If $J = \{0,1\}$ then we simply write C(I) instead of C(I,2).

(b) The algebras $C(\alpha)$. For an arbitrary ordinal α, the finite functions $f \in F(\alpha,2)$ determine a clopen base of the topological space ${}^{\alpha}2$. It follows that the complete Boolean algebra $C(\alpha)$ is isomorphic to the algebra of regular open sets $RO({}^{\alpha}2)$ and this is the completion of the algebra of clopen sets $CO({}^{\alpha}2)$.

C(0) is the trivial two–element algebra. For a natural number n, the agebra C(n) is finite and has 2^{2^n} elements. For infinite α the algebra $C(\alpha)$ is atomless. By III.1.24 the Suslin number $c(F(\alpha,2))$ equals ω, hence sat $(C(\alpha)) = \omega_1$. In particular, $C(\omega)$ is the algebra of regular open sets of the Cantor discontinuum.

(c) Subsets of the algebra $C(\alpha)$. Consider a fixed α. If $I \subseteq \alpha$, then $F(I,2) \subseteq F(\alpha,2)$ and the identity $k : F(I,2) \to F(\alpha,2)$ can be extended uniquely to a complete embedding K of the algebra $C(I)$ into $C(\alpha)$. Here the image $K[C(I)]$ consists of all elements $u \in C(\alpha)$ which, in $C(\alpha)$, can be expressed as $u = \bigvee X$ for some $X \subseteq F(I,2)$. If we identify $C(I)$ with $K[C(I)]$, then $C(I)$ is a complete subalgebra of the algebra $C(\alpha)$. In particular for $\beta, \gamma < \alpha$, $C(\beta)$ and $C(\gamma)$ are complete subalgebras of the algebra $C(\alpha)$ and from $\beta < \gamma$ it follows that $C(\beta) \subset C(\gamma)$.

We shall show that if the cofinality of the ordinal α is uncountable then $C(\alpha) = \bigcup \{C(\beta) : \beta < \alpha\}$, which means that $C(\alpha)$ is the union of an increasing chain of complete subalgebras. Let $u \in C(\alpha)$. By 2.11 (ii) there exists a set $X \subseteq F(\alpha,2)$ of disjoint elements such that $u = \bigvee X$. Since sat $(C(\alpha)) \le \omega_1$, the set X is at most countable. It follows that the size of the set $D = \bigcup \{\mathrm{Dom}(f) : f \in X\}$ is at most ω and hence bounded in α. This means that for some $\beta < \alpha$ we have $D \subseteq \beta$ and hence $X \subseteq F(\beta,2)$ and $u \in C(\beta)$.

(d) $C(\alpha)$ as a free product of subalgebras. Let $\beta < \alpha$. If we assign to every function $f \in F(\beta,2)$ and $g \in F(\alpha-\beta,2)$ the function $f \cup g$, we get an isomorphism of the Cartesian product $F(\beta,2) \times F(\alpha-\beta,2)$ onto the ordered set $F(\alpha,2)$. It is not difficult to verify that $C(\alpha)$ is the completion of the free product of the subalgebras $C(\beta)$ and $C(\alpha-\beta)$.

(e) Collapsing algebras. For any infinite cardinal κ, the complete Boolean algebra $C(\omega,\kappa)$ determined by the ordered set $F(\omega,\kappa)$ is called a collapsing algebra. This because every generic extension determined by the algebra $C(\omega,\kappa)$ adds a one–to–one mapping of the cardinal κ onto ω and hence collapses the cardinal κ which in the extension becomes a countable ordinal. From this fact originates the idea of the proof of the following statement (Solovay 1966).

Every collapsing algebra has a countable set of complete generators.

Proof. It is obvious that $C(\omega,\kappa)$ is isomorphic with the algebra $RO({}^{\omega}\kappa)$ where ${}^{\omega}\kappa$ is the topological product of ω copies of the set κ with discrete topology. For arbitrary $m < \omega$ and $\alpha < \kappa$ let us set

$$b(m,\alpha) = \{f \in {}^{\omega}\kappa : f(m) = \alpha\}.$$

Each set $b(m,\alpha)$ is clopen and their finite intersections form a clopen base of the space ${}^{\omega}\kappa$. It follows that every regular open set is the join of the clopen sets of the base which it contains. This means that $\{b(m,\alpha) : m < \omega, \alpha < \kappa\}$ is a family of complete generators of the algebra RO $({}^{\omega}\kappa)$, however of size κ. Now for any natural numbers m and n we define

$$a(m,n) = \{f \in {}^{\omega}\kappa : f(m) < f(n)\}.$$

Every set $a(m,n)$ and its complement can be expressed as a union of open sets

$$a(m,n) = \bigcup_{\alpha<\beta} \{f \in {}^{\omega}\kappa : f(m) = \alpha \ \& \ f(n) = \beta\}.$$

$${}^{\omega}\kappa - a(m,n) = a(n,m) \cup \bigcup_{\alpha<\kappa} \{f\in{}^{\omega}\kappa : f(m) = \alpha \ \& \ f(n) = \alpha\}.$$

therefore the sets $a(m,n)$ are clopen and belong to $RO({}^{\omega}\kappa)$. We shall show that every set $b(m,\alpha)$ can be obtained from the family $S = \{a(m,n) : m,n < \omega\}$ using infinite Boolean operations. From this it follows that S is a countable family of complete generators of the algebra $RO({}^{\omega}\kappa)$.

It suffices to verify that for every m and $\alpha < \kappa$ we have

$$b(m,\alpha) = (-\bigvee_{\beta<\alpha} b(m,\beta)) \wedge \bigwedge_{n<\omega} (\bigvee_{\beta<\alpha} b(n,\beta) \vee (-a(n,m))). \tag{7}$$

From the definition of the sets $b(m,\beta)$ we get

$$\bigcup_{\beta<\alpha} b(m,\beta) = \{f \in {}^{\omega}\kappa : f(m) < \alpha\}$$

and this is a clopen set. Hence

$$\bigvee_{\beta<\alpha} b(m,\beta) = \{f \in {}^{\omega}\kappa : f(m) < \alpha\}. \tag{8}$$

$$-\bigvee_{\beta<\alpha} b(m,\beta) = \{f \in {}^{\omega}\kappa : f(m) \geq \alpha\}. \tag{9}$$

This means that in the second member of the right hand side of (7) we can replace joins by unions and complements by set–theoretic complements. For every $n < \omega$ we have

(10) $$\bigvee_{\beta<\alpha} b(n,\beta) \vee (-a(n,m)) =$$

$$= \{f \in {}^{\omega}\kappa : f(n) \geq f(m) \text{ or } f(n) < \alpha\}.$$

From (9) and (10) it follows that in relation (7) we have the inclusion $\subseteq$. Suppose that the inclusion is strict. Then there exists a nonempty clopen set

$$c = \{f \in {}^{\omega}\kappa : f(i_1) = \gamma_1 \ \& \ .... \ \& \ f(i_k) = \gamma_k\}.$$

which is disjoint with $b(m,\alpha)$ and is a subset of the right hand side in (7). We may assume that $i_1 = m$. Because $c \subseteq \{f \in {}^{\omega}\kappa : f(m) \geq \alpha\}$ and for no $f \in c$, $f(m) = \alpha$ since c is disjoint with $b(m,\alpha)$, we have $\gamma_1 > \alpha$. Let us choose a natural number n different from all i_j's. Then for $g \in c$ such that $g(n) = \alpha$ we have $g(n) < g(m)$ and also $g(n) \geq \alpha$. By (10) this means that

$$g \notin \bigvee_{\beta<\alpha} b(n,\beta) \vee (-a(n,m)),$$

and this contradicts the assumption on the set c.. We proved (7) and hence also the whole statement.

(f) For sets I, J and an infinite cardinal λ let us denote by $F(I,J,\lambda)$ the set of all functions f such that Dom (f) $\subseteq$ I, $|\text{Dom}(f)| < \lambda$ and Rng (f) $\subseteq$ J, ordered by reverse inclusion. We can see that $F(I,J,\omega)$ is the same as $F(I,J)$ from Example (a).

The complete Boolean algebra $C(I,J,\lambda)$ determined by the ordering $F(I,J,\lambda)$ is isomorphic with the algebra of regular open sets of the space $\bigtimes^{(\lambda)}\{J_i : i \in I\}$ from Example III.4.74(c), where each $J_i = J$ and J is a topological space with discrete topology.

(g) $C(\omega_1, 2, \omega_1) \cong C(\omega_1, 2^{\omega}, \omega_1)$. It is sufficient to verify that the algebras have isomorphic dense subsets. We partition ω_1 into countable subsets $\{A_\alpha : \alpha < \omega_1\}$. For every $\alpha < \omega_1$ let $\{f_{\alpha,\beta} : \beta < 2^{\omega}\}$ be a one–to–one enumeration of all functions defined on A_α with values in $\{0,1\}$. Let us assign to an arbitrary function $h \in F(\omega_1, 2^{\omega}, \omega_1)$ the function

$$\bigcup \{f_{\alpha,\, h(\alpha)} : \alpha \in \text{Dom}(h)\}.$$

We thus get an isomorphic mapping of the set $F(\omega_1, 2^{\omega}, \omega_1)$ onto a subset $P \subseteq F(\omega_1, 2, \omega_1)$.

It is obvious that P is a dense set of the algebra $C(\omega_1, 2. \omega_1)$.

(h) For a weakly inaccessible cardinal κ let us take an increasing sequence of infinite cardinals $< \kappa_\alpha : \alpha < \kappa >$ converging to κ. If $< A_\alpha : \alpha < \kappa >$ is a family of sets such that $|A_\alpha| = \kappa_\alpha$ for every $\alpha < \kappa$, then the set

$P = \{f : f$ is a finite function, $Dom (f) \subseteq \kappa$ and $(\forall_i \in Dom (f)) (f(i) \in A_i)\}$

is separatively ordered by reverse inclusion and from III.1.23 it follows that for the algebra C determined by the set P we have $c(P) = sat (B) = \kappa$.

<u>2.26. Refinements of partitions.</u> In I.5.37 we were dealing with partitions of a set. Every partition of a nonempty set X is in fact a partition of the unit element in the power set algebra $\mathscr{P}(X)$. Now consider partitions of unity, in short partitions, in an arbitrary Boolean algebra. The following definition is analogous to the definition of refinement of a partition of a set.

<u>2.27. Definition. Refinements.</u> Let P and Q be partitions of B. We say that P is a <u>refinement</u> of Q or that P refines Q, denote by $P << Q$, if for every $x \in P$ there exists a $y \in Q$ with $x \leq y$.

If $P << Q$ then for every $y \in Q$ the set $P_y = \{x \in P : x \leq y\}$ is a partition of the element y. On the other hand, if we choose for every $y \in Q$ a partition P_y of the element y then $P = \bigcup\{P_y : y \in Q\}$ is a partition of unity and $P << Q$. If H is a dense set in B then by 2.11(ii) every partition Q has a refinement P which consists solely of elements of the set H. If the algebra B is atomless then every partition Q has a refinement P such that every $y \in Q$ is partitioned into two elements in P. For an arbitrary partition P of B, the set $B < P > = \{ u \in B : (\exists X \subseteq P) (u = \bigvee X)\}$ is an atomic subalgebra of the algebra B and P is the set of its atoms.

<u>2.28. Theorem.</u> All countable atomless Boolean algebras are isomorphic.

Proof. Let B be a countable atomless algebra. First we shall construct an increasing sequence of finite subalgebras B_n such that

$$B = \bigcup_{n<\omega} B_n$$

Let us set $B_0 = \{0, 1\}$. Let $x_1, x_2, x_3, \ldots$ be an enumeration of the elements of $B-\{0, 1\}$. By recursion we construct a sequence $P_1, P_2, P_3, \ldots$ of finite partitions refining each other, so that $x_n \in B < P_n >$. Let us set $P_1 = \{x_1, -x_1\}$, obviously $x_1 \in B < P_1 >$. The partition P_{n+1} is constructed in such a way that every element $y \in P_n$ is partitionned into exactly two elements of P_{n+1}. If for $y \in P_n$ both elements $y \wedge x_{n+1}$ and $x_{n+1} - y$ are nonzero, then we put them in P_{n+1}. Otherwise we choose arbitrarily two elements which partition the element $y \in P_n$ and these we put in P_{n+1}. It is obvious that $P_{-1} << P_n$, $x_{n+1} \in B < P_{n+1} >$ and $|P_{n+1}| = 2^{n+1}$. If we set $B_n = B < P_n >$ then $B = \bigcup_{n<\omega} B_n$.

Suppose that C is another countable atomless algebra. By the construction given above we obtain partitions Q_n and subalgebras C_n. The isomorphism f_n of the algebras B_n and C_n is uniquely determined by a one–to–one mapping φ_n of the partition P_n onto Q_n. Let f_o be the isomorphism of the trivial algebras B_o and C_o. Let φ_1 be an arbitrary one–to–one mapping of P_1 onto Q_1. If we have already constructed φ_n, we construct the mapping φ_{n+1} so that for any $x \in P_{n+1}$ and $y \in P_n$ we have

$$x \leq y \quad - \quad \varphi_{n+1}(x) \leq \varphi_n(y).$$

Then the isomorphism f_{n+1} extends f_n and $f = \bigcup f_n$ is an isomorphism of B onto C.

2.29. Corollary. All complete atomless Boolean algebras with countable dense subsets are isomorphic.

Proof. Let H be a countable dense set of a complete atomless Boolean algebra B. Then the algebra $C \subseteq B$ generated by the set H is countable, atomless and the set $C - \{0\}$ is dense in B. From this it follows by Theorem 2.28 that the complete algebras under consideration have isomorphic dense subsets and hence by 2.15 are isomorphic.

From this it follows that the algebras of regular open sets of all separable metric spaces without isolated points are isomorphic. In particular, the algebras $RO(\mathbb{R})$ of the real line, $RO(\mathbb{R}^2)$ of the plane and $RO({}^{\omega}\mathbb{I})$ of the Hilbert cube are all isomorphic with the algebra $C(\omega)$ from Example (b).

<u>2.30. Distributivity of algebras.</u> The infinite distributive laws stated in 1.19 are equivalent with the fact that any two partitions P and Q of unity in an arbitrary algebra have a common refinement, which is

$$\{x \wedge y \neq \mathbf{0} : x \in P, y \in Q\}.$$

However, a common refinement for infinitely many partitions need not exist.

<u>2.31. Definition.</u> Let κ and μ be cardinals, possibly finite. A Boolean algebra B is said to be <u>(κ, μ)–distributive</u> if for any collection of partitions $< P_\alpha : \alpha < \kappa >$ such that for every $\alpha < \kappa$, $|P_\alpha| \leq \mu$, there exists a partition P refining each P_α.

If an algebra is (κ,μ)–distributive for every cardinal μ, then we say that it is (κ,∞)–distributive.

It is obvious that a (κ,μ)–distributive algebra is also (λ,η)–distributive for every $\lambda < \kappa$ and $\eta < \mu$. An algebra B which is (κ,μ)–distributive and has sat $(B) \leq \mu^+$ is (κ,∞)–distributive. If the algebra B is atomic then the partition At(B) refines all partitions in B and hence B is (κ,∞)–distributive for every κ.

Let us assume that B is atomless and let κ be the minimal cardinality of a dense set H in

B. We shall show that B is not $(\kappa,2)$–distributive. We can assume that $1 \notin H$. For every $u \in H$ we set $Q(u) = \{u,-u\}$. We get a family $S = \{Q(u) : u \in H\}$ consisting of κ two–element partitions. Moreover, for every nonzero $v \in B$ there is a $u \in H$ such that $u < v$, hence v is compatible with both u and $-u$. From this it follows that the family S has no refinement.

In particular, $C(\omega)$ is a complete algebra which is not $(\omega,2)$–distributive.

2.32 Examples. (a) An algebra B is $(\kappa,2)$–distributive if and only if it is (κ,κ)–distributive.

(b) A complete algebra is $(\kappa,2)$–distributive if and only if it is $(\kappa,2^\kappa)$–distributive.

If a complete Boolean algebra B is $(\kappa,2^\kappa)$–distributive then it is also $(\kappa,2)$–distributive. Suppose that B is $(\kappa,2)$–distributive. Given a partition P of size 2^κ, we shall show that P can be replaced by a family of κ two–element partitions. The elements of the partition P can be enumerated by mappings $f : \kappa \to 2$, hence $P = \{p_f : f \in {}^\kappa 2\}$. Next, for every $\alpha < \kappa$ let $u_\alpha = \bigvee \{p_f : f(\alpha) = 0\}$ and $P = \{u_\alpha, -u_\alpha\}$. We have a family $\{P_\alpha : \alpha < \kappa\}$ of two–element partitions and it is easy to check that a common refinement of all partitions P_α is also a refinement of the partition P. If we have κ partitions of size 2^κ, we replace each one of them by a family of κ two–element partitions. From the $(\kappa,2)$–distributivity of the algebra B we get a refinement for the original family.

(c) A complete Boolean algebra B is (κ,λ)–distributive if and only if for every matrix $< u(\alpha,\beta) : \alpha < \kappa,\ \beta < \lambda >$ of elements of B we have

$$\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} u(\alpha,\beta) = \bigvee_{f\in{}^\kappa\lambda} \bigwedge_{\alpha<\kappa} u(\alpha,f(\alpha)).$$

2.33. A criterion for distributivity. Let λ be an infinite cardinal. An ordered set Q is said to be λ–closed if for any $\xi < \lambda$ and any decreasing sequence $< q_\alpha : \alpha < \xi >$ of elements of Q there exists a $q \in Q$ such that $q \leq q_\alpha$ for every $\alpha < \xi$. In other words, Q is λ–closed if every chain in Q of size less than λ has a lower bound.

The ordered set $F(I,J,\lambda)$ from Example 2.25 (f) is $cf(\lambda)$–closed.

2.34. Theorem. If a Boolean algebra B has a dense λ–closed subset then B is (κ,∞)–distributive for every $\kappa < \lambda$.

Proof. Let $\kappa < \lambda$ and let $< P_\alpha : \alpha < \kappa >$ be a collection of partitions. Let H be a dense λ–closed set in B. We shall verify that for every $u \in B^+$ there exists an $x \leq u$ such that

(11) $x \in H$ and x is below exactly one element from each partition P_α.

For any $u \in B^+$ we construct by recursion a decreasing sequence $< x_\alpha : \alpha < \kappa >$ of elements of H such that $x_0 \leq u$ and each x_α is below one of the elements of the partition P_α. If we have already constructed a sequence $< x_\alpha : \alpha < \beta >$ for $\beta < \kappa$, then from λ–closedness of the set H we get a $z \in H$ which is below all x_α's. Moreover, z is compatible with some element $y \in P_\beta$ because $z \neq \mathbf{0}$. Let us choose $x_\beta \in H$ so that $x_\beta \leq z \wedge y$. In this way we construct a decreasing sequence $< x_\alpha : \alpha < \kappa >$ of elements of H. Now any lower bound of this sequence satisfies (11). We showed that the set of all elements which satisfy (11) is dense. It is obvious that any partition of unity consisting of such elements refines all partitions P_α. Therefore the algebra B is (κ, ∞)–distributive.

2.35. Three–parameter distributivity. Let κ, μ and ν be cardinals and let $\nu \geq 2$. An algebra B is said to be (κ, μ, ν)–distributive if for every collectionin B such that for every $\alpha < \kappa$ $|P_\alpha| \leq \mu$, there exists a partition P such that for every $x \in P$ and every $\alpha < \kappa$

(12) $|\{y \in P_\alpha : y \wedge x \neq \mathbf{0}\}| < \nu$.

If an algebra is (κ, μ, ω)–distributive then we say that it is weakly (κ, μ)–distributive.

Let us notice that $(\kappa, \mu, 2)$–distributivity is the same as (κ, μ)–distributivity. Hence three–parameter distributivity is a generalization of the notion of distributivity.

2.36. Completeness and distributivity. An algebra A is (κ, ∞)–distributive if and only if its completion cm(A) is (κ,∞)–distributive. Let us realize that every partition of unity in A is also a partition of unity in cm(A), and that every partition of unity in cm(A) has a refinement in the algebra A. On the other hand, (κ,μ)–distributivity is not carried over from an algebra to its completion. In Example 2.45(b) we shall describe an algebra that is $(\omega,2)$–distributive but its completion is not even $(\omega,\omega_1,\omega_1)$–distrubutive.

In the next paragraph we shall see that distributivity of complete Boolean algebras determines some properties of generic extensions of models of set theory. Therefore we shall above all be concerned with the distributivity of complete Boolean algebras.

2.37. Lemma. If a complete Boolean algebra B is (κ,μ,ν)–distributive then every complete subalgebra of B is also (κ,μ,ν)–distributive.

Proof. Let $C \subseteq B$ be a complete subalgebra and $< P_\alpha : \alpha < \kappa >$ a collection of partitions of unity in the algebra C such that $|P_\alpha| \le \mu$ for every α. This is at the same time a collection of partitions of unity in B, and therefore there exists a partition P in B such that for every $x \in P$ and every $\alpha < \kappa$ the relation (12) holds. From this it follows that the set

$$H = \{x \in B^+ : (\forall\, \alpha < \kappa)\, (|\, \{\, y \in P_\alpha : y \wedge x \neq \mathbf{0}\,\} \,| < \nu)\}$$

is dense in B. For every $x \in H$ let us set $C(x) = \bigwedge \{u \in C : x \le u\}$. Since C is a complete subalgebra, we have $C(x) \in C$ and therefore $C(H) = \{C(x) : x \in H\}$ is a dense set in C. For any C(x) and each $\alpha < \kappa$ we have

$$C(x) \le \bigvee \{y \in P_\alpha : y \wedge x \neq \mathbf{0}\}.$$

This means that C(x) is compatible with less than ν elements of each partition P_α. Any partition $Q \subseteq C(H)$ is a partition as desired in C.

2.38. (κ,μ,ν)–nondistributivity. Suppose that a complete algebra B is not (κ,μ,ν)–distributive.

This means that there exists a collection $< P_\alpha : \alpha < \kappa >$ of partitions of unity in B such that $|P_\alpha| \leq \mu$ for every $\alpha < \kappa$ and, moreover, the set

$$X = \{ x \in B^+ : (\forall\alpha<\kappa) \left(|\{y \in P_\alpha : x \wedge y \neq \mathbf{0} \}| < \nu \right) \}$$

is not dense in B. For such a collection of partitions there can be two possibilities. Either X is an empty set, this means that every nonzero element $x \in B$ is compatible with at least ν elements of some partition P_α, and then the collection $< P_\alpha : \alpha < \kappa >$ ensures the negation of (κ,μ,ν)–distributivity everywhere. Or $X \neq \phi$, and then $u = -\bigvee X$ is a nonzero element and $u \neq 1$, since X is not dense in B. If we now take the restrictions $P_\alpha | u = \{u \wedge y : y \in P_\alpha\}$ of partitions P_α to the element u, we get a collection of partitions of unity in the algebra $B|u$ which ensures the negation of (κ,μ,ν)–distributivity everywhere in $B|u$. This leads us to the following strenghtening of the negation of (κ,μ,ν)–distributivity.

2.39. Definition. A complete Boolean algebra B is said to be everywhere (κ,μ,ν)–nondistributive if there exists a collection $< P_\alpha : \alpha < \kappa >$ consisting of partitions of size at most μ such that for every $x \in B^+$ there exists an $\alpha < \kappa$ such that x is compatible with at least ν elements of the partition P_α.

2.40. Definition. A Boolean algebra B is called homogeneous if B is isomorphic with every factor $B|u$.

From the above it follows that a complete homogeneous Boolean algebra is not (κ,λ,μ)–distributive if and only if it is everywhere (κ,λ,μ)–nondistributive.

2.41. Example. The complete algebra $C(\omega_1,2,\omega_1)$ from Example 2.25.(f) is (ω,∞)–distributive and is not $(\omega_1,2)$–distributive. The (ω,∞)–distributivity of the algebra follows from Theorem 2.34, since the dense subset $F(\omega_1, 2, \omega_1)$ of it is ω_1–closed. If it would be $(\omega_1, 2)$–distributive, then by 2.32 (b) it would also be $(\omega_1, 2^{\omega_1}, 2)$–distributive.

We shall show, however, that it is everywhere $(\omega_1, 2^\omega, 2^\omega)$–nondistributive. By 2.25(g), the algebra $C(\omega_1, 2, \omega_1)$ is isomorphic with $C(\omega_1, 2^\omega, \omega_1)$. Moreover, the set H of all functions defined on some countable ordinal with values in 2^ω is dense in $C(\omega_1, 2^\omega, \omega_1)$. If we set

$$P_\alpha = \{f \in H : \text{Dom}\,(f) = \alpha\}$$

for every $\alpha < \omega_1$, then the partitions of unity P_α, $\alpha < \omega_1$, refine each other. Also each $f \in P_\alpha$ is partitioned into 2^ω elements of $P_{\alpha+1}$. Moreover, $H = \bigcup \{P_\alpha : \alpha < \omega_1\}$ is a dense set. So the collection $< P_\alpha : \alpha < \omega_1 >$ ensures that the algebra $C(\omega_1, 2^\omega, \omega_1)$ is everywhere $(\omega_1, 2^\omega, 2^\omega)$–nondistributive and the same holds for the algebra $C(\omega, 2, \omega_1)$.

<u>2.42. A characterization of collapsing algebras.</u> For an infinite cardinal κ let us consider the complete algebra $C(\omega,\kappa)$ defined in 2.25(e). For every natural number n, the set $P_n = {}^n\kappa$ is a partition of unity in $C(\omega, \kappa)$ and each $f \in P_n$ is partitioned in P_{n+1} into κ elements

$$\{f \cup \{< n, \alpha >\} : \alpha < \kappa\}.$$

Moreover, the set ${}^{<\omega}\kappa = \bigcup P_n$ is of size κ and is dense in $C(\omega,\kappa)$. From this it follows that the family $< P_n : n < \omega >$ ensures that $C(\omega,\kappa)$ is everywhere (ω,κ,κ)–nondistributive.

<u>2.43. Theorem. (McAloon).</u> A complete Boolean algebra C which is everywhere (ω, κ,κ)–nondistributive and has a dense subset of size κ is isomorphic with the collapsing algebra $C(\omega,\kappa)$.

<u>Proof.</u> Let $H \subseteq C$ be a dense set of size κ and let the collection $< Q_n : n < \omega >$ ensure that C is everywhere (ω, κ, κ)–nondistributive. We may assume that each Q_n has cardinality κ. First we construct partitions Q_n' so that $\bigcup\{Q_n' : n < \omega\}$ is a dense set in C. For every n, let us set

$$H_n = \{x \in H : |\{y \in Q_n : y \wedge x \neq 0\}| = \kappa\}.$$

From the (ω, κ,κ)–nondistributivity we get $H = \bigcup \{H_n : n < \omega\}$. Also $|H_n| \leq \kappa$ for every

$n < \omega$ hence there exists a one–to–one mapping $\varphi_n : H_n - Q_n$ such that $\varphi_n(x) \wedge x \neq \mathbf{0}$ for every $x \in H_n$. For every n let us set

$$Q'_n = (Q_n - \mathrm{Rng}\,(\varphi_n)) \cup \{x \wedge \varphi_n(x) : x \in H_n\} \cup \cup(\{-x \wedge \varphi_n(x) : x \in H_n\} - \{\mathbf{0}\}).$$

Then each Q'_n is a partition of unity and the set $\bigcup \{Q'_n : n < \omega\}$ is dense in C.

From the $(\omega.\kappa.\kappa)$–nondistributivity it follows that every nonzero element y can be partitioned into κ elements. For every y let us choose one such partition P(y) of size κ. By recursion we define a refining sequence of partitions $< P_n : n < \omega >$. Let us set $P_0 = \{1\}$. For $n > 0$ we first take a common refinement P'_n of the partitions $P_0, \ldots, P_{n-1}, Q'_{n-1}$ and we set $P_n = \bigcup \{P(y) : y \in P'_n\}$. Obviously the P'_ns are partitions of unity. Because $P_{n+1} << Q'_n$ for every n, the set $\bigcup \{P_n : n < \omega\}$ is dense in C. Moreover every $y \in P_n$ is partitioned into κ elements of P_{n+1}. From this it follows that the dense set $\bigcup \{P_n : n < \omega\}$ together with the canonical ordering is isomorphic with the dense set ${}^{<\omega}\kappa$ of the algebra $C(\omega.\kappa)$. This means that the complete algebra C and $C(\omega.\kappa)$ are isomorphic.

<u>2.44. Theorem.</u> (Kripke). Every Boolean algebra which has a dense subset of size $\leq \kappa$ can be completely embedded into the collapsing algebra $C(\omega.\kappa)$. Hence every algebra can be completely embedded into some complete algebra with a countable set of complete generators.

<u>Proof</u>. Let B be the completion of the free product $A \odot C(\omega.\kappa)$. We know that the algebra A is completely embedded into B and we may assume that $C(\omega.\kappa)$ is a complete subalgebra of the algebra B. Since $C(\omega.\kappa)$ is everywhere $(\omega.\kappa.\kappa)$–nondistributive, by 2.37 the same holds for the algebra B. If $H \subseteq A$ is a dense subset of size κ then the Cartesian product $H \times F(\omega.\kappa)$ also has cardinality κ and determines a dense set in B. We proved that B has the characteristic properties of the collapsing algebra $C(\omega.\kappa)$ and therefore the two algebras are isomorphic. From this it follows that the algebra A is completely embedded into $C(\omega.\kappa)$.

<u>2.45. Examples.</u> (a) For any tree T let us denote by T^* the set T with the reverse ordering. Then two nodes x and y are incomparable in T if and only if they are disjoint in T^*. Moreover, T^* is a separative ordering provided every node of the tree splits.

The following statements are equivalent:

(i) There exists a complete atomless Boolean algebra B which is (ω,∞)–distributive and $\mathrm{sat}(B) = \omega_1$.

(ii) There exists a Suslin ω_1–tree.

Let B be an algebra satisfying (i). For every non–zero element y let us choose a countable partition Q(y) of the element y. By recursion we construct a sequence $< P_\alpha : \alpha < \omega_1 >$ of partitions refining each other. We set $P_0 = \{1\}$. If we have constructed the sequence $< P_\alpha : \alpha < \beta >$ and β is limit, let P_β be any common refinement of all partitions P_α. From the (ω, ∞)–distributivity it follows that such a partition exists. In case $\beta = \gamma + 1$ we set $P_\beta = \bigcup \{Q(y) : y \in P_\gamma\}$. It is easy to check that the set $T = \bigcup \{P_\alpha : \alpha < \omega_1\}$ together with the reverse canonical ordering is a Suslin ω_1–tree.

Conversely, let T be a Suslin ω_1–tree. We may assume that every node splits and that T has no short branches. Let B be the complete Boolean algebra determined by the separative ordering T^*. Then B is atomless since every node of T splits, and $\mathrm{sat}(B) = \omega_1$ since every antichain in T is at most countable. T has no short branches, hence every level T_α determines a partition of unity in B such that $T_\beta << T_\alpha$ for $\alpha < \beta < \omega_1$. Any element $x \in B$ can be expressed as a join of some at most countable subset $X \subseteq T$. If $\alpha < \omega_1$ is such that $X \subseteq T \upharpoonright \alpha$ then $x \in B < T_\alpha >$. This means that B is a union of an increasing sequence of ω_1 complete Boolean algebras. Each countable set of partitions in B lies in some complete subalgebra $B < T_\alpha >$ and T_α is the common refinement of these partitions in B. Hence B is (ω,∞)–distributive.

(b) Let us recall that a special Aronszajn tree is an ω_1–tree without cofinal branches which is a union of countably many antichains. By III.3.41 and 3.48 a special Aronszajn tree exists and cannot be a Suslin tree.

Let $< T, \leq >$ be a special Aronszajn tree. We may assume that every node in T splits and that T has no short branches. We shall show that the complete algebra B determined by the separative ordered set T^* is isomorphic with the collapsing algebra $C(\omega, \omega_1)$. Let T be the union of countably many antichains Q_n. This means that each Q_n is a set of disjoint elements in B which can be extended to a partition of unity P_n in B. We shall show that the collection of partitions $< P_n : n < \omega >$ guarantees that the algebra B is everywhere $(\omega, \omega_1, \omega_1)$–nondistributive. Let us take an arbitrary $y \in T$. Since T has no short branches, the set $Y = \{x \in T : y < x\}$ is uncountable, and hence there exists an n such that $Y \cap Q_n$ has cardinality ω_1. This means that x is compatible with ω_1 elements of the partition P_n. The algebra B is thus everywhere $(\omega, \omega_1, \omega_1)$–nondistributive and is isomorphic with $C(\omega, \omega_1)$ by 2.43, since T is a dense set in B of size ω_1.

Every level T_α of the tree T is a partition of unity in B and $B < T_\alpha >$ is a complete subalgebra. The union of all algebras $B < T_\alpha >$ for $\alpha < \omega_1$ is a subalgebra $C \subseteq B$ and in a similar way as in Example (a) we can prove that C is $(\omega, 2)$–distributive. The subalgebra C is dense in B since $T \subseteq C$. This means that the completion of C is the whole algebra B, which, however, is not $(\omega, 2)$–distributive. Hence C is an example of an algebra which is $(\omega, 2)$–distributive such that the completion of it is not $(\omega, 2)$–distributive.

<u>2.46</u>. Ideals and filters in a Boolean algebra are direct generalizations of the notions of ideal and filter in the power set algebra of sets which we studied in detail in I.8 (ideals and filters on a set).

<u>Definition</u>. Let B be a Boolean algebra. An ideal in B is a nonempty set $J \subseteq B$ such that $J \neq B$, $a \in J \;\&\; b \leq a - b \in J$ and $a, b \in J - a \vee b \in J$.

A filter in B is a nonempty set $F \subseteq B$ such that $F \neq B$, $a \in F \;\&\; b \geq a - b \in F$ and $a, b \in F - a \wedge b \in F$.

A filter in B is called an ultrafilter if for every $a \in B$ either $a \in F$ of $-a \in F$.

A principal filter is a set of the form $\{b \in B : b \geq a\}$ where a is a nonzero element of B.

Obviously the notions of ideal and filter are dual notions. If J is an ideal in B then we say that $J^* = \{-a : a \in J\}$ is the dual filter of J. Similarly, $F^* = \{-a : a \in F\}$ is the dual ideal of the filter F.

2.47. Lemma. (i) A filter in B is an ultrafilter if and only if it is a maximal filter in B with respect to inclusion.

(ii) A principal filter $\{b \in B : b \geq a\}$ is an ultrafilter if and only if a is an atom of the algebra B.

The proof of (i) and (ii) are analogous to the proofs of I.8.17 and I.8.11(a).

From the Principle of Maximality follows easily the following more general formulation of the fundamental theorem on ultrafilters (I.8.18).

2.48. Theorem. Let $S \subseteq B$ be a centered family of elements of an algebra B, i.e. for every finite $S_0 \subseteq S$ we have $\bigwedge S_0 \neq \mathbf{0}$. Then there exists an ultrafilter F in B such that $S \subseteq F$.

In particular, for every nonzero element $a \in B$ there exists an ultrafilter F such that $a \in F$.

2.49. Corollary. For distinct elements $a, b \in B$ there exists an ultrafilter in B which contains exactly one of the elements a, b.

Proof. For distinct $a, b \in B$ either $u = a-b \neq \mathbf{0}$ or $v = b-a \neq \mathbf{0}$. Suppose that $u \neq \mathbf{0}$. Let us take an ultrafilter $F \supseteq \{u\}$ whose existence is guaranteed by Theorem 2.48. Since $u \in F$ and $u \leq a$, $u \leq -b$, we have $a \in F$ and $b \notin F$.

<u>2.50. Factorizations, quotients.</u> We shall show how to construct from a given Boolean algebra B and an ideal I in B a new algebra B/I.

For arbitrary a, b ∈ B we define

$$a \sim b \leftrightarrow (a-b) \vee (b-a) \in I.$$

The relation $a \sim b$ says that the elements a and b differ little from each other with respect to the ideal I. In other words there exist u, v ∈ I such that $a = (b-u) \vee v$. Since I is an ideal, the relation ~ is an equivalence relation on B which respects the Boolean operations, i.e. for any a, b, c, d ∈ B the following holds

$$a \sim b \rightarrow -a \sim -b$$

$$(a \sim b) \,\&\, (c \sim d) \rightarrow (a \wedge c) \sim (b \wedge d).$$

The equivalence class which contains an element a ∈ B is denoted by [a]. Let us notice that $[0] = I$ and $[1] = I^*$, therefore $[0] \neq [1]$.

On the quotient set B/~ we define the Boolean operations by

$[a] \wedge_I [b] = [a \wedge b]$, $\quad [a] \vee_I [b] = [a \vee b]$,

$-_I [a] = [-a]$, $\quad 0_I = [0]$, $\quad 1_I = [1]$.

The natural projection $h : B \rightarrow B/\sim$ which assigns to every a ∈ B the equivalence class $h(a) = [a]$, preserves the Boolean operations and therefore

$$B/I = \langle B/\sim, \wedge_I, \vee_I, -_I, 0_I, 1_I \rangle$$

is a Boolean algebra. We call it the factorization or the quotient algebra of the algebra B by the ideal I.

<u>2.51. Examples.</u> (a) The factorization B/I is a two–element algebra if and only if I is a maximal ideal in B (the dual filter of I is an ultrafilter).

(b) If I is a principal ideal in B determined by an element a, then the factorization B/I is isomorphic to the factor B/(−a).

(c) Suppose that $X \neq o$ is a Baire topological space, this means that every nonempty open subset is not a meager set. Then the factorization of the σ–algebra of sets Baire (X), the sets having the Baire property, by the ideal of meager set is isomorphic with the complete Boolean algebra of regular open sets RO(X).

(d) (Smith and Tarski, 1957). If B is a σ–complete Boolean algebra then for every ideal I in B such that sat $(B/I) \leq \omega_1$ the factorization B/I is a complete Boolean algebra.

In particular, if μ is a measure on $\mathscr{P}(\omega)$, see I.8.29, and if I_μ is the ideal of sets of measure zero then sat $(B/I_\mu) \leq \omega_1$, and hence the factorization B/I_μ is a complete Boolean algebra.

(e) In general the completeness of an algebra is not preserved by factorization. The factorization of the power set algebra $\mathscr{P}(\omega)$ by the ideal I_F of all finite subsets of ω is not a σ – complete algebra. Moreover sat $(\mathscr{P}(\omega)/I_F) = (2^\omega)^+$, which follows from III.1.14.

<u>2.52. Notation.</u> The set of all ultrafilters in a Boolean algebra B is denoted by Ult(B). Let us notice that βX from I.8.19 is the same as Ult($\mathscr{P}$(X)).

The following statement shows that Boolean algebras are closely related to algebras of sets.

<u>2.53. Representation Theorem.</u> (Stone 1934). Every Boolean algebra is isomorphic with an algebra of sets.

<u>Proof.</u> If B is an atomic Boolean algebra and if X is the set of all atoms of B then by 2.14 we know that the mapping f defined for every $b \in B$ by $f(b) = \{a \in X : a \leq b\}$ is an isomorphism of the algebra B onto the algebra of subsets $\{f(b) : b \in B\}$ of the set X.

There exist, however, Boolean algebras which are not atomic. We are looking for objects similar to atoms which can serve as elements of the underlying set of the algebra of sets. There exists a one–to–one relation between atoms and principal ultrafilters. The set of all ultrafilters is an extension of the set of all atoms and serves as an underlying set for the desired algebra of sets.

Let B be an arbitrary Boolean agebra. For every $a \in B$ we define $S(a) = \{F \in \text{Ult}(B) : a \in F\}$. By 2.49, S is a one–to–one mapping of the algebra B into

$\mathcal{P}(\text{Ult}(B))$. Since **0** does not belong to any ultrafilter and **1** is in every ultrafilter, we have $S(\mathbf{0}) = \emptyset$ and $S(\mathbf{1}) = \text{Ult}(B)$. Moreover, for any a, b ∈ B

$$S(a \wedge b) = S(a) \cap S(b),$$

$$S(-a) = \text{Ult}(B) - S(a).$$

This means that S is an isomorphism between the algebra B and the algebra of sets $\{S(a) : a \in B\}$.

2.54. Let us remark that the just proved Stone Theorem says that finite Boolean operations can be represented by set–theoretic operations. For infinite Boolean operations such a representation is not possible in general. A complete atomless Boolean algebra which has a countable dense subset is not isomorphic with any σ–algebra of sets.

2.55. Stone Duality. (Stone 1934, 1936, 1937). Stone Duality connects Boolean algebras and compact totally disconnected topological spaces (1.2.(e)), which are also called Boolean spaces.

We shall assign to a Boolean algebra a topological space. Let us take the set Ult (B) of all ultrafilters in B and the mapping $S : B \to \mathcal{P}(\text{Ult}(B))$ such that

$$S(a) = \{F \in \text{Ult}(B) : a \in F\}.$$

From the proof of 2.53 we know that S[B] is a family which is closed under finite intersections. This means that S[B] forms a base of some topology on the set Ult (B) which is called the Stone topology. The set Ult (B) with this topology is called the Stone space or the dual space of the algebra B.

2.56. Theorem. For every Boolean algebra B, Ult (B) is a Boolean space and S is an isomorphism of the algebra B onto CO(Ult(B)).

Proof. If F and G are different ultrafilters then there exists an element a ∈ B such that a ∈ F, −a ∈ G. Then S(a) and S(−a) are clopen sets separating F and G. We proved that the space is totally disconnected. We shall prove compactness. Let Q be an open cover of the space. We may assume that Q consists of basic sets, hence $Q = \{S(a) : a \in Z\}$ for some $Z \subseteq B$. Suppose that there is no finite $Q_0 \subseteq Q$ covering the whole space. Then $\{-\bigvee Y : Y \subseteq Z, Y \text{ finite}\}$ is a centered family of elements from B and any ultrafilter in B which extends it, does not belong to any set from Q, a contradiction. Hence Ult(B) is a Boolean space.

Since S[B] is an algebra of subsets of the space Ult(B), all sets from S[B] are clopen. On the other hand, let X be a clopen set. From the compactness it follows that it is a union of a finite number of basic sets, for example

$$X = S(a_1) \vee \ldots\ldots \vee S(a_n).$$

Then $X = S(b)$, where $b = a_1 \vee \ldots\ldots \vee a_n$, and hence $X \in S[B]$. The uniqueness of the mapping S was proved in 2.53.

We assigned to every Boolean algebra a Boolean space. Conversely, given a Boolean space X, consider the Boolean algebra $B = CO(X)$. We shall show that the Stone space of the algebra B is homeomorphic to the space X.

Ultrafilters in B correspond uniquely to points of the space X. For $x \in X$ define $D(x) = \{A \in B : x \in A\}$, which is an ultrafilter in B. From the total disconnectedness of the space X it follows that D is a one–to–one mapping. Let an arbitrary ultrafilter F in B be given. Then F is a centered family of closed sets in the space X. From the compactness it follows that $x \in \bigcap F$ for some $x \in X$, and hence $F = D(x)$. The mapping D is a one–to–one mapping of X onto the set Ult (B) and it is easily seen that D is a homeomorphism.

The Stone duality enables us to use topological methods to investigate Boolean algebras.

§ 3 Generic extensions of models of set theory.

In the preceding part of the book we several times met principles, hypotheses and statements which are unprovable in the theory ZFC. We shall become acquainted with a method due to P. Cohen, to demonstrate their unprovability: the method of generic extensions of models of set theory. In the explanation we exploit the theory of complete Boolean algebras and generic ultrafilters on them. Using this technique we shall obtain among other things the consistency of CH, $\diamond, \neg$ CH. After that we become familiar with Martin's Axiom. A proof of the Generic Extension Theorem forms the end of the paragraph.

3.1. Let us recall that Zermelo–Fraenkel set theory (ZF) is a theory with equality, with a special predicate symbol $\in$ and with the axioms of Extensionality, Union, Power Set, Infinity, the Replacement Axiom Scheme and with the Axiom of Foundation (I.2.27). If we add the Axiom of Choice, we speak of the theory ZFC. Unless stated otherwise we work in ZF.

3.2. Relativization. Let M be an arbitrary class. For any formula φ of the language of set theory we define its relativization φ^M to the class M:

(i) $(x \in y)^M$ is $x \in y$. $(x = y)^M$ is $x = y$.

(ii) $(\neg\varphi)^M$ is $\neg(\varphi^M)$. $(\varphi \& \psi)^M$ is $\varphi^M \& \psi^M$ and similarly for disjunction, implication, and equivalence.

(iii) $(\forall x\, \varphi)^M$ is $(\forall x \in M)\varphi^M = (\forall x)\,(x \in M \rightarrow \varphi^M)$.

$(\exists x\, \varphi)^M$ is $(\exists x \in M)\, \varphi^M = (\exists x)\,(x \in M \,\&\, \varphi^M)$.

Thus, the formula φ^M results from the formula φ by restricting all quantifiers to the class M, where the predicates of equality and membership remain unchanged. We can say that the formula φ^M says the same for the set universe M as the formula φ for the universe V.

3.3. Transitive models. (i) We say that a closed formula φ holds in a class M if the formula φ^M is provable in set theory.

(ii) We say that a class M is a model of ZF if each axiom φ of the theory ZF holds in M. If, moreover, M is a transitive class (II.1.1) then we say that M is a transitive model of ZF or an inner model of set theory (II.7.13).

The universal class V and the class L of all constructible sets are transitive models of set theory. Moreover, by II.7 we get that L is a model of ZFC in which the generalized continuum hypothesis holds. So L is a model of ZFC + GCH.

<u>3.4. Examples.</u> (a) The Axiom of Extensionality holds in every transitive class M. The relativization of the Axiom of Extensionality to the class M is the following formula

(1) $(\forall x, y \in M)\,((\forall u \in M)\,(u \in x \leftrightarrow u \in y) \rightarrow x = y).$

From the transitivity of the class M it follows that for $x, y \in M$, $x = x \cap M$ and $y = y \cap M$, and therefore from $x \cap M = y \cap M$ we get $x = y$. We proved formula (1), hence the Axiom of Extensionality holds in M.

(b) The Axiom of Foundation holds in every class M. Its relativization is the following formula

(2) $(\forall a \in M)\,((\exists\, x \in M)\,(x \in a) \rightarrow$

$\rightarrow (\exists\, x \in M)\,(x \in a \;\&\; \neg\,(\exists y \in M)\,(y \in a \;\&\; y \in x))).$

Because we assume the Axiom of Foundation, for each $a \in M$, if $a \cap M$ is nonempty, there is an $\in$–minimal element x in the set $a \cap M$. This means that (2) holds.

(c) We work in ZFC. For any cardinal λ let $H(\lambda)$ be the family of all sets whose transitive closure (II.6.9) has cardinality $< \lambda$. It is clear that the transitive closure of every element of $H(\lambda)$ is a subset of $H(\lambda)$ and hence $H(\lambda)$ is a transitive set. In particular, $H(\omega) = V_\omega$ and the set $H(\omega)$ is a model of the theory of finite sets. i.e.. the theory which is obtained from ZF if we replace the Axiom of Infinity by its negation. If λ is an uncountable regular cardinal then all axioms of ZFC except possibly the Power Set Axiom hold in $H(\lambda)$. If, moreover, λ is inaccessible then also the Power Set Axiom holds in $H(\lambda)$ and $H(\lambda)$ is a transitive model of ZFC which is a set.

If there is an inaccessible cardinal then there is also a countable transitive model of ZFC.

3.5. Our goal is not the metamathematical study of the existence of models of set theory. We shall concentrate on the description of a construction which is due to P. Cohen and which enables us to obtain from a given ground model a new model which extends the ground model. We want the extended model to contain more sets than the ground model. This sometimes enables us to verify that, in the extended model, some hypothesis or a combinatorial principle hold, whic.. in the ground model either do not hold or we are not able to verify their validy there. It is clear that the universal class V is not an appropriate ground (=starting) model. If we want to add new sets our ground model must be smaller.

The construction of an extension of a model of set theory remotely resembles the construction of an algebraic extension of a field. Given a field T and a polynomial p(x) over T which does not have a root in T, one constructs an extension T' of the field T in which there is a root of the polynomial p(x).

3.6. By choosing a model M we determine a new universe of set theory in which the role of the universal class is played by the class M. The elements of the class M are the sets of the model and the membership relation among them is the original relation $\in$. For every formula φ we have its relativization φ^M, therefore with every notion, which we introduced before in the book, is associated some notion in the class M. It is important to know what the translated notion means from the point of view of the universal class.

Membership and equality among the sets of the model M are the same as in the whole universe V, because the formula $(x \in y)^M$ is $x \in y$ and the formula $(x = y)^M$ is $x = y$. We say that membership and equality are absolute for M. We want to know what other set–theoretic notions and operations (intersection and difference of sets, ordinals) are absolute. And if some operation is not absolute we want to know its values.

<u>3.7. Absoluteness.</u> Let φ be a set–theoretic formula all of whose free variables are among $x_1, \ldots, x_n$. Let M and N be classes such that $M \subseteq N$.

(i) We say that φ is <u>absolute for M and N</u>, if

$$(\forall x_1, \ldots, x_n \in M)\,(\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi^N(x_1, \ldots, x_n)).$$

(ii) We say that φ is <u>absolute for M</u>, if it is absolute for M and V, i.e.

$$(\forall x_1, \ldots, x_n \in M)\,(\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n)).$$

From the definition of absoluteness we get

<u>3.8. Lemma</u>. Let $M \subseteq N$. If the formulas φ and ψ are absolute for M and N, then also the formulas $\neg\varphi$, $\varphi \vee \psi$, $\varphi \,\&\, \psi$, $\varphi \rightarrow \psi$ and $\varphi \leftrightarrow \psi$ are absolute for M and N.

If φ is absolute for M and also absolute for N then φ is absolute for M and N.

<u>3.9. Examples</u>. We assume that M is a transitive class.

(a) <u>Absoluteness of inclusion.</u> The relation $x \subseteq y$ is an abbreviation of the formula

$$(\forall z)\,(z \in x \rightarrow z \in y)$$

therefore $(x \subseteq y)^M$ means

(3) $$(\forall z \in M)\,(z \in x \rightarrow z \in y).$$

For $x, y \in M$, (3) is equivalent with $x = x \cap M \subseteq y$ or with $x \subseteq y$. We verified that for every $x, y \in M$ we have $(x \subseteq y)^M \leftrightarrow x \subseteq y$, hence the notion of inclusion is absolute for M.

(b) The <u>Power Set Axiom</u> holds in M iff

(4) $(\forall x \in M)\,(\exists y \in M)\,(\forall z \in M)\,((z \subseteq y)^M \rightarrow z \in y).$

From Example (a) it follows that (4) is equivalent with the following: for every $x \in M$, $\mathcal{P}(x) \cap M \in M$. We shall see that the power set operation need not be absolute even for transitive models of ZF.

3.10. Transitive models have a big advantage because of the fact that many notions are absolute for them so that it is easy to understand what is happening inside them. This is the reason why we restrict our attention to transitive models of ZF or ZFC.

The following set–theoretic operations are absolute for every transitive model M:

(5)	$\{x,y\}$,	$x - y$,	$\mathrm{Rng}(x)$,
	$\langle x,y \rangle$,	$x \cup \{x\}$,	$r''x$,
	$x \cap y$,	$x \times y$,	$\bigcup x$.
	$x \cup y$,	$\mathrm{Dom}(x)$,	

That is to say, for each operation $F(x,y)$ above the formula $z = F(x,y)$ is absolute. Let us mention that for every operation $F(x,y)$ which is defined in ZF we must show that for every x,y there is exactly one z such that $z = F(x,y)$. The same also holds in the model M. From this and from absoluteness it follows that the class M is closed under all operations (5).

We shall show a way to prove absoluteness of formulas for transitive models.

3.11. Lemma. If M is a transitive class and if φ is absolute for M then also the formulas $(\exists x \in y)\varphi$ and $(\forall x \in y)\varphi$ are absolute.

Proof. Let $x_1, \dots, x_n$ be all free variables of φ. For the transitive class M and $y \in M$, $(\exists x)(x \in y)$ is equivalent with $(\exists x \in M)(x \in y)$. From this and from the absoluteness of φ it follows that for every $y, x_1, \dots, x_n \in M$

$$((\exists x \in y)\varphi)^M \leftrightarrow (\exists x \in M)(x \in y \,\&\, \varphi^M) \leftrightarrow$$

$\leftrightarrow (\exists x)(x \in y \;\&\; \varphi^M) \leftrightarrow (\exists x)(x \in y \;\&\; \varphi).$

Similarly we can prove that $(\forall x \in y)\varphi$ is absolute.

The expressions $(\exists x \in y)$ and $(\forall x \in y)$ are called *bounded quantifiers*. We say that a formula is *bounded* if all quantifiers in it are bounded. Because all formulas without quantifiers are absolute we get

3.12. Corollary. If M is transitive and φ is bounded then φ is absolute for M.

The same notion can be defined by different equivalent formulas. If the equivalence $\varphi \leftrightarrow \psi$ is provable in ZF (we write ZF $\vdash \varphi \leftrightarrow \psi$), then in every model M of ZF $\varphi^M \leftrightarrow \psi^M$ holds. If, moreover, ψ is absolute for M, then also φ is absolute for M. This can be used to prove that some formula φ is absolute. If we are able to find a bounded formula ψ and prove in ZF the equivalence $\varphi \leftrightarrow \psi$, then we know that φ is also absolute for the model M. In this way we can verify the absolutenes of all operations from (5). For example for the union we use the equivalence

$$z = \bigcup x \leftrightarrow (\forall y \in x)\,(y \subseteq z) \;\&\; (\forall w \in z)\,(\exists y \in x)\,(w \in y).$$

The formula on the right hand side is equivalent to a bounded formula, which can be obtained by replacing $y \subseteq z$ by $(\forall w \in y)\,(w \in z)$.

3.13. Lemma. For a transitive model M the following notions are absolute:

(i) z is an ordered pair.

(ii) r is a relation.

(iii) r is a function

(iv) r is a linear ordering on a.

We shall sketch the proof of (i) and (iv). It is clear that z is an ordered pair if and only if $(\exists x, y \in \bigcup z)\,(z = \langle x, y \rangle)$ and this formula is equivalent to a bounded formula. Similarly, property (iv) can be expressed using ordered pairs and bounded quantifiers.

<u>3.14. Ordinals in a model.</u> The following notions are absolute for a transitive model, because, in ZF, they are equivalent to some bounded formula:

(i) x is an ordinal,

(ii) x is a limit ordinal,

(iii) x is a successor ordinal,

(iv) x is a natural number,

(v) $0, \omega$.

<u>Proof</u>. (i) From the Axiom of Foundation it follows that x is an ordinal if and only if x is a transitive set linearly ordered by the relation $\in$. Both these properties are equivalent to a bounded formula. (v). $x = 0 \leftrightarrow (\forall y \in x)(y \neq y)$ and $x = \omega$ if and only if x is a limit ordinal and $(\forall y \in x)$ (y is not limit). (iv). x is a natural number if and only if x and each $y \in x$ is either 0 or a successor ordinal.

Absoluteness to the formula "x is an ordinal" means that the ordinals of M are exactly all ordinals which are in M, i.e. $On^{M} = On \cap M$. Because M is transitive, $On \cap M$ is a transitive part of On and so $On \cap M$ is either the whole On, or it is an ordinal. The second case happens if M is a set. Moreover, $0 \in M$ and for each $x \in M$, $x \cup \{ x \} \in M$. It follows that $\omega \subseteq M$ and $On \cap M$ is a limit ordinal. In M, the Axiom of Infinity holds, hence $\omega \in M$. In a transitive model the natural numbers are the same as in V.

3.15. Example. In a transitive model the following notions are absolute:

(i) $< a,r >$ is a well–ordering.

(ii) α is the order type of the well–ordering $< a,r >$.

Proof. Let $a, r \in M$. We shall show

(6) $(< a,r > \text{ is a well ordering })^M \rightarrow < a,r >$ is a well–ordering.

We already proved that every well–ordering is isomorphic to an ordinal (II.1.19). This theorem holds also in the model M and hence there exist $\alpha, f \in M$ such that

$(\alpha$ is an ordinal and f is an isomorphism of $< a,r >$ onto $\alpha)^M$.

This formula is absolute for M because α is a real ordinal and f is a real isomorphism of $< a,r >$ onto α.

Hence $< a,r >$ is really a well–ordering of order type α. The proof of the reverse implication in (6) is easy.

3.16. Power set operation in a model. The Power Set Axiom holds in the transitive model M, so from Example 3.9(b) we get

(i) $(\mathcal{P}(x))^M = \mathcal{P}(x) \cap M$ for all $x \in M$.

(ii) $(V_\alpha)^M = V_\alpha \cap M$ for all $\alpha \in On \cap M$.

From this it follows that the rank function is absolute for M.

The operation $\mathcal{P}^M$ and the sets $(V_\alpha)^M$ are in general not absolute. In particular it is obvious in the case when M is a countable model. Then for each infinite $x \in M$, $\mathcal{P}^M(x) \neq \mathcal{P}(x)$ because $\mathcal{P}(x)$ is uncountable but $\mathcal{P}(x) \cap M$ is a countable set. Also $V_\alpha^M \neq V_\alpha$ for every $\alpha \in M$ and $\alpha > \omega$.

A cardinal is another notion which need not be absolute. If the model M is countable then

$On \cap M$ is a countable ordinal. That is why every uncountable cardinal in M is a countable ordinal in V.

The notion "B is a Boolean algebra" is absolute for M. But the completeness of a Boolean algebra is not an absolute notion. If an algebra $B \in M$ is complete in the model M then we say that B is M–complete.

<u>3.17. Extensions</u>. If M and N are transitive models, we say that N is an extension of the model M (or that M is extended to the model N) if both models have the same ordinals, hence $On \cap M = On \cap N$, and $M \subseteq N$.

<u>3.18. Generic sets – a motivation</u>. Suppose a transitive ground model M and its extension to a transitive model N are given. If $N - M \neq o$, consider an $\in$–minimal set σ from the class $N - M$. For it we have $\sigma \in N$ and $\sigma \subseteq M$. We shall show that for every set $\sigma \in N$ which is a subset of M, there is a set $a \in M$ such that $\sigma \subseteq a$. It is sufficient to set $a = (V_\alpha)^M$, where α is the rank of the set σ (with respect to $\in$). The rank function is absolute and for $\sigma \in N$, $\text{rank}(\sigma) \in On^M$, hence $\sigma \subseteq V_\alpha^M \in M$.

Let us take a nonempty set $\sigma \in N$ with $\sigma \subseteq M$. We know that there is an $a \in M$ with $\sigma \subseteq a$. Consider the images of the set σ under all relations from the ground model

$$Ob(\sigma, M) = \{ r''\sigma : r \text{ is a relation}, r \in M\}.$$

For arbitrary $r \in M$ the image $r''\sigma$ is an element of N and a subset of M. Moreover, any $x \in M$ can be obtained as an image of the set σ, it suffices to take the relation $r = \{y\} \times x$ for some $y \in \sigma$. Obviously $\sigma \in Ob(\sigma,M)$, because the identity on a is in M. We get

$$M \subseteq Ob(\sigma,M) \subseteq \{\rho \in N : \rho \subseteq M\}.$$

The need for simplicity forces us to ask the question whether there is an extension N of a given model M which is determined by the ground model M and one set $\sigma \in N$, $\sigma \subseteq M$ such that

$$Ob(\sigma, M) = \{\rho \in N : \rho \subseteq M\}.$$

In this case all subsets of the ground model which are elements of the extension can be obtained as images of one set σ. Let us have a look at what conditions such a set σ should satisfy.

The class $\{\rho \in N : \rho \subseteq M\}$ is closed under unions and differences of sets. For any σ the class $Ob(\sigma, M)$ is closed under unions because $r_1'' \sigma \cup r_2'' \sigma = (r_1 \cup r_2)'' \sigma$. In general, the class $Ob(\sigma, M)$ is not closed under differences. This leads to the definition of a generic set.

3.19. Generic sets. Let M be a transitive model of ZF. If $\sigma \subseteq a \in M$, then we say that σ is a generic set over M if the class $Ob(\sigma, M)$ is closed under differences.

Generic sets are closely connected to ultrafilters on Boolean algebras.

3.20. Definition. Let B be a Boolean algebra in a transitive model M. We say that an ultrafilter G in B is generic over M if G has a nonempty intersection with every set H of the model M which is a dense subset of the algebra B.

Notice that an ultrafilter G in an M–complete algebra $B \in M$ is generic over M iff for every set $c \subseteq B$ from M

(7) $\quad c \cap G \neq o \quad - \quad \bigvee c \in G$

or dually

$$c \subseteq G \quad - \quad \bigwedge c \in G.$$

If it is clear what our ground model is we shall simply say a generic set or a generic ultrafilter.

3.21. Similar sets. The sets $\sigma, \rho \subseteq M$ are said to be similar if there are relations $r, s \in M$ such that $r'' \sigma = \rho$ and $s'' \rho = \sigma$.

Using composition of relations we get:

3.22. Lemma. Every set ρ which is similar to a generic set σ is also generic, and $Ob(\sigma,M) = Ob(\rho,M)$.

3.23. Theorem. (Vopěnka, Balcar). Canonical representation of generic sets. Let M be a transitive model of ZF and let $\sigma \subseteq a \in M$. Then σ is a generic set over M if and only if it is similar to a generic ultrafilter G in an M–complete Boolean algebra $B \in M$.

Proof. Suppose that $\sigma \subseteq a \in M$ and that σ is a generic set over M. The set

$$\rho = \{ u \in M : u \subseteq a \quad \& \quad u \cap \sigma \neq 0\}$$

lies in $Ob(\sigma,M)$, because for the relation $q = \{ <x,u> : x \in u \quad \& \quad u \in \mathcal{P}^M(a)\}$ we have $q''\sigma = \rho$. It is obvious that $\mathcal{P}(a - \sigma) \cap M = \mathcal{P}^M(a) - \rho$. Because $\mathcal{P}^M(a) \in M$ and the class $Ob(\sigma,M)$ is closed under differences, we have $\mathcal{P}(a-\sigma) \cap M \in Ob(\sigma,M)$. Let us choose a relation $r \in M$ such that

$$r''\sigma = \mathcal{P}(a-\sigma) \cap M.$$

We can assume that $Dom(r) \subseteq a$ and $Rng(r) \subseteq \mathcal{P}^M(a)$. From the relation r we shall build in M a symmetric and antireflexive relation $s \subseteq a \times a$ such that

(i) for each $x \in \sigma$ we have $s''\{x\} \cap \sigma = 0$.

(ii) for every set $c \subseteq a - \sigma$ from the model M there is an $x \in \sigma$ such that $c \subseteq s''\{x\}$.

Let s_1 be the relation on the set a defined by

$$s_1 = \{<x,y> : x,y \in a \;\&\; x \neq y \;\&\; y \in \bigcup r''\{x\}\}.$$

Let $s = s_1 \cup s_1^{-1}$. Obviously, s is a symmetric and antireflexive relation and $s \in M$. Let $z \in \sigma$ be arbitrary. Then $r''\{z\} \subseteq \mathcal{P}(a-\sigma) \cap M$ and hence $s_1''\{z\} \subseteq a - \sigma$. So for every $z \in \sigma$, the set $s_1''\{z\}$ is disjoint from σ. Suppose that for the relation s and for some $x \in \sigma$ condition (i) does not hold. Then for $y \in s''\{x\} \cap \sigma$ we have $y \in \sigma$ and either $<x,y> \in s_1$ or $<y,x> \in s_1$. The sets $s_1''\{x\}$ and $s_1''\{y\}$ are disjoint from σ, therefore x and y cannot be both in σ. We showed (i). Now we prove (ii). Because $\mathcal{P}(a-\sigma) \cap M = r''\sigma$, for any set $c \in M$ such that $c \subseteq a - \sigma$, there is $x \in \sigma$ with $c \in r''\{x\}$. It follows that $c \subseteq s_1''\{x\} \subseteq s''\{x\}$, and hence (ii) holds. Using the relation s we define a quasiordering $\leq$ on the set a by

$$x \leq y \rightarrow s''\{x\} \supseteq s''\{y\}.$$

The relation $\leq$ lies in M and we shall show that the following holds

(8) for every $x,y \in \sigma$ there is an $z \in \sigma$ such that $z \leq x$ and $z \leq y$,

(9) if $x \in \sigma$ and $x \leq y$ then $y \in \sigma$.

(10) if the set $c \in M$ is such that $c \subseteq a - \sigma$ then there is an $x \in \sigma$ which is disjoint (with respect to the quasiordering) with all elements of the set c.

If (8)–(10) hold then we say that σ is a generic filter over M in the quasiordered set $< a, \leq >$.

If $x,y \in \sigma$, then by (i) we have $s''\{x\} \cup s''\{y\} \subseteq a - \sigma$, and hence by (ii) there is an $z \in \sigma$ with $s''\{z\} \supseteq s''\{x\} \cup s''\{y\}$. This means that $z \leq x$ and $z \leq y$, so (8) holds. Let $x \in \sigma$ and $x \leq y$. Then $s''\{y\} \subseteq s''\{x\} \subseteq a - \sigma$. If $y \notin \sigma$ then by (ii) there exists an $z \in \sigma$ such that $s''\{z\} \supseteq s''\{x\} \cup \{y\}$. So $< z,y > \in s$ and from the symmetry follows $< y,z > \in s$, hence $z \in a - \sigma$ which is a contradiction. It remains to verify (10). Let $c \in M$ and $c \subseteq a - \sigma$. By (ii) there is $z \in \sigma$ such that $c \subseteq s''\{z\}$. We shall show that z is disjoint from all elements of the set c, i.e. for no $x \in c$ is there a $y \in a$ such that

(11) $s''\{y\} \supseteq s''\{z\} \cup s''\{x\}$.

Suppose that for some $x \in c$ and $y \in a$ (11) holds. Because $< z,x > \in s$, by (11) also $< y, x > \in s$ and by symmetry we get $< x,y > \in s$. From this and from (11) it follows that $< y,y > \in s$, and this is a contradiction because s was antireflexive.

We verified that σ is a generic filter in the quasiordered set $< a,\leq >$. Now, from this quasiordering we pass to a Boolean algebra. It is easily seen that Theorem 2.17 also holds for quasiordered sets. So every nonempty quasiordered set determines a complete Boolean algebra. It follows that in the model M there is an M–complete Boolean algebra B and a mapping $j \in M$ such that j maps the set a onto a dense subset of B and j preserves (quasi) ordering and disjointness. We shall show that

$$G = \{u \in B : (\exists x \in \sigma)\ j(x) \leq u\}$$

is a generic ultrafilter in the algebra B. From property (8) of the set σ it follows that G is a filter. If $H \in M$ is a dense set in B, let

$$c = \{x \in a : j(x) \leq u \text{ for some } u \in H\}.$$

The set c lies in M. We shall show that $c \cap \sigma \neq o$. If not, then by (10) there is a $y \in \sigma$ disjoint with all elements of c. Because the mapping j preserves ordering and disjointness, $j(y)$ is disjoint with all elements of H, and this is impossible. Let $x \in \sigma \cap c$ and $u \in H$ be such that $j(x) \leq u$, then $u \in G$. We showed that every set $H \in M$ which is dense in B has a common element with G. Now it follows that G is an ultrafilter in B, because for any $v \in B$ the set

$$\{u \in B - \{\mathbf{0}\} : u \leq v \text{ or } u \leq -v\}$$

in dense, hence v or $-v$ lies in G. The sets σ and G are similar, because $\sigma = j^{-1}{''}G$ and $G = q''\sigma$, where $q = \{< x,u > : j(x) \leq u\}$. We showed that every generic set $\sigma \subseteq M$ is similar to some generic ultrafilter.

It remains to show that every generic ultrafilter in an M–complete algebra $B \in M$ is a generic set over M. We shall prove it in the next paragraph.

<u>3.24. Names of subsets.</u> Let B be a Boolean algebra and a an arbitrary set. Every function

(12) $\qquad f : a \longrightarrow B$

is called a <u>name of a subset</u> of the set a.

A name of a subset is in fact a generalized characteristic function of a set. An element

$x \in a$ belongs to a subset of a with probability not only 0 and 1, but its value can be any element $0_B \leq f(x) \leq 1_B$. Given an ultrafilter G in B the name (12) determines uniquely the following set

$$f_G = \{ x \in a : f(x) \in G\} = f^{-1}"G.$$

It is obvious that the set f_G depends on the choice of the ultrafilter G. The function f is therefore a name of a subset of the set a and its meaning is determined only by the ultrafilter G as the set f_G. We introduced names of subsets generally in the whole universal class, but we shall use them usually inside some transitive model. For an M–complete Boolean algebra $B \in M$ and a set $a \in M$ we shall consider only names (12) which are elements of the model M.

Suppose that G is a generic ultrafilter in an M–complete Boolean algebra $B \in M$. We shall show

(13) $$\mathrm{Ob}\ (G,M) = \{f_G : f \text{ is a name}, f \in M\}.$$

Let $\rho = r"G$, r is a relation from M. We can assume that $\mathrm{Dom}\ (r) \subseteq B$. Let $a = \mathrm{Rng}\ (r)$. From the M–completeness of the algebra B it follows that for every $x \in a$, $f(x) = \bigvee r^{-1}"\{x\}$ is an element of B. This defines a function $f : a \to B$ and $f \in M$.
By (7), $x \in r"G$ if and only if $f(x) \in G$. This means that $f^{-1}"G = r"G$ or $\rho = f_G$. We showed that in (13) the inclusion $\subseteq$ holds; the opposite inclusion is obvious.

Now it is easy to show that for any generic ultrafilter G in B, the class of images Ob(G,M) is closed under differences, and so by 3.19 G is really a generic set over M. (13) implies that instead of relations we can consider names. Let $f : a - B$ and $g : b - B$ be from M. For every $x \in a$ let us set $h(x) = f(x) -_B g(x)$. Because G is an ultrafilter, for every $x \in a$ we have

$$h(x) \in G \leftrightarrow f(x) \in G \ \&\ g(x) \notin G.$$

This means that $h_G = f_G - g_G$, and so the class Ob(G,M) is closed under differences. The proof of Theorem 3.23 is complete.

<u>3.25. Example.</u> Let B be an M–complete algebra in a model M. A generic ultrafilter G in B lies in M iff $G = \{x \in B : u \leq x\}$ for some atom u of the algebra B. This follows from (7). If B is atomless then no generic ultrafilter in B is an element of M.

<u>3.26. Generic Extension Theorem.</u> If M is a transitive model of ZF and G a generic set over M then there is an extension N of the model M such that

(i) $Ob(G,M) = \{\rho \in N : \rho \subseteq M\}$,

(ii) if N_1 is a transitive model of ZF such that $M \subseteq N_1$ and $G \in N_1$ then $N \subseteq N_1$.

Moreover, if M is a model of ZFC then also N is a model of ZFC.

<u>3.27. Generic extensions.</u> An extension N with properties (i) and (ii), which exists by Theorem 3.26, is called a generic extension of the model M and is denoted by M[G].

If G is a generic set over M, the extension M[G] is determined uniquely by the ground model M and the generic set G, because M[G] is the smallest extension of the model M in which G lies. By 3.23 we can assume that the generic set G is a generic ultrafilter in an M–complete Boolean algebra from M.

The Generic Extension Theorem has two aspects. The first one is metamathematical and it ensures that for every generic set G over M there is an extension M[G] which is a model of ZF or ZFC. The proof of this fact is the same for all generic sets and we shall do it later. The second aspect is more mathematical. It says that the choice of a generic set can influence whether some set–theoretic principle (hypothesis) holds in the generic extension or not. In this way the question whether some hypothesis is consistent with the axioms of set theory is transfered to a combinatorial problem, whether, in the ground model M, there is a Boolean algebra with some structural properties. The second aspect we shall illustrate later.

First we have to deal with the existence of generic sets, in particular generic ultrafilters in Boolean algebras. For countable models it is easy to obtain many generic sets.

3.28. Theorem. If M is a countable transitive model and $B \in M$ is a Boolean algebra then for every nonzero element $v \in B$ there exists a generic ultrafilter $G \subseteq B$ over M such that $v \in G$.

The theorem follows from the following result on Boolean algebras.

3.29. Theorem. (Rasiowa, Sikorski, 1950). For any countable family $< H_n : n \in \omega >$ of dense subsets of a Boolean algebra B and for any nonzero $v \in B$ there is an ultrafilter F in B such that F has common elements with each set H_n and $v \in F$.

Proof. By recursion we construct a sequence $< v_n : n \in \omega >$ of elements from B such that $v = v_0 \geq v_1 \geq \dots$ and for every n, $v_{n+1} \in H_n$. The set $\{v_n : n \in \omega\}$ has the finite intersection property and every ultrafilter in B which extends this set is as required.

Let us notice that by Stone Duality, Theorem 3.29 is an algebraic dual to Baire Category Theorem (1.29(c)) for Boolean topological spaces.

Proof of Theorem 3.28. Recall that every Boolean algebra in M is a Boolean algebra in V. The model M is countable and so it follows that there are only countably many dense subsets of B which lie in M. Let us enumerate them as $< H_n : n \in \omega >$. By 3.29 there is an ultrafilter G in B such that $v \in G$ and $H_n \cap G \neq \emptyset$ for every n. The ultrafilter G is generic over M.

3.30. What holds in a generic extension M[G], if an algebra $B \in M$ and in it a generic ultrafilter G are given, depends essentially on properties of the algebra B and on the fact what elements the ultrafilter G contains. We shall show how we can ensure, by choosing a Boolean algebra, that in the generic extension one of the following principles holds:

(i) the continuum hypothesis,

(ii) the diamond principle (III.2.40).

(iii) the negation of the continuum hypothesis.

(iv) every almost disjoint family on ω_1 has cardinality at most ω_2 and $2^{\omega_1} = \omega_3$.

The usual proof of the consistency of the continuum hypothesis uses the universe of constructible sets in which GCH holds. We shall use a generic extension in order to obtain a model in which the continuum hypothesis holds. Moreover, we shall show something of the basic technique.

We shall assume that our ground model M is a countable transitive model of ZFC. Condition 3.28 then guaranties that for any M–complete Boolean algebra $B \in M$ there exists a generic ultrafilter G in B and that M[G] is a model of ZFC, too.

3.31. Consistency of the continuum hypothesis. If M is a countable transitive model of ZFC then there is a generic extension $N = M[G]$ in which $2^{\omega} = \omega_1$.

Proof. Since M is a transitive model the set of all natural numbers is absolute, and hence $\omega^M = \omega$. In M the Axiom of Choice holds and therefore it is possible to enumerate all subsets of the natural numbers of the model M by ordinals smaller than $(2^{\omega})^M$ and some enumeration $< A_{\alpha} : \alpha < (2^{\omega})^M >$ exists in M. We have $(2^{\omega})^M \geq \omega_1^M$. We shall use the fact that cardinals need not be absolute for M, N. It is sufficient to find a generic set G so that in the generic extension M[G] we have

(i) there is a 1–1 mapping of the ordinal $(2^{\omega})^M$ onto ω_1^M.

(ii) there is no new subset of the natural numbers, i.e. there is no $X \subseteq \omega$ such that $X \in M[G]-M$; in other words $\mathcal{P}^M(\omega) = \mathcal{P}^{M[G]}(\omega)$.

(iii) ω_1^M remains a cardinal, i.e. $\omega_1^M = \omega_1^{M[G]}$.

From conditions (i) – (iii) it follows that in M[G] we can enumerate all subsets of the natural numbers by ordinals smaller than $\omega_1^{M[G]}$, and hence in M[G] holds $2^{\omega} = \omega_1$.

Let us notice that by (iii), condition (i) is equivalent to the following: in M[G] there exists a mapping ρ of the cardinal ω_1^M onto $(2^{\omega})^M$. Approximations of such a mapping ρ in the ground model M are all functions $f \in M$ which are countable in M and with $\mathrm{Dom}(f) \subseteq \omega_1^M$

and $\mathrm{Rng}(f) \subseteq (2^\omega)^M$. This leads us to the following ordered set $F(\omega_1, 2^\omega, \omega_1)$ from 2.25(f) and to the complete Boolean algebra $C(\omega_1, 2^\omega, \omega_1)$ associated to it, which both are considered inside the model M. Hence $C = C^M(\omega_1, 2^\omega, \omega_1)$ is an M–complete algebra in M and $F = F^M(\omega_1, 2^\omega, \omega_1)$ is a dense subset. Let G be an ultrafilter in C which is generic over M. In the generic extension M[G] we define

$$\rho = \bigcup \{f \in F : f \in G\}.$$

The set ρ is defined by $F \in M$ and $G \in M[G]$, therefore $\rho \in M[G]$. We shall show that ρ is a mapping of $\omega_1{}^M$ onto $(2^\omega)^M$. Because G is a filter and the set F is ordered by the reverse inclusion, any two functions $f_1, f_2 \in F \cap G$ must be compatible and $f_1 \wedge f_2 = f_1 \cup f_2 \in G$, hence $f_1 \cup f_2$ is again in $F \cap G$. It follows that ρ is a mapping. We shall use the fact that G is generic to show that ρ is defined on the whole $\omega_1{}^M$ and that $\mathrm{Rng}(\rho) = (2^\omega)^M$. First we verify that for every $\alpha < \omega_1{}^M$ the set

$$D_\alpha = \{f \in F : \alpha \in \mathrm{Dom}(f)\}$$

is a dense subset of the algebra C. Given α, it is sufficient to show that every function $g \in F$ has an extension in D_α. If $\alpha \in \mathrm{Dom}(g)$ then immediately $g \in D_\alpha$. If $\alpha \notin \mathrm{Dom}(g)$ then the extension $g \cup \{<\alpha, 0>\}$ is in D_α. The set D_α is dense in C. Hence for every $\alpha < \omega_1{}^M$ there is an $f \in D_\alpha \cap G$ and this means that $f \in \rho$ and $\alpha \in \mathrm{Dom}(\rho)$. It follows $\mathrm{Dom}(\rho) = \omega_1{}^M$.

Similarly we shall verify that for every $\beta < (2^\omega)^M$, the set

$$R_\beta = \{ f \in F : \beta \in \mathrm{Rng}(f)\}$$

is a dense subset of C. Each $g \in F$ is a countable function in M, hence $\omega_1{}^M - \mathrm{Dom}(g) \neq \emptyset$. Whenever $g \notin R_\beta$ we take the least $\alpha \in \omega_1{}^M - \mathrm{Dom}(g)$ and set $f = g \cup \{<\alpha, \beta>\}$. This extension of g is in R_β and therefore the set R_β is dense. Hence for every $\beta < (2^\omega)^M$ there exists an $f \in R_\beta \cap G$ and this means that $f \subseteq \rho$ and $\beta \in \mathrm{Rng}(\rho)$. We have shown that ρ maps $\omega_1{}^M$ onto $(2^\omega)^M$. From this and from the fact that $\omega_1{}^M \leq (2^\omega)^M$ it follows that in M[G] there is a one–to–one mapping of $(2^\omega)^M$ onto ω_1. We have proved (i).

To prove (ii) and (iii) we shall use the distributivity of the algebra C. The ordered set $F(\omega_1, 2^\omega, \omega_1)$ is ω_1–closed (2.33) and this implies that the complete Boolean algebra $C(\omega_1, 2^\omega, \omega_1)$ is (ω,∞)–distributive. Hence the algebra C is (ω,∞)–distributive in M. The following statement 3.32 implies that in M[G] there is no new mapping of ω into any ordinal $\alpha < On^M$. Hence in M[G] there is no new subset of the natural numbers (i.e. no function from ω into $\{0,1\}$) either, nor a mapping from ω onto $\omega_1{}^M$. Hence $\omega_1{}^M$ remains a cardinal in the extension M[G]. We have proved (ii) and (iii) and hence in M[G] the continuum hypothesis holds.

<u>3.32. Theorem</u>. Let κ and $\lambda \geq 2$ be cardinals in a model M. Let in M, B be a complete (κ,λ)–distributive Boolean algebra. If G is an ultrafilter in B which is generic over M then in M[G] there exists no new mapping from κ to λ. In particular no new subset of the cardinal κ is added to M[G], hence $\mathcal{P}^M(\kappa) = \mathcal{P}^{M[G]}(\kappa)$ and all cardinals less than or equal to κ^+ in the model M are also cardinals in the extension M[G].

<u>Proof</u>. Let B be a complete (κ,λ)–distributive algebra in M and let G be a generic ultrafilter over M. Let $\rho : \kappa \to \lambda$ be any mapping in the generic extension M[G]. We want to show that $\rho \in M$. Because $\rho \subseteq \kappa \times \lambda \in M$, the set ρ is a subset of the ground model M, and by (13) there is therefore a name $f : (\kappa \times \lambda) \to B$ such that $f \in M$ and $\rho = f_G$.

The fact that f is a name of a mapping enables us to construct a new name $v : \kappa \times \lambda \to B$ in M such that $v_G = \rho$ and such that for every $\alpha < \kappa$ the family $< v(\alpha,\beta) : \beta < \lambda >$ consists of pairwise disjoint elements of the algebra B and its join is 1. We construct the name v in M. For each $\alpha < \kappa$ and $\beta < \lambda$ let us set

$$v(\alpha, \beta) = f(\alpha,\beta) - \bigvee \{f(\alpha,\gamma) : \gamma < \beta\}.$$

It is obvious that $v(\alpha,\beta) \leq f(\alpha,\beta)$. Because $\rho = f_G$ is a mapping, for every $\alpha < \kappa$ there exists a unique $\beta_\alpha < \lambda$ such that $f(\alpha,\beta_\alpha) \in G$. From this and from Property (7) of generic ultrafilters it follows that

$$\bigvee \{f(\alpha,\gamma) : \gamma < \beta_\alpha\} \notin G \text{ and } v(\alpha,\beta_\alpha) \in G.$$

We have verified that for any $\alpha < \kappa$ and $\beta < \lambda$

$$(14) \qquad v(\alpha,\beta) \in G \quad — \quad f(\alpha,\beta) \in G.$$

hence $v_G = f_G = \rho$. For every $\alpha < \kappa$ the family $< v(\alpha,\beta) : \beta < \lambda >$ consists of pairwise disjoint elements. We may assume that $\bigvee \{v(\alpha,\beta) : \beta < \lambda\} = 1$. If not, then the complement of the join can be added to $v(\alpha,0)$. This spoils neither the disjointness nor (14). We get a (κ,λ)–matrix $< v(\alpha,\beta) : \alpha < \kappa, \beta < \lambda >$ of elements of the algebra B such that the nonzero elements of each row form a partition of unity. We have κ partitions and each of them is of size at most λ. From the distributivity of the algebra B it follows that there exists a common refinement $Q \in M$ which is also a partition. By (7), $Q \cap G \neq o$ and because the elements of Q are pairwise disjoint there is exactly one element $u \in Q$ which belongs to G. For this element we define

$$h = \{< \alpha,\beta > : \alpha < \kappa, \beta < \lambda, u \leq v(\alpha,\beta)\}.$$

It is easily seen that h is a mapping of κ to λ and $h \in M$, because it is defined inside M without using a generic ultrafilter. Because $h = \rho$, ρ lies in the ground model, too.

We have showed that, in the extension M[G], no new mapping of κ into λ is added. It follows that for any set $x \in M$ of size κ (in M) no mapping of x into λ is added in M[G]. In particular no function from κ into $\{0,1\}$ is added and this means that no new subset of the cardinal κ is added. Hence no new relation on κ is added either.

Let us note that every cardinal of the extension M[G] is a cardinal of the ground model M. If κ is a cardinal in M, then it follows that between the cardinals κ and $(\kappa^+)^M$ there is no cardinal in M[G]. We shall show that every $\nu \leq \kappa^+$, which is a cardinal in μ, also is a cardinal in M[G]. Suppose that some cardinal $\nu \leq \kappa^+$ in M is not a cardinal in M[G]. This means that there exists an ordinal $\alpha < \nu$ and a one–to–one mapping $\sigma \in M[G]$ of the ordinal α onto ν, with $\alpha \leq \kappa$. The relation

$$R = \{ < \xi,\eta > : \xi, \eta < \alpha \ \& \ \sigma(\xi) < \sigma(\eta)\}$$

is a well–ordering of the set α in type ν in $M[G]$ and $R \subseteq \kappa \times \kappa$. Because the type of well–orderings is absolute and $\alpha < \nu$, R cannot lie in M, and this is a contradiction. The proof is finished.

The absoluteness of other cardinals is given by the saturatedness of the Boolean algebra (2.4).

3.33. Theorem. Let B be a complete Boolean algebra in M and let B be κ–saturated in M. If G is an ultrafilter in B which is generic over M, then all cardinals of the model M which are bigger than or equal to κ remain cardinals in the extension $M[G]$.

In particular, if sat $(B) \leq \omega_1^M$ then all cardinals and their cofinalities are absolute for M and $M[G]$.

Proof. Let in M sat $(B) = \kappa$. Let λ be a cardinal in M and $\lambda \geq \kappa$. Consider any mapping $\rho : \nu \to \lambda$ from the generic extension $M[G]$ defined on some $\nu < \lambda$. To show that λ is also a cardinal in $M[G]$ it is sufficient to show that ρ does not map ν onto the whole of λ. Because ρ is a mapping, $\rho \subseteq \nu \times \lambda \in M$, and from the proof of Theorem 3.32 we know that there is a name $v : (\nu \times \lambda) \to B$ such that the rows of the matrix $< v(\alpha,\beta) : \alpha < \nu, \beta < \lambda >$ consist of pairwise disjoint elements and for every $\alpha < \lambda$ we have $\rho(\alpha) = \beta$ if and only if $v(\alpha,\beta) \in G$. For every $\alpha < \nu$ let $A_\alpha = \{\beta < \lambda : v(\alpha,\beta) \neq 0\}$.

The family $< A_\alpha : \alpha < \nu >$ and the set $A = \bigcup \{A_\alpha : \alpha < \nu\}$ lie in M. From the saturatedness of the algebra B it follows that $|A_\alpha| < \kappa$ for every $\alpha < \nu$. Because $\rho(\alpha) \in A_\alpha$ we can say that $< A_\alpha : \alpha < \nu >$ is a pipe in M with diameter everywhere smaller than κ, through which the mapping ρ runs. If $\nu \geq \kappa$ then, in M, we have $|A| \leq \nu$, $\kappa \leq \nu < \lambda$, and therefore $Rng(\rho) \subseteq A \neq \lambda$. If $\nu < \kappa$ then from the regularity of κ (see 2.5) we get $|A| < \kappa \leq \lambda$, hence again $Rng(\rho) \neq \lambda$. We showed that λ remains a cardinal in $M[G]$.

Now suppose sat (B) $\le \omega_1^{M}$. This means that every disjoint set in B which lies in M is at most countable in M. Let us realize that absoluteness of cofinality implies absoluteness of cardinals. If ξ is a limit ordinal with countable cofinality in M, then its cofinality is countable in M[G], too. Suppose that $cf^{M}(\xi) > \omega$ and $\nu < cf^{M}(\xi)$. Every mapping $\rho : \nu \to \xi$ in M[G] goes through some pipe $< A_\alpha : \alpha < \nu > \in M$, which has everywhere at most countable diameter in M. Because $|A| \le \omega$, $\nu < cf^{M}(\lambda)$ neither the set A nor $Rng(\rho) \subseteq A$ are cofinal subsets of the ordinal ξ. We have showed $cf^{M}(\xi) = cf^{M[G]}(\xi)$.

The methode used in the proofs of the two theorems above gives something more.

<u>3.34 Corollary</u>. Let B be a complete Boolean algebra in M with (in M) : sat (B) = κ and B is $(\nu,2)$–distributive for every $\nu < \kappa$. If G is a generic ultrafilter in B, then all cardinals and their cofinalities are absolute for M and M[G].

<u>3.35. Examples</u>. (a) Suppose that κ is an infinite cardinal in M. Consider the collapsing algebra $C = C(\omega,\kappa)$ in M, in which the set $F = \{f : n \to \kappa : n \in \omega\}$ ordered by reverse inclusion is dense (see 2.25(e)). If G is an ultrafilter in C which is generic over M then in M[G]

$$\rho = \bigcup \{f \in F : f \in G\}$$

is a mapping of ω onto κ. Hence, in M[G], ω can be mapped onto each infinite cardinal $\lambda \le \kappa$ in M. We say that λ is collapsed onto ω. Because $sat(C) = \kappa^{+}$ in M, all cardinals $\lambda \ge \kappa^{+}$ in M remain cardinals in the extension. In particular $\omega_1^{M[G]} = (\kappa^{+})^{M}$.

(b) Consider in M the algebra $C(\omega_2, 2^{\omega_1}, \omega_2)$ and let G be a generic ultrafilter on it. Similarly as in 3.31 we can prove that in M[G] we have $2^{\omega_1} = \omega_2$. Moreover, $\mathcal{P}^{M}(\omega_1) = \mathcal{P}^{M[G]}(\omega_1)$, $(2^{\omega_1})^{M}$ is collapsed onto ω_2^{M}, and all cardinals λ in M such that $\lambda \le \omega_2^{M}$ or $\lambda \ge ((2^{\omega_1})^{+})^{M}$ are absolute. If, moreover, the continuum hypothesis holds in M, then it holds in M[G], too.

<u>3.36. Consistency of the diamond principle $\diamond$</u>. Any countable transitive model M of ZFC has a generic extension N in which $\diamond$ holds.

<u>Proof</u>. In the extension N we want the following to hold: there exists a sequence of sets $A = < A_\alpha : \alpha < \omega_1 >$ such that for every α, $A_\alpha \subseteq \alpha$ and for any $X \subseteq \omega_1$ and any closed unbounded set C in ω_1 there exists $\alpha \in C$ for which $A_\alpha = X \cap \alpha$.

The usual idea is to construct the required object by its approximations in the ground model. Consider in M the set P of all sequences $p = < A_\xi : \xi < \alpha >$ of at most countable length $\alpha < \omega_1$ such that $A_\xi \subseteq \xi$ for all $\xi < \alpha$. The set P is ordered by reverse inclusion. Let B be the complete Boolean algebra in M determined by the partially ordered set P. Choose an ultrafilter G in B which is generic over M. We shall show that in the generic extension $N = M[G]$ the following holds:

(i) $\omega_1^{M} = \omega_1^{M[G]}$.

(ii) there is no new subset of any ordinal $\alpha < \omega_1$.

(iii) $A = \bigcup \{p \in P : p \in G\}$ is a $\diamond$–sequence.

The parts (i) and (ii) follow from 3.32 because the partially ordered set P is ω_1–closed in M, and B is therefore (ω,∞)–distributive.

We shall show (iii). Because G is a filter, any two sequences p, $q \in G$ must be compatible and this means that one is an extension of the other. It follows that A is a sequence. G is generic and so the length of A is ω_1, because for every $\alpha < \omega_1$ the set $\{p \in P : \alpha \in Dom(p)\}$ is dense in B. Next, it is clear that for every $\xi < \omega_1$ we have $A_\xi \subseteq \xi$. Let $X \subseteq \omega_1$ be a set from M[G], let us choose its name $f : \omega_1 \to B$ in M, then $f_G = X$. The algebra B is (ω,∞)–distributive and hence there is in M a family $< Q(\alpha) : \alpha < \omega_1 >$ of partitions of unity such that each $Q(\alpha)$ refines $\{\{f(\xi), -f(\xi)\} : \xi < \alpha\}$ and also for each $\alpha < \beta < \omega_1$, $Q(\beta)$ refines $Q(\alpha)$. Hence for each $u \in Q(\alpha)$ and for each $\xi < \alpha$ either $u \leq f(\xi)$ or u is disjoint from $f(\xi)$. Let $C \subseteq \omega_1$ be a closed unbounded set from M[G], let $g : \omega_1 \to B$ be its name in M, i.e. $g_G = C$. The set C is unbounded and this tells us that for every $\beta < \omega_1$ there exists a $\gamma > \beta$ such that $\gamma \in C$. In Boolean values it means by (7) that for every $\beta < \omega_1$

(15) $$w_\beta = \bigvee \{g(\gamma) : \beta < \gamma < \omega_1\} \in G.$$

It follows that

$$w = \bigwedge \{w_\beta : \beta < \omega_1\} \in G.$$

The element w is non–zero because it lies in G. From (15) it follows that for every $p \in P$, with $p \leq w$ and for every $\beta < \omega_1$ there is a $\gamma > \beta$ such that p and $g(\gamma)$ are compatible.

We are looking for an $\alpha \in C$ such that for the α–th element A_α of the sequence A we have $A_\alpha = X \cap \alpha$. To get this we shall construct a suitable dense set in the algebra $B|w$.

Choose a $p \in P$, $p \leq w$ of length α_0. By resursion on $n \in \omega$ we construct a sequence of extensions

(16) $$p = p_0 > p_1 > p_2 \ldots\ldots \qquad \text{from } P,$$

an increasing sequence of countable ordinals

(17) $$\alpha_0 < \gamma_0 < \alpha_1 < \gamma_1 < \alpha_2 < \gamma_2 < \ldots$$

and a decreasing sequence

(18) $$u_0 \geq u_1 \geq u_2 \geq \ldots. \qquad \text{of elements } u_n \in Q(\alpha_n).$$

If we have p_n and α_n, then there are $\gamma_n > \alpha_n$ and $u_n \in Q(\alpha_n)$ such that $p_n \wedge g(\gamma_n) \wedge u_n$ is nonzero: since $Q(\alpha_n)$ is a partition of unity, there exists $u_n \in Q(\alpha_n)$ such that $p_n \wedge u_n \neq 0$.

and because $p_n \leq p \leq w$, also $p_n \wedge u_n \leq w$ and hence it is compatible with some $g(\gamma_n)$ for $\gamma_n > \alpha_n$. Pick $p_{n+1} \in P$ such that the length α_{n+1} of the sequence p_{n+1} is bigger than γ_n and $p_{n+1} \leq p_n \wedge g(\gamma_n) \wedge u_n$. In this way we get (16), (17), (18). Let us set

$$\alpha = \sup \alpha_n = \sup \gamma_n, \qquad p_\omega = \bigcup\{p_n : n \in \omega\}, \quad Y_\alpha = \{\xi < \alpha : f(\xi) \geq p_\omega\}$$

and define

$$\varphi(p) = p_\omega \cup \{ < \alpha, Y_\alpha > \}.$$

The set $H = \{\varphi(p) : p \in P, p \leq w\}$ is obviously dense in $B|w$ and so $H \cap G \neq \emptyset$. Let $p \in P$ be such that $\varphi(p) \in G \cap H$. This means that $\varphi(p)$ is an initial segment of the sequence A. Let $\alpha + 1$ be the length of $\varphi(p)$, then $Y_\alpha = A_\alpha$.

Since, for every γ_n from (17), $g(\gamma_n) \geq p_{n+1} \geq \varphi(p)$, we have $\gamma_n \in C$. The set C is closed and so $\alpha = \sup \gamma_n \in C$. It remains to show that for the last member A_α of the sequence $\varphi(p)$ we have $A_\alpha = X \cap \alpha$. Let $\xi \in A_\alpha$, then $\xi < \alpha$ and $f(\xi) \geq p_\omega \geq \varphi(p) \in G$ and hence $\xi \in X \cap \alpha$. If $\xi < \alpha$ and $\xi \notin A_\alpha$ then from the definition of the elements u_n it follows that $f(\xi)$ is disjoint from $\varphi(p)$ and therefore $f(\xi) \notin G$. This means that $\xi \notin X \cap \alpha$. We have showed that $A_\alpha = X \cap \alpha$.

<u>3.37. Remark</u>. By III.2.41 $\diamond$ implies CH. Hence we have just showed the consistency of the continuum hypothesis for the second time. On the other hand, $\diamond$ holds also in our first model from 3.31. Given a model M, the algebras $C = C^M(\omega_1, 2^\omega, \omega_1)$ and B from 3.36, constructed in M, are isomorphic in M. A generic ultrafilter G_1 in C is by the isomorphism of the algebras transformed into a generic ultrafilter G_2 in B, hence G_1 and G_2 are similar generic sets. Hence $M[G_1] = M[G_2]$.

<u>3.38. The consistency of the negation of the continuum hypothesis</u>. Any countable transitive model M of ZFC has a generic extension N in which $2^\omega > \omega_1$. In particular, if in M we have $2^\omega = \omega_1$ then there exists a generic extension in which $2^\omega = \omega_3$.

<u>Proof</u>. We are looking for a generic extension N of the model M which adds many new subsets

of the set of all natural numbers. The characteristic functions of subsets of the set ω are approximated by finite sequences of zeros and ones.

Let us choose in M an infinite cardinal κ and consider in M the set $F = F(\kappa,2)$ from 2.25 (a), i.e. the set

$F = \{p: X \to \{0,1\} : X \subseteq \kappa, X \text{ finite}\}$

ordered by reverse inclusion. Let $C(\kappa)$ denote the complete Boolean algebra in M determined by the set F. For any generic ultrafilter G in $C(\kappa)$ we have a generic extension $N = M[G]$. Because in M we have sat $(C(\kappa)) = \omega_1$, by 3.33 all cardinals and their cofinalities are in M[G] the same as in M. We shall verify that in the extension $N = M[G]$ the following holds:

(i) $2^{\omega} \geq \kappa$,

(ii) $2^{\omega} = (\kappa^{\omega})^{M}$; the cardinality of the continuum in M[G] is equal to the cardinal κ^{ω} of the model M.

In M[G], let $\rho = \bigcup \{f \in F : f \in G\}$. Every finite sequence $p \in F$ has for any $\alpha < \kappa$ an extension $g \in F$ such that $\alpha \in Dom(g)$, therefore ρ is a function defined on the whole κ with values in $\{0,1\}$. To show (i) we have to show that in N there are at least κ subsets of ω. For every $\alpha < \kappa$ we define in N a function $\rho_\alpha : \omega \to \{0,1\}$ so that for every natural number n, $\rho_\alpha(n) = \rho(\alpha + n)$, where $\alpha + n$ means the sum of the ordinals α and n. We shall show that the functions ρ_α and ρ_β are different for different $\alpha, \beta < \kappa$. Let $\alpha, \beta < \kappa$ and let us construct a dense subset $H \in M$ of the algebra $C(\kappa)$. Let $p \in F$, $p : X \to \{0,1\}$ be arbitrary. Because X is a finite subset of a limit ordinal, there is a natural number n_p such that $\alpha + n_p$ and $\beta + n_p$ do not lie in X. We have $\alpha \neq \beta$, hence $\alpha + n_p \neq \beta + n_p$. If we set

$\varphi(p) = p \cup \{<\alpha + n_p, 0>, <\beta + n_p, 1>\}$,

then $p \subseteq \varphi(p) \in F$. Hence the set $H = \{\varphi(p) : p \in F\}$ is dense in $C(\kappa)$ and is defined in M, hence $H \cap G \neq \emptyset$. Let $p \in F$ be such that $\varphi(p) \in G$. Then $\varphi(p) \subseteq \rho$ and for n_p we have $\rho_\alpha(n_p) = 0$ and $\rho_\beta(n_p) = 1$. Hence ρ_α and ρ_β are different functions. We showed that in M[G] there are at least κ different characteristic functions on ω and therefore in M[G] we have $2^\omega \geq \kappa$.

To show (ii) consider the set

$$J = \{f \in M : f : \omega \to C(\kappa)\}$$

of all names in M of subsets of ω. We compute in M the cardinality of the set J. Since $|F| = \kappa$ and each element of the algebra $C(\kappa)$ can be expressed as a join of at most countably many elements of F, we get $|C(\kappa)| \leq \kappa^\omega$, hence $|J| \leq (\kappa^\omega)^\omega = \kappa^\omega$. From this and from the absoluteness of cardinals and from the equality $\mathcal{P}^{M[G]}(\omega) = \{f_G : f \in J\}$ we get $(2^\omega)^{M[G]} \leq (\kappa^\omega)^M$. By (i) and by the obvious fact that $(\kappa^\omega)^M \leq (\kappa^\omega)^{M[G]}$ we also get the opposite inequality; we showed (ii).

In particular, if in the ground model we have $2^\omega = \omega_1$, then in M we also have $\omega_3{}^\omega = \omega_3$. Hence in the extension $N = M[G]$ by the algebra $C(\omega_3)$ we have $2^\omega = \omega_3$.

3.39. From the construction of generic extensions for the negation of CH it is obvious that by choosing different cardinals κ we can obtain different extensions in which the continuum is arbitrarily large. In other words, the theory ZFC gives no upper bound for the cardinality of the set of all reals.

3.40. We shall come back to families of almost disjoint sets on ω, i.e. to AD families. By III.1.13 we know that there is always an AD family on ω_1 of size ω_2. If we assume CH then there exists an AD family on ω_1 of maximal possible cardinality 2^{ω_1} (III.1.15). We shall show that without any other assumptions, i.e. only in ZFC, one cannot prove this. To show it, we shall use a suitable generic extension and from combinatorics we shall use the arrow

$$(19) \qquad (2^{\omega_1})^+ \to (\omega_1^+)^2_{\omega_1}$$

from the Erdös – Rado Theorem (III.4.73).

3.41. Theorem. (Baumgartner 1976). Let M be a countable model of ZFC in which $2^{\omega_1} = \omega_2$. Suppose that N is a generic extension of the model M by an M–complete Boolean algebra $B \in M$ for which in M we have sat (B) $\leq \omega_1$. Then in the extension N the following holds:

(20) every AD family on ω_1 is of size at most ω_2.

In particular, if we choose in M the algebra $C(\omega_3)$ from 3.38 and in it a generic ultrafilter G then in M[G] we have (20) and $2^{\omega} = 2^{\omega_1} = \omega_3$.

Proof. Let B be an M–complete Boolean algebra in M, for which in M sat (B) $\leq \omega_1$. Let $N = M[G]$ where G is a generic ultrafilter on B. By 3.33 all cardinals and cofinalities are absolute for M and N. Let A be an arbitrary AD–family on ω_1 in the extension N. We want to show that in N we have $|A| \leq \omega_2$. We shall prove it by contradiction. Suppose the family A is of size ω_3 in N. We work in $N = M[G]$. Let us choose a one–to–one enumeration $< A_\alpha : \alpha < \omega_3>$ of the family A. Every set A_α is uncountable and $A_\alpha \subseteq \omega_1$. For $\alpha \neq \beta$ the intersection $A_\alpha \cap A_\beta$ is at most countable, therefore there exists a $\gamma < \omega_1$ such that $A_\alpha \cap A_\beta \subseteq \gamma$. We define a mapping $\rho : [\omega_3]^2 \to \omega_1$ such that

$$\rho(\{\alpha,\beta\}) = \min \{\gamma : A_\alpha \cap A_\beta \subseteq \gamma\}.$$

Here $\rho \subseteq [\omega_3]^2 \times \omega_1 \in M$, so there is a name $v : ([\omega_3]^2 \times \omega_1) \to B$ for ρ such that $v \in M$ and for every $\{\alpha,\beta\} \in [\omega]^3$ the family

(21) $$< v(\{\alpha,\beta\}, \gamma) : \gamma < \omega_1 >$$

consists of pairwise disjoint elements of the algebra B. We shall use the fact that the saturatedness of B is small. In the family (21) there are at most countably many nonzero elements and this means that for $\alpha \neq \beta$ there exists $\delta = f(\{\alpha,\beta\})$ such that for every $\gamma \geq \delta$ we have $v(\{\alpha,\beta\},\gamma) = \mathbf{0}$. Thus we obtained a mapping $f : [\omega_3]^2 \to \omega_1$ which is in M. By the assumption in M we have $2^{\omega_1} = \omega_2$ and hence by (19) $\omega_3 \to (\omega_2)^2_{\omega_1}$. This means that in M there are a set $X \subseteq \omega_3$ of size ω_2 and an ordinal $\xi < \omega_1$ such that for any two different $\alpha,\beta \in X$, $f(\{\alpha,\beta\}) = \xi$.

So, in $M[G]$ we have : ρ is everywhere smaller than f, because $\rho = v_G$ and for different $\alpha, \beta \in X$, $A_\alpha \cap A_\beta \subseteq \xi$. Consider the family $D = \{A_\alpha - \xi : \alpha \in X\}$. The sets A_α are uncountable and $\xi < \omega_1$, therefore D consists of ω_2 nonempty pairwise disjoint subsets of the cardinal ω_1, and this is a contradiction.

We proved that, in $M[G]$, ω_2 is an upper bound for the cardinality of families of almost disjoint sets on ω_1. In particular, the algebra $C(\omega_3)$ has in M saturatedness ω_1 and by 3.38 in the generic extension by this algebra C we have $2^\omega = \omega_3$. We also have $2^{\omega_1} = \omega_3$ because there are at most $\omega_3^{\omega_1} = \omega_3$ many names of subsets of the cardinal ω_1 in M.

3.42. By the Rasiowa – Sikorski Theorem, on every Boolean algebra there exists a filter (ultrafilter) which meets a given countable family of dense subsets. We know that whenever the Boolean algebra B is atomless we cannot demand that there exists (in V) a filter which meets all dense subsets of the algebra B. One of the combinatorial principles which ensures the existence of filters which meet also uncountably many dense subsets is Martin's axiom.

3.43. Definition. Martin's axiom. (i) Let κ be an uncountable cardinal. Martin's axiom for κ, in short MA_κ, is the following statement: If B is a Boolean algebra with sat (B) $\leq \omega_1$ then for every family $\{H_\alpha : \alpha < \kappa\}$ of at most κ dense subsets of the algebra B there exists a filter on B which meets every set H_α.

(ii) Martin's axiom MA is the assertion: $(\forall \kappa < 2^\omega)\ (MA_\kappa)$.

Let us notice that MA_ω is a particular case of Theorem 3.29, and so MA is a consequence of the continuum hypothesis. More interesting is MA_{ω_1} or MA + $\neg$ CH. In 3.47 we shall prove

$$MA_\kappa - \kappa < 2^\omega \text{ and } 2^\omega = 2^\kappa.$$

Hence from MA it follows that for every infinite $\kappa < 2^\omega$

$$(2^\omega)^\kappa = 2^\omega \text{ and } 2^\omega \text{ is a regular cardinal.}$$

3.44. The consistency of MA with the negation of CH was proved by Solovay and Tennenbaum (1971).

Theorem. Let M be a transitive model of ZFC and λ an uncountable cardinal in M for which in M we have $\lambda^{<\lambda} = \lambda$. Then there exists an M–complete Boolean algebra $B \in M$ such that in M we have sat $(B) = \omega_1$ and for every ultrafilter G in B which is generic over M in the extension M[G] we have MA + $2^{\omega} = \lambda$.

The proof of this statement uses iteration of generic extensions and can be found in the book of T. Jech (1978).

We shall show a few consequences of MA + $\neg$ CH.

3.45. Theorem. (MA_κ). Let $< A_\alpha : \alpha < \kappa >$ be a family of countable subsets of a given set X such that for any $\alpha < \beta < \kappa$, the intersection $A_\alpha \cap A_\beta$ is finite. Then for any $I \subseteq \kappa$ there is a set $S \subseteq X$ such that $|S| = |X|$ and for every $\alpha < \kappa$ we have

$$|A_\alpha \cap S| < \omega — \alpha \in I.$$

First we give an application of this theorem, its proof is in 3.50.

3.46. Lemma. (i) (MA_κ) Every infinite MAD family on ω has size bigger than κ.

(ii) (MA) Every infinite MAD family on ω has size 2^{ω}.

Proof. (i) Let $A = < A_\alpha : \alpha < \lambda >$ be an arbitrary infinite MAD family on ω. If $\lambda > \kappa$ then we are done. Suppose $\lambda \leq \kappa$. Because MA_κ implies MA_λ, we can replace in Theorem 3.45 the cardinal κ by the cardinal λ. If we set $I = \lambda$, $X = \omega$, then the set S from the theorem just mentioned is infinite, $S \subseteq \omega$ and is almost disjoint from all A_α's. Hence A is not maximal, a contradiction.

(ii) follows from (i).

3.47. Lemma. MA_κ implies $\kappa < 2^\omega$ and $2^\kappa = 2^\omega$.

Proof. The previous Lemma 3.46 implies that there exists an AD family $<A_\alpha : \alpha < \kappa>$ on ω of size κ. For every set $I \subseteq \kappa$ let us choose a set $S_I \subseteq \omega$ whose existence follows from Theorem 3.45. If $I, J \subseteq \kappa$ are different subsets then for α from the symmetric difference $I \Delta J$ the set A_α has a finite intersection with one of the sets S_I, S_J and an infinite intersection with the other one. This means that $S_I \neq S_J$. We obtained a one–to–one mapping of $\mathscr{P}(\kappa)$ into $\mathscr{P}(\omega)$ and hence $2^\kappa \leq 2^\omega$. From this and because $\omega \leq \kappa < 2^\kappa$, we have $\kappa < 2^\omega$ and $2^\kappa = 2^\omega$.

3.48. Strongly almost disjoint sets on ω_1. If $\mathscr{D}$ is an AD family on ω_1 then all sets from $\mathscr{D}$ are uncountable and any two different sets from $\mathscr{D}$ have an at most countable intersection. We ask whether there can exist an AD family on ω_1 which has cardinality at least ω_2 and any two sets from $\mathscr{D}$ are strongly almost disjoint, i.e. they have a finite intersection.

CH gives a negative answer. If we assume CH then there exists an enumeration $< X_\xi : \xi < \omega_1>$ of all countable subsets of the cardinal ω_1. If $< Y_\alpha : \alpha < \omega_2 >$ is any family of infinite subsets of the cardinal ω_1 then we can choose for every set Y_α a $\xi = f(\alpha)$ such that $X_\xi \subseteq Y_\alpha$. Then f cannot be a one–to–one mapping from ω_2 into ω_1 hence there are different α and β such that Y_α and Y_β contain the same countable subset and so the intersection $Y_\alpha \cap Y_\beta$ is infinite.

On the other hand, MA_{ω_1} gives an affirmative answer.

3.49. Theorem. (i) (MA_{ω_1}). There is a family of strongly almost disjoint subsets on ω_1 of size ω_2.

(ii) ($MA + \neg CH$). There is a family of strongly almost disjoint sets on ω_1 of size $\geq \omega_2$ which is a maximal AD family on ω_1.

Proof. (i) Let us take an AD family $< C_\alpha : \alpha < \omega_2 >$ on ω_1 of size ω_2. By III.1.13 such a family always exists. By recursion we shall construct a family $D = < D_\alpha : \alpha < \omega_2>$ so that for every $\alpha < \omega_2$, $D_\alpha \subseteq C_\alpha$. For any $\alpha < \omega_1$ let us set $D_\alpha = C_\alpha - \bigcup \{C_\beta : \beta < \alpha\}$.

Then $\{D_\alpha : \alpha < \omega_1\}$ consists of uncountable and pairwise disjoint sets. Suppose that for given ξ, $\omega_1 \leq \xi < \omega_2$, we have already constructed sets D_α for all $\alpha < \xi$ and we now construct the set D_ξ. Let us observe that $C_\xi \cap D_\alpha \subseteq C_\xi \cap C_\alpha$, and hence $C_\xi \cap D_\alpha$ is at most countable. Let

$$A = \{C_\xi \cap D_\alpha : \alpha < \xi \text{ and } |C_\xi \cap D_\alpha| = \omega\}.$$

It is obvious that $|A| \leq \omega_1$ and A consists of countable pairwise almost disjoint subsets of the set $X = C_\xi$. From MA_{ω_1} by 3.45 there exists an uncountable set $S \subseteq C_\xi$ which has a finite intersection with every set from A. If we set $D_\xi = S$, then for every $\alpha < \xi$, $D_\xi \cap D_\alpha$ is a finite set. The proof of (i) is finished.

Proof of (ii) is an easy modification of the proof of (i). We shall use a fixed enumeration $<Y_\alpha : \alpha < 2^\omega>$ of the set $[\omega_1]^{\omega_1}$ and MA_κ for every $\kappa < 2^\omega$. The set D_α is found as a subset of Y_α only in the case when $\{Y_\alpha\} \cup \{D_\beta : \beta < \alpha\}$ form an almost disjoint family.

By Theorem 3.44 we can obtain a model of $MA + 2^\omega = \omega_3$ using an algebra with saturatedness ω_1. So by 3.41 it is consistent to assume

$$MA + \neg CH + \text{"there is no MAD on } \omega_1 \text{ of size } 2^{\omega_1}\text{"}.$$

Therefore Theorem 3.49 (ii) does not hold if we replace "cardinality $\geq \omega_2$" by "cardinality 2^{ω_1}".

<u>3.50. Proof of Theorem 3.45</u>. Let $A = < A_\alpha : \alpha < \kappa >$ be a family of pairwise almost disjoint subsets of a set X. We may assume that $X = \bigcup \{A_\alpha : \alpha < \kappa\}$, and so $|X| \leq \kappa$. Consider the following set of functions

$$P = \{p : \mathrm{Dom}\,(p) = \bigcup \{A_\alpha : \alpha \in J\} \cup Y, \text{ where } J \subseteq I$$
$$\text{and } Y \subseteq X \text{ are finite sets, } \mathrm{Rng}\,(p) \subseteq \{0,1\} \text{ and } p^{-1}[\{1\}] \text{ is finite}\}$$

ordered by reverse inclusion. Let us observe that for every $p \in P$, $X - \mathrm{Dom}(p)$ is infinite and

Dom(p) is an at most countable set. Let $V(p) = p^{-1}[\{1\}]$, it is a finite set. Let B be the complete Boolean algebra determined by the ordered set P. We shall verify that $sat(B) \leq \omega_1$. If not then there is a family $< p_\alpha : \alpha < \omega_1 >$ of pairwise disjoint elements from P. Then $< V(p_\alpha) : \alpha < \omega_1 >$ is an uncountable family of finite sets. By the Δ–System Theorem (III.1.18) there exists an uncountable set of indices $L \subseteq \omega_1$ such that $< V(p_\alpha) : \alpha \in L >$ is a Δ–system with some root W. Let us pick $\alpha \in L$. Because $|Dom(p_\alpha)| \leq \omega$ and L is uncountable, there exists a $\beta \in L$, $\beta \neq \alpha$ such that $Dom(p_\alpha) \cap V(p_\beta) = W$. It follows that $p_\alpha \cup p_\beta \in P$ and $p_\alpha \cup p_\beta$ is an extension of both p_α and p_β. But this is a contradiction because p_α and p_β are disjoint. We showed that $sat(B) \leq \omega_1$ and so we can use MA_κ for the algebra B.

We are looking for a family C of dense subsets of the algebra B such that $|C| \leq \kappa$ and such that for a given ultrafilter G in B which meets all sets from C the mapping

$$\rho = \bigcup \{ p \in P : p \in G \}$$

is the characteristic function of the set $S = \rho^{-1}[\{1\}] \subseteq X$ we are looking for. Let us put into C every set

$$H(\alpha,n) = \{ p \in P : |A_\alpha \cap V(p)| \geq n \}$$

for $\alpha \in \kappa - I$ and natural number n. We shall verify that $H(\alpha,n)$ is dense. If $q \in P$ with $Dom(q) = \bigcup \{A_\beta : \beta \in J\} \cup Y$ then $\alpha \notin J$. Because the sets from A are almost disjoint, $A_\alpha - Dom(q)$ is infinite. Let $Y_n \subseteq A_\alpha - Dom(q)$ be an arbitrary set with n elements, then the extension $q \cup (Y_n \times \{1\})$ is in $H(\alpha,n)$ and so $H(\alpha,n)$ is a dense subset of the algebra B. The ultrafilter G meets all sets from C so there exists $p \in G \cap H(\alpha,n)$ and $p \subseteq \rho$. It follows that $|S \cap A_\alpha| \geq n$. Because n was arbitrary, the intersection $S \cap A_\alpha$ is infinite.

Next, we put into C the sets

$$H(\alpha) = \{p \in P : A_\alpha \subseteq Dom(p) \}$$

for every $\alpha \in I$. We shall show that $H(\alpha)$ is dense in B.

Let $q \in P$ with $\mathrm{Dom}(q) = \bigcup \{A_\beta : \beta \in J\} \cup Y$. If $\alpha \in J$ then $q \in H(\alpha)$.
If $\alpha \notin J$ then $A_\alpha \cap \mathrm{Dom}(q)$ is a finite set and $p = q \cup ((A_\alpha - \mathrm{Dom}(q)) \times \{0\})$ is an extension of q which lies in $H(\alpha)$. Hence $H(\alpha)$ is dense and $p \in H(\alpha) \cap G$ ensures that the intersection $S \cap A_\alpha \subseteq V_p$ is finite.

It remains to show that $|S| = |X|$. We distinguish two cases.

(i) The set X is countable. Let $\prec$ be a well ordering of type ω. For every $x \in X$ we put into C the set

$$D(x) = \{p \in P : (\exists y \succ x)(y \in V(p))\}.$$

The set D(x) is dense in B and $p \in D(x) \cap G$ ensures that in S there is an element bigger than x. It means that S is infinite and $|S| = |X|$.

(ii) The set X is uncountable. $\omega < \lambda = |X| \le \kappa$. We decompose $X = \bigcup \{X_\xi : \xi < \lambda\}$ into λ parts, each of cardinality ω_1. We put also into C all sets

$$D_\xi = \{ p \in P : V(p) \cap X_\xi \neq \emptyset \}$$

for $\xi < \lambda$. For any $q \in P$ and $\xi < \lambda$, X_ξ is uncountable and $X_\xi - \mathrm{Dom}(q)$ is nonempty. If we pick $x \in X_\xi - \mathrm{Dom}(q)$ then $q \cup \{< x, 1 >\} \in D(\xi)$ and so $D(\xi)$ is dense in B. It follows that S has common elements with every set of the partition X_ξ, hence $|S| = \lambda = |X|$.

The family C of dense sets constructed during the proof has cardinality at most κ and so all assumptions of Martin's axiom for κ are satisfied. End of the proof.

<u>3.5.1. Theorem</u>. MA_{ω_1} implies the Suslin hypothesis.

<u>Proof</u>. By III.3.64 the Suslin hypothesis is equivalent to the assertion that there is no Suslin ω_1–tree. To prove the theorem, we assume the contrary. Let $< T, \le >$ be a Suslin ω_1–tree. We may assume that T has no short branches and that it splits everywhere. Consider the complete Boolean algebra determined by the ordered set $< T, \ge >$. By 2.45 we know that

sat(B) = ω_1. For every $\alpha < \omega_1$ $H(\alpha) = \bigcup \{T_\beta : \beta \geq \alpha\}$ is a dense subset of the algebra B. Let G be a filter which meets all sets $H(\alpha)$; its existence follows from MA_{ω_1}. For every $\alpha < \omega_1$ there is exactly one element $x_\alpha \in T_\alpha \cap G$. If $\alpha < \beta < \omega_1$ then the elements x_α and x_β are compatible, which means that they are comparable in the tree T. We get an uncountable branch $< x_\alpha : \alpha < \omega_1 >$ in the tree T, a contradiction.

From now on we are working towards the proof of the Generic Extension Theorem 3.26.

3.52. Boolean universe. Suppose that M is an arbitrary transitive model of ZF and that $B \in M$ is an M–complete Boolean algebra. In 3.24 we introduced names of subsets and by (13) we know that for a generic ultrafilter G in B the set $x \in M[G]$ is the interpretation of some name from M if and only if $x \subseteq M$. If $M \neq M[G]$ then there must be sets in M[G] which are not subsets of the ground model M, for example the singleton {G}, and these sets cannot be obtained using the names introduced till now.

The Boolean universe $M^{(B)}$ consists of all functions $f \in M$ such that $Dom(f) \subseteq M^{(B)}$ and $Rng(f) \subseteq B$. Its recursive definition is similar to the construction of the cumulative hierarchy of sets. For $\alpha < On^M$ we define

$$M_0^{(B)} = \emptyset.$$

$$M_{\alpha+1}^{(B)} = \{a \in M : Dom\,(a) \subseteq M_\alpha^{(B)} \ \&\ Rng\,(a) \subseteq B\}.$$

$$M_\alpha^{(B)} = \bigcup \{M_\beta^{(B)} : \beta < \alpha\} \quad \text{for } \alpha \text{ limit}.$$

$$M^{(B)} = \bigcup \{M_\alpha^{(B)} : \alpha < On^M\}.$$

The (Boolean) rank of a function $a \in M^{(B)}$ is the ordinal

$$\mathrm{rank}_B\,(a) = \min\,\{\, \alpha : \mathrm{Dom}\,(a) \subseteq M_\alpha^{(B)}\}.$$

If $b \in \mathrm{Dom}(a)$ then $\mathrm{rank}_B\,(b) \leq \mathrm{rank}_B\,(a)$.

There exists a canonical embedding of the class M into the universe $M^{(B)}$ which is defined recursively on the cumulative rank by

$$\check{x} = \{< \check{y}, 1 > : y \in x\}.$$

Let us observe that the universe $M^{(B)}$ and the mapping $^{\vee} : M \to M^{(B)}$ are classes of the model M, and their recursive definition is not changed by relativization to M.

<u>3.53. Definition of M[G]</u>. The elements of the Boolean universe $M^{(B)}$ are suitable names for all sets of arbitrary generic extensions of the model M by the algebra B.

Let G be an ultrafilter on B which is generic over M. On the class $M^{(B)}$ we define a mapping i_G recursively on the Boolean rank of elements $a \in M^{(B)}$:

(22) $$i_G\,(o) = o.$$

$$i_G\,(a) = \{\, i_G(b) : b \in \mathrm{Dom}(a)\ \&\ a(b) \in G\}.$$

The set $i_G(a)$ is <u>the interpretation of the name a</u> determined by the ultrafilter G. We define

$$M[G] = \{i_G(a) : a \in M^{(B)}\}.$$

For every $x \in M[G]$ there exists a name $a \in M^{(B)}$ such that $x = i_G(a)$. It remains to show that M[G] is really a generic extension of the model M.

The following property of the class M[G] can be proved easily. By (22) we get that M[G] is a transitive class. For any $x \in M$ we have a canonical name $\check{x} \in M^{(B)}$ and by induction we can show that $i_G(\check{x}) = x$. Hence $M \subseteq M[G]$. We have $G \in M[G]$ because $B \in M$ and for the name $a = \{< \check{u}, u > : u \in B\}$, $i_G(a) = \{i_G(\check{u}) : u \in G\} = G$. The classes M and M[G] have the same ordinals. Because $M \subseteq M[G]$, $M \cap On \subseteq M[G] \cap On$. On the other hand, for any $a \in M^{(B)}$, $rank_B(a)$ is bigger than or equal to the cumulative rank of the set $i_G(a)$. Hence we have also the reverse inclusion.

Let N be a model of ZF such that $M \subseteq N$ and $G \in N$. For every $\alpha \in On^M$, $M_\alpha^{(B)} \in M$ and hence also $M_\alpha^{(B)} \subseteq N$. By induction we can show that $i_G | M_\alpha^{(B)} \in N$, and this means that for any $a \in M^{(B)}$, $i_G(a) \in N$, i.e. $M[G] \subseteq N$. We proved 3.26 (ii).

To show that M[G] is a model of ZF, eventually of ZFC, we need finer tools.

<u>3.54. Boolean values of formulas</u>. For any formula $\varphi(x_1, \ldots, x_n)$ of set theory we define in M a mapping $\|\varphi\|_M$, in short $\|\varphi\|$, which assigns to every n–tuple $a_1, \ldots, a_n \in M^{(B)}$ an element $\|\varphi(a_1, \ldots, a_n)\|$ of the algebra B. Here, the logical connectives ¬, & and ∨ will correspond to the Boolean operations of complement, meet and join in the algebra B, respectively. The Boolean operation corresponding to the implication is the following operation

$$u \Rightarrow v = -u \vee v \qquad (\varphi \to \psi \text{ is equivalent to } \neg\varphi \vee \psi).$$

and the quantifiers ∀ and ∃ correspond to the infinite operations $\bigwedge$ and $\bigvee$.

For any $a, b, a_1, \ldots, a_n \in M^{(B)}$ we define

(i) $$\|a=b\| = \Big(\bigwedge_{c \in Dom(a)} (a(c) \Rightarrow \bigvee_{d \in Dom(b)} b(d) \wedge \|d = c\|)\Big) \wedge$$

$$\wedge \Big(\bigwedge_{d \in Dom(b)} (b(d) \Rightarrow \bigvee_{c \in Dom(a)} a(c) \wedge \|c = d\|)\Big).$$

(ii) $\|a \in b\| = \bigvee_{d \in Dom(b)} b(d) \wedge \| d = a\|.$

(iii) $\|\neg\, \varphi(a_1, \ldots, a_n)\| = -\|\varphi(a_1, \ldots, a_n)\|.$

$\|(\varphi \,\&\, \psi)(a_1, \ldots, a_n)\| = \|\varphi(a_1, \ldots, a_n)\| \wedge \|\psi(a_1, \ldots, a_n)\|.$

$\|(\varphi \vee \psi)(a_1, \ldots, a_n)\| = \|\varphi(a_1, \ldots, a_n)\| \vee \|\psi(a_1, \ldots, a_n)\|.$

(iv) $\|(\exists x)\, \varphi(x, a_1, \ldots, a_n)\| = \bigvee \{\|\varphi(b, a_1, \ldots, a_n)\| : b \in M^{(B)}\}.$

$\|(\forall x)\, \varphi(x, a_1, \ldots, a_n)\| = \bigwedge \{\|\varphi(b, a_1, \ldots, a_n)\| : b \in M^{(B)}\}.$

Obviously the definition of $\|\varphi\|$ is recursive on the complexity of the formula φ. First we define a mapping

$$\| x = y \| : M^{(B)} \times M^{(B)} \to B$$

by a well–founded recursion with respect to the relation R on $M^{(B)} \times M^{(B)}$, where $< c,d > R < a,b >$ if and only if $c \in Dom(a)$ and $d \in Dom(b)$. It is clear that R is a well–founded relation and for every $< a,b >$, $\{ < c,d > : < c,d > R < a,b > \}$ is a set, because $c \in Dom(a)$ implies $rank_B(c) < rank_B(a)$. The mapping $\| x \in y \|$ is defined from the mapping $\| x = y \|$ explicitly.

<u>3.55. Lemma</u>. For every $a \in M^{(B)}$, $\| a = a \| = 1$.

<u>Proof</u>. By induction on the rank:

$$\|a = a\| = \bigwedge_{b \in Dom(a)} (a(b) \Rightarrow$$

$$\Rightarrow \bigvee_{c \in Dom(a)} a(c) \wedge \| c = b \|) \geq \bigwedge_{b \in Dom(a)} (-a(b) \vee (a(b) \wedge \| b = b \|)) =$$

$$= \bigwedge_{b \in Dom(a)} (-a(b) \vee a(b)) = 1.$$

Now it is obvious that for $a,b \in M^{(B)}$ we have $b(a) \leq \|a \in b\|$ and $\|a = b\| = \|b = a\|$.

3.56 Example. For every $a,b,c \in M^{(B)}$

$\|a = b\| \wedge \|b = c\| \leq \|a = c\|$.

$\|a \in b\| \wedge \|a = c\| \leq \|c \in b\|$.

$\|a \in b\| \wedge \|b = c\| \leq \|a \in c\|$.

For a given formula $\varphi(x_1, \ldots, x_n)$ we introduced the mapping $\| \varphi \| : (M^{(B)})^n \to B$. The mapping $\| \varphi \|$ determines a relation $\Vdash$ between elements of the algebra B and n–tuples of names from $M^{(B)}$ defined by

$$u \Vdash \varphi(a_1, \ldots, a_n) \quad \text{iff} \quad u \leq \| \varphi(a_1, \ldots, a_n)\|.$$

If $u \Vdash \varphi(a_1, \ldots, a_n)$ then we say that u forces φ for $a_1, \ldots, a_n$.

Let us notice that the Boolean values of formulas and the relation of forcing are notions defined in M, but the class M[G] is defined by a generic ultrafilter which need not be in M. The following important theorem shows the connection between the validity of a formula in M[G] and the forcing relation in M.

3.58. Forcing Theorem. Let G be an ultrafilter in B which is generic over M. Then for any formula $\varphi(x_1, \ldots, x_n)$

$$(\forall a_1, \ldots, a_n \in M^{(B)}) \, (\varphi^{M[G]}(i_G(a_1), \ldots, i_G(a_n)) \leftrightarrow$$
$$\leftrightarrow \quad \|\varphi(a_1, \ldots, a_n)\|_M \in G).$$

In other words, if we want to show that a formula φ holds in M[G] for sets $X_1, \ldots, X_n \in M[G]$, it is sufficient to take some names $a_1, \ldots, a_n \in M^{(B)}$ which correspondend to the sets $X_1, \ldots, X_n$, compute in M the value $\| \varphi(a_1, \ldots, a_n)\|$ and prove that this value lies in the generic ultrafilter G.

Proof. Instead of the interpretation i_G we write only i. The heart of the proof is in the atomic formulas. By well–founded induction for a.b $\in M^{(B)}$ we prove

(i) $i(a) = i(b) \rightarrow \|a = b\| \in G.$

It suffices to show

$$i(a) \subseteq i(b) \rightarrow \Big(\bigwedge_{c\in Dom(a)} (a(c) \Rightarrow \bigvee_{d\in Dom(b)} b(d) \wedge \|c = d\|)\Big) \in G.$$

The following relations are equivalent

$$\Big(\bigwedge_{c\in Dom(a)} (a(c) \Rightarrow \bigvee_{d\in Dom(b)} b(d) \wedge \|c = d\|)\Big) \in G.$$

$$(\forall c \in Dom(a))((a(c) \Rightarrow \bigvee_{d\in Dom(b)} b(d) \wedge \|c = d\|) \in G).$$

$$(\forall c \in Dom(a))(a(c) \in G \rightarrow (\bigvee_{d\in Dom(b)} b(d) \wedge \|c = d\|) \in G.$$

$$(\forall c \in Dom(a))(a(c) \in G \rightarrow (\exists d \in Dom(b))(b(d) \wedge \| c = d \|) \in G).$$

$$(\forall c \in Dom(a))(a(c) \in G \rightarrow (\exists d \in Dom(b))(b(d) \in G \;\&\; i(c) = i(d))).$$

$$(\forall c \in Dom(a))(a(c) \in G \rightarrow i(c) \in \{i(d) : b(d) \in G\}).$$

$$((\forall c \in Dom(a))(i(c) \in i(a) \rightarrow i(c) \in i(b)).$$

$$i(a) \subseteq i(b).$$

(ii) $i(a) \in i(b)$ if and only if $\| a \in b\| \in G$. which follows from the following equivalent expressions

$$\| a \in b \| = \bigvee_{d \in Dom(b)} b(d) \wedge \| d = a \| \in G.$$

$$(\exists d \in Dom(b))(b(d) \in G \ \& \ \| d = a \| \in G.$$

$$(\exists d \in Dom(b))(i(d) \in i(b) \ \& \ i(d) = i(a)).$$

$$i(a) \in i(b).$$

(iii) $\| \neg \varphi \| \in G \longleftrightarrow \| \varphi \| \notin G \longleftrightarrow \neg (\varphi^{M[G]}) = (\neg \varphi)^{M[G]}$.

The proof is similar for other logical connectives.

(iv) Let $\psi(y_1, \dots, y_n) = \exists x \, \varphi(x, y_1, \dots, y_n)$. Then

$$\| \psi(a_1, \dots, a_n) \| = \| \exists x \, \varphi(x, a_1, \dots, a_n) \| \in G,$$

if and only if

$$(\exists b \in M^{(B)}) (\| \varphi(b, a_1, \dots, a_n) \| \in G).$$

By the inductive hypothesis we have

$$(\exists b \in M^{(B)}) (\varphi^{M[G]} (i(b), i(a_1), \dots, i(a_n))$$

and this is equivalent to

$$(\exists x \in M^{(G)}) (\varphi^{M[G]} (x, i(a_1), \dots, i(a_n)),$$

which is $\psi^{M[G]}(i(a_1), \dots, i(a_n))$. The proof is similar for the universal quantifier.

<u>3.59. Final part of the proof of Generic Extension Theorem 3.26</u>. We know by 3.53 that $N = M[G]$ is a transitive class and $M \subseteq M[G]$. It follows that the axioms of Extensionality, Foundation and Infinity hold in $M[G]$. Using the Forcing Theorem we shall prove 3.26(i) and we show that the axioms of Power set, Union and the Replacement scheme hold in $M[G]$. Again instead of i_G we just write i.

Let $X \in M[G]$ be a set and let $a \in M^{(B)}$ be a name of X, i.e. $X = i(a)$. First we show that any $Y \subseteq X$ with $Y \in M[G]$ has a name of a special type. Let b be a name for $Y \subseteq X$. Define a name b' by

(22) $\quad \mathrm{Dom}(b') = \mathrm{Dom}(a) \quad \& \quad b'(c) = a(c) \wedge \| c \in b \|.$

So for every c we have $b'(c) \leq a(c)$. Since $Y \subseteq X$ and $i(b') = \{i(c) : (a(c) \wedge \| c \in b \| \in G\}$, by the Forcing Theorem we get

$$i(b') = \{i(c) : a(c) \in G \;\&\; i(c) \in i(b)\ \} = X \cap Y = Y.$$

Hence b' is a name for Y.

To prove 3.26(i) we shall use (13). Let $\rho \subseteq x \in M$ and $\rho \in N = M[G]$. Consider the canonical name $a = \check{x}$ for x. By (22) there exists a name b' for ρ such that $\mathrm{Dom}(b') = \{\check{y} : y \in x\}$. If we define $f(y) = \| \check{y} \in b' \|$ for every $y \in x$ we get a function $f : x \to B$, $f \in M$ such that

$$y \in \rho \leftrightarrow b(\check{y}) \in G \leftrightarrow f(y) \in G.$$

Hence $\rho = f^{-1}{}''G$, $\rho \in \mathrm{Ob}(G,M)$. In a similar way we can find a name $b' \in M^{(B)}$ for $\rho \subseteq x$, if we know $f : x \to B$, $f \in M$, for which $\rho = f^{-1}{}''G$. Hence 3.26(i) is true.

<u>Power Set Axiom</u>: Let $a \in M^{(B)}$ be a name of the set $X \in M[G]$. By 3.9(b) we have to show that $\mathcal{P}(X) \cap M[G] \in M[G]$. Consider the following set of names

$$w = \{b \in M^{(B)} : \mathrm{Dom}(b) = \mathrm{Dom}(a) \;\&\; (\forall c \in \mathrm{Dom}(a))(b(c) \leq a\,(c))\}.$$

Then $s = w \times \{ 1_B \} \in M^{(B)}$ is a name of the set $\mathcal{P}(X) \cap M[G]$, because for every $b \in w$ we have $i(b) \subseteq X$, and every $Y \subseteq X$ with $Y \in M[G]$ has by (22) some name in w. Hence $i(s) = \mathcal{P}(X) \cap M[G]$.

Similarly we can verify the Axiom of Union.

<u>Axiom of Replacement</u>. Let us recall that this axiom for the formula $\varphi(u,v)$ says: If the class $F = \{< u,v > : \varphi(u,v)\}$ is a mapping then for every set X, $F''X$ is again a set.

We want to show that the Axiom of Replacement for φ holds in M[G]. Suppose that $\varphi^{M[G]}$ defines a mapping, i.e.

$$F = \{< x,y > : x,y \in M[G] \;\&\; \varphi^{M[G]}(x,y)\}$$

is a mapping. From this and from the Forcing Theorem for any $b,c,c' \in M^{(B)}$ we have

$(\| \varphi(b,c) \| \in G \ \& \ \| \varphi(b,c') \| \in G) \to$

$\to (\varphi^{M[G]}(i(b), i(c)) \ \& \ \varphi^{M[G]}(i(b), i(c')) \to i(c) = i(c'))$.

Let $X \in M[G]$ have a name a. We want to find a name d for F"X. For every $b \in Dom(a)$ there is only a set of names J_b such that

(23) $$\bigvee \{\| \varphi(b,c) \| : c \in M^{(B)}\} = \bigvee \{\| \varphi(b,c) \| : c \in J_b\}.$$

because $B \in M$ and we can consider only names c of minimal rank. Now define $d \in M^{(B)}$ by

$$Dom(d) = \bigcup \{J_b : b \in Dom(a)\}$$

and

$$d(c) = \bigvee \{\| \varphi(b,c) \| \wedge a(b) : b \in Dom(a)\}.$$

We now show that d is a name for F"X. If $z \in i(d)$ then $z = i(c)$ for some c with $d(c) \in G$. By the genericity of G there exists $b \in Dom(a)$ for which $\| \varphi(b,c) \| \in G$ and $a(b) \in G$, so for $x = i(b) \in X$ we have $z = F(x)$. Conversely, if $a = F(x)$ for some $x \in X$, take a name $b \in Dom(a)$ for x and a name c' for z. We also have $\| \varphi(b,c') \| \in G$ and $a(b) \in G$. From (23) it follows that there exists a $c \in J_b \subseteq Dom(d)$ for which $\| \varphi(b,c) \| \in G$, and therefore $d(c) \in G$. Since $i(c') = z$, also $i(c) = z$, hence $z \in i(d)$.

The proof of the main part of Theorem 3.26 is finished. It remains to show that the Axiom of Choice holds in M[G] too, in case M was a model of ZFC. If $X \in M[G]$ with a name a then for $S = Dom(a)$ we have $i_G | S \in M[G]$ because by 3.53 for every $\alpha \in M \cap On$ we have $i_G | M_\alpha^{(B)} \in M[G]$. AC holds in M, therefore there is a one–to–one mapping $h : S \to On^M$. We define a mapping g from X in On by

$$g(x) = \min \{h(b) : b \in S \ \& \ i_G(b) = x\}.$$

Since $g \in M[G]$, g ensures the well ordering of the set X in M[G]. This is all.

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