SET THEORY

2021-11-01

WE ASSUME AC THROUGHOUT

WHAT WE KNOW:

$$\kappa+\lambda = \kappa\cdot\lambda = \max\{\kappa,\lambda\}$$

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha\cdot\aleph_\beta = \aleph_{\max\{\alpha,\beta\}}$$

WHAT WE DO NOT KNOW

WHAT γ GOES WITH

$$\aleph_\alpha^{\aleph_\beta} \; ?$$

$$2^{\aleph_\alpha} = \aleph_\alpha^{\aleph_\alpha} = \aleph_{???}$$

WE WILL STUDY $\kappa \longmapsto 2^\kappa$

CONTINUUM FUNCTION

AND EXPONENTIATION

$$(\kappa,\lambda) \longmapsto \kappa^\lambda$$

A BIT MORE ABOUT COFINALITY

$\mathrm{cf}(\kappa) = \min\{|C| : C\subseteq\kappa \text{ cofinal}\}$

EQUIVALENT

$\mathrm{cf}(\kappa) = \min\{\gamma :$ THERE AN INCR. COF MAP $f:\gamma\to\kappa\}$

$\mathrm{cf}(\kappa) = \min\{\lambda :$ THERE IS A FAMILY $\{S_\alpha : \alpha<\lambda\}$ OF SUBSETS OF κ SUCH THAT

- $\bigcup_{\alpha<\lambda} S_\alpha = \kappa$
- $|S_\alpha| < \kappa$

- $CF(CF(\kappa)) = CF(\kappa)$
- $CF(\kappa)$ IS REGULAR.
- κ IS REGULAR IFF $\kappa = CF(\kappa)$

  SINGULAR $\kappa > CF(\kappa)$

WE ALREADY HAVE

— $2 \le \kappa \le 2^\lambda$ IMPLIES $\underline{\kappa}^\lambda = 2^\lambda$

— MORE INTERESTING WHAT IF $2^\lambda < \kappa$.

— $\kappa \le \kappa^\lambda$ --- CONSTANT FUNCTIONS

— $\kappa^\lambda \le \kappa^\kappa = 2^\kappa$

— SO $\underline{\kappa \le \kappa^\lambda \le 2^\kappa}$

IS THAT IT? NO, THERE IS MORE.

3.11 IF κ IS INFINITE THEN $\kappa < \underline{\underline{\kappa}}^{CF\kappa}$.

CARDINAL

PROOF DOM: $CF\kappa$ CODOM: κ

LET $\boxed{\xi \mapsto \alpha_\xi}$ BE INCR. COFINAL FROM $CF\kappa$ TO κ.

LET $F \subseteq \underline{\underline{\kappa^{CF\kappa}}}$ BE OF CARDINALITY κ.

SET OF FUNCTIONS

CONSTRUCT $f \in \kappa^{CF\kappa} \setminus F$

ENUMERATE F AS $\{f_\beta : \beta < \kappa\}$

FOR $\xi \in CF\kappa$ DEFINE

$$f(\xi) = \min \kappa \setminus \underbrace{\{f_\beta(\xi) : \beta < \alpha_\xi\}}_{\text{CARD} < \kappa}$$

$f \ne f_\beta$ BECAUSE

$f(\xi) \ne f_\beta(\xi)$ IF $\alpha_\xi > \beta$.

SO $\underline{\underline{f \notin F}}$

$(2^\lambda < \kappa)$

MORE INFORMATION:
IF $cf\kappa \leq \lambda$ THEN $\kappa < \kappa^\lambda$

IF $|A| \geq \lambda$ THEN
$\to$ $[A]^\lambda = \{X \subseteq A : |X| = \lambda\}$
$[A]^{\leq\lambda}$ — — — — — — $\leq$ —
$[A]^{<\lambda}$ — — — — $<$ —

5.7 IF $\kappa \geq \lambda$ THEN $|[\kappa]^\lambda| = \kappa^\lambda$

PROOF:
SETS OF FUNCTIONS:
$\underset{\text{FUNCTIONS}}{\kappa^\lambda} \subseteq [\lambda \times \kappa]^\lambda \underset{\text{BIJECTION}}{\approx} [\kappa]^\lambda$; $\underset{\text{CARD}}{\kappa^\lambda} \leq |[\kappa]^\lambda|$

$\kappa^\lambda \geq |[\kappa]^\lambda|$ BLATANT CHOICE !!
$X \in [\kappa]^\lambda \leadsto$ THERE IS $f_X : \lambda \to \kappa$
SUCH THAT RAN $f_X = X$
EVEN INJECTIVE
AC GIVES INJECTION: $[\kappa]^\lambda \to \underset{\text{FUNCTIONS}}{\kappa^\lambda}$

IF λ IS A LIMIT CARDINAL
THEN
$$\kappa^{<\lambda} = \sup\{\kappa^\mu : \mu < \lambda;\ \mu \text{ CARDINAL}\}$$
WE SHALL SEE MOMENTARILY:
$$\kappa^{<\lambda} = |[\kappa]^{<\lambda}|$$

PRODUCTS AND SUMS OF ARBITRARY FAMILIES OF CARDINALS.

We have $\{\kappa_i : i \in I\}$
an <u>indexed</u> set of cardinals
($i \neq j$ and $\kappa_i = \kappa_j$ is allowed)

$$\sum_{i\in I}\kappa_i = \left|\bigcup_{i\in I}\{i\}\times\kappa_i\right|$$

$$\underbrace{\prod_{i\in I}\kappa_i}_{\text{cardinal}} = \Big|\underbrace{\prod_{i\in I}\kappa_i}_{\text{functions}}\Big|$$

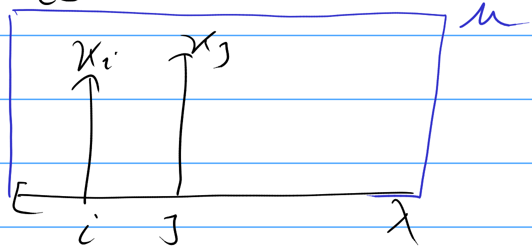
the set choice functions for $\{\kappa_i : i\in I\}$

5.8 If λ is infinite and $\kappa_i>0$ for all $i<\lambda$ then

$$\sum_{i<\lambda}\kappa_i = \lambda\cdot\underbrace{\sup_{i<\lambda}\kappa_i}_{\mu}$$

Proof

$\le$: $\bigcup_{i<\lambda}\{i\}\times\kappa_i \subseteq \underbrace{\lambda\times\mu}_{\text{set}}$



$\ge$ • $\lambda = \sum_{i<\lambda} 1 \le \sum_{i<\lambda}\kappa_i$

• for every j: $\kappa_j \le \sum_{i<\lambda}\kappa_i$

then $\mu \le \sum_{i<\lambda}\kappa_i$.

$\sum_{i<\lambda}\kappa_i$ is an upper bound of $\{\kappa_j : j\in\lambda\}$

μ is the <u>least</u> upper bound

$$\lambda * \mu = \max\{\lambda,\mu\} \le \sum_{i<\lambda}\kappa_i$$

κ IS SINGULAR: $\kappa = \sum_{i \in \lambda} \kappa_i$

WITH $\lambda < \kappa$

AND $\kappa_i < \kappa$

FOR ALL i

NOW SHOW

$$|[\kappa]^{<\lambda}| = \kappa^{<\lambda}$$

— COMMUTATIVITY / ASSOCIATIVITY:

IF $I = \bigcup_{j \in J} A_j$ (PAIRWISE DISJ.)

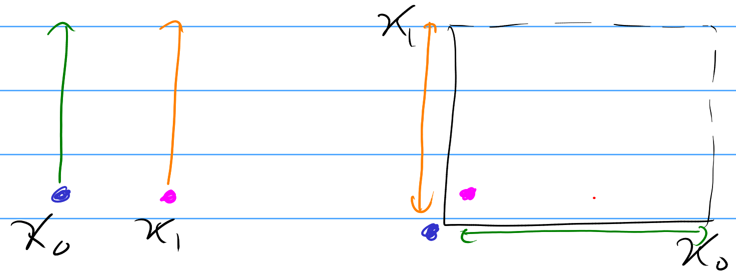
THEN $\prod_{i \in I} \kappa_i = \prod_{j \in J} \left(\prod_{i \in A_j} \kappa_i \right)$

EASY BIJECTION BETWEEN THE SETS OF FUNCTIONS.

- IF $\kappa_i \geq 2$ FOR ALL $i \in I$ THEN

$$\sum_{i \in I} \kappa_i \leq \prod_{i \in I} \kappa_i$$

$|I| = 2$



$|I| \geq 3$ $\quad f : \bigcup_{i \in I} \{i\} \times \kappa_i \longrightarrow \prod_{i \in I} \kappa_i$

$$\alpha > 0 \qquad f(i,\alpha)(j) = \begin{cases} \alpha & \text{IF } j = i \\ 0 & \text{IF } j \neq i \end{cases}$$

$$\alpha = 0 \qquad f(i,0)(j) = \begin{cases} 0 & \text{IF } j = i \\ 1 & \text{IF } j \neq i \end{cases}$$

f IS INJECTIVE!

NOTE $\quad 1 + 1 + 1 > 1 \cdot 1 \cdot 1$

λ INFINITE

5.9 IF $\langle \kappa_i : i \in \lambda \rangle$ IS NON-DECREASING

THEN $\prod_{i\in\lambda} \kappa_i = (\sup_{i\in\lambda} \kappa_i)^\lambda$ $\kappa_i \neq 0$

FOR EXAMPLE

$$\prod_{n\in\omega} \aleph_n = \aleph_\omega^{\aleph_0}$$

USE $\lambda = \lambda \cdot \lambda$

TO WRITE λ AS A DISJOINT UNION $\bigcup_{\alpha<\lambda} A_\alpha$ WITH $|A_\alpha| = \lambda$

NOTE

$$\sup_{i\in A_\alpha} \kappa_i = \sup_{i\in\lambda} \kappa_i$$

$$\prod_{i\in\lambda} \kappa_i = \prod_{\alpha\in\lambda} \left(\prod_{i\in A_\alpha} \kappa_i\right) \qquad \prod_{i\in A_\alpha} \kappa_i \geq \sup_{i\in A_\alpha} \kappa_i$$

$$\geq \prod_{\alpha\in\lambda} \left(\sup_{i\in\lambda} \kappa_i\right)$$

$$= (\sup_{i\in\lambda} \kappa_i)^\lambda$$

CLEARLY $\prod_{i\in\lambda} \kappa_i \leq \prod_{i\in\lambda} (\sup_{j\in\lambda} \kappa_j)$

DONE!

EXAMPLE

— $\kappa_0 = \aleph_0$ $\kappa_n = 1$ $n \geq 1$ $(n\in\omega)$

$\sup = \aleph_0^\lambda$

$\prod_n \kappa_n = \aleph_0$ $(\sup_n \kappa_n)^{\aleph_0} = 2^{\aleph_0}$

THEOREM [KÖNIG]

IF $\kappa_i < \lambda_i$ FOR ALL $i \in I$

THEN $$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$$

PROOF : WLOG $\kappa_i \geq 1$

$J = \{i : \kappa_i \geq 1\}$

$$\sum_{i\in I} \kappa_i = \sum_{i\in J} \kappa_j \leq \prod_{i\in J} \lambda_i \leq \prod_{i\in I} \lambda_i$$

$i \notin J : \lambda_j \geq 1$

NOW $\lambda_i \geq 2$ FOR ALL i

So

$$\sum_{i\in I} \kappa_i \underset{\uparrow}{\leq} \sum_{i\in I} \lambda_i \underset{\uparrow}{\leq} \prod_{i\in I} \lambda_i$$

JUST SHOWN

NOW $\leq$

LET $F : \bigcup_{i\in I} \{i\}\times\kappa_i \longrightarrow \underbrace{\prod_{i\in I}\lambda_i}_{\text{FUNCTIONS!}}$

WE MUST SHOW F IS NOT ONTO

TAKE $i \in I$

$$\{ \underbrace{\underbrace{F(i,\alpha)}_{\text{FUNCTION}}(i)}_{\text{VALUE AT } i} : \alpha < \kappa_i \} \subseteq \lambda_i$$

$\neq$

LET $g(i) = \min \lambda_i \setminus \{ \;\; \}$

THEN $g \neq F(i,\alpha)$ FOR ALL (i,α).

CONSEQUENCES

1. $\kappa_i = 1 \quad \lambda_i = 2 : \quad |I| < 2^{|I|}$

2. $CF(2^\kappa) > \kappa$. κ INFINITE

ASSUME $\kappa_i < 2^\kappa$ FOR $i \in \kappa$.

$\lambda_i = 2^\kappa$ FOR $i \in \kappa$

$$\sum_{i\in\kappa} \kappa_i < \prod_{i\in\kappa} \lambda_i = (2^\kappa)^\kappa = 2^\kappa$$

$2^{\aleph_0} \neq \aleph_\omega$!!!

3. $\kappa^{CF\kappa} > \kappa \Leftarrow$

$\kappa = \sum_{i\in CF\kappa} \kappa_i \qquad \kappa_i < \kappa$

$\lambda_i = \kappa \quad i \in CF\kappa$

GCH : $(\forall\alpha)\left(2^{\aleph_\alpha} = \aleph_{\alpha+1}\right)$

CONSISTENT WITH ZFC

5.15 IF κ AND λ ARE INFINITE

- $\kappa \le 2^\lambda$: $\kappa^\lambda = 2^\lambda = \lambda^+$
- $\mathrm{cf}\,\kappa \le \lambda < \kappa$: $\kappa^\lambda = \kappa^+$ $(\kappa < \kappa^\lambda \le 2^\kappa)$
- $\lambda < \mathrm{cf}\,\kappa$: $\kappa^\lambda = \kappa$

AS SETS $\kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda$

SO $\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\lambda = \kappa \cdot \sup_{\alpha<\kappa} |\alpha|^\lambda = \kappa$

BUT IF $\alpha < \kappa$

THEN $|\alpha|^\lambda \le 2^{|\alpha|\cdot\lambda} = (|\alpha|\cdot\lambda)^+ \le \kappa$

WITHOUT GCH ?

IF F: CARD $\longrightarrow$ CARD

HAS
- $\kappa < \lambda \rightarrow F(\kappa) \le F(\lambda)$
- $\mathrm{cf}(F(\kappa)) > \kappa$

THEN THERE IS A MODEL OF ZFC WITH $2^\kappa = F(\kappa)$ FOR ALL REGULAR κ.

FOR SINGULAR κ LIFE IS MORE COMPLICATED.

5.16
(i) $\kappa < \lambda \longrightarrow 2^\kappa \le 2^\lambda$

(ii) $\mathrm{cf}\, 2^\kappa > \kappa$

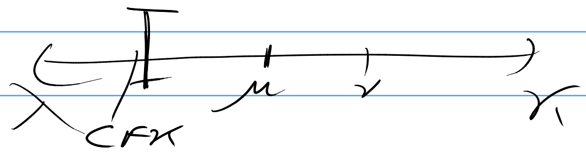
(iii) κ A LIMIT $2^\kappa = \left(2^{<\kappa}\right)^{\mathrm{cf}\,\kappa}$

$\kappa = \sum_{i<\mathrm{cf}\,\kappa} \kappa_i$

$$2^\kappa = 2^{\sum_i \kappa_i} = \prod_{i<\mathrm{cf}\,\kappa} 2^{\kappa_i} \le \prod_{i<\mathrm{cf}\,\kappa} 2^{<\kappa}$$
$$= \left(2^{<\kappa}\right)^{\mathrm{cf}\,\kappa}$$
$$\le \left(2^\kappa\right)^\kappa$$
$$= 2^\kappa$$

IF κ IS SINGULAR AND 2^{μ} IS CONSTANT ON AN INTERVAL OF THE FORM (λ, κ)

"EVENTUALLY CONSTANT BELOW κ" $2^{\mu} = 2^{\nu}$



2^{κ} IS EQUAL TO THAT CONSTANT VALUE

(WE CAN HAVE $2^{\aleph_0} = 2^{\aleph_1} = \dots = 2^{\aleph_n} = \dots$ FOR ALL n)

WE HAVE $\mu > cf\kappa$ AND τ SUCH THAT $2^{\nu} = \tau$ $\quad \mu \le \nu < \kappa$

$$2^{<\kappa} = 2^{\mu} = \tau$$

$$2^{\kappa} = (2^{\mu})^{cf\kappa} \le (2^{\mu})^{\mu} = 2^{\mu} = \tau$$

GIMEL $\quad \gimel(\kappa) = \kappa^{cf\kappa}$

κ A LIMIT AND 2^{μ} NOT EVENTUALLY CONSTANT BELOW κ.

$(\forall \mu < \kappa \ \exists \nu < \kappa \quad 2^{\mu} < 2^{\nu})$

THEN $2^{<\kappa} = \sup_{\nu} 2^{\nu}$

AND $cf(2^{<\kappa}) = cf(\kappa)$

$$2^{\kappa} = (2^{<\kappa})^{cf\kappa} = \gimel(2^{<\kappa})$$

κ REG: $\quad (2^{<\kappa})^{cf\kappa} = (2^{<\kappa})^{\kappa} = 2^{\kappa}$

Corollary 5.18.

(i) *If κ is a successor cardinal, then $2^\kappa = \beth(\kappa)$.*
(ii) *If κ is a limit cardinal and if the continuum function below κ is eventually constant, then $2^\kappa = 2^{<\kappa} \cdot \beth(\kappa)$.*
(iii) *If κ is a limit cardinal and if the continuum function below κ is not eventually constant, then $2^\kappa = \beth(2^{<\kappa})$.* □

κ REGULAR: $\kappa^{\mathrm{cf}\kappa} = \kappa^\kappa = 2^\kappa$

(ii) $2^\kappa \geq 2^{<\kappa}$ $\quad 2^\kappa = \kappa^\kappa \geq \kappa^{\mathrm{cf}\kappa}$

κ SINGULAR $\quad 2^\kappa = 2^{<\kappa}$

κ REGULAR $\quad 2^\kappa = \kappa^\kappa = \kappa^{\mathrm{cf}\kappa}$

(iii) SEE ABOVE

ON TO EXPONENTIATION

κ REGULAR: $\quad \kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda$ (SETS)

IF $\lambda < \kappa$

$$\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\lambda$$

$\kappa = \aleph_{\alpha+1} \qquad \lambda = \aleph_\beta$

$$\kappa^\lambda = \aleph_{\alpha+1}^{\aleph_\beta} = \sum_{\gamma<\omega_{\alpha+1}} |\gamma|^{\aleph_\beta}$$

$$= \aleph_{\alpha+1} \cdot \underbrace{\sup\{|\gamma|^{\aleph_\beta} : \gamma < \aleph_{\alpha+1}\}}_{\aleph_\alpha^{\aleph_\beta}}$$

$$\boxed{\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}}$$

HAUSDORFF

κ A LIMIT CARDINAL $\quad \lambda < \kappa$, $\lambda \geq \mathrm{cf}\kappa$

THEN $\quad \kappa^\lambda = (\sup_{\alpha<\kappa} \alpha^\lambda)^{\mathrm{cf}\kappa}$

$$\kappa = \sum_{i<\mathrm{cf}\kappa} \kappa_i$$

$$\kappa^\lambda = \left(\sum_{i<\mathrm{cf}\,\kappa} \kappa_i\right)^\lambda \le \left(\prod_{i<\mathrm{cf}\,\kappa} \kappa_i\right)^\lambda = \prod_{i<\mathrm{cf}\,\kappa} \kappa_i^\lambda \le \prod_{i<\mathrm{cf}\,\kappa} \left(\sup_{\alpha<\kappa} |\alpha|^\lambda\right) = \left(\sup_{\alpha<\kappa} |\alpha|^\lambda\right)^{\underline{\underline{\mathrm{cf}\,\kappa}}} \le (\kappa^\lambda)^{\mathrm{cf}\,\kappa} = \underline{\underline{\kappa^\lambda}}$$

Theorem 5.20. *Let λ be an infinite cardinal. Then for all infinite cardinals κ, the value of κ^λ is computed as follows, by induction on κ:*

(i) *If $\kappa \le \lambda$ then $\kappa^\lambda = 2^\lambda$.*
(ii) *If there exists some $\mu < \kappa$ such that $\mu^\lambda \ge \kappa$, then $\kappa^\lambda = \mu^\lambda$.*
(iii) *If $\kappa > \lambda$ and if $\mu^\lambda < \kappa$ for all $\mu < \kappa$, then:*
  (a) *if* $\mathrm{cf}\,\kappa > \lambda$ *then* $\kappa^\lambda = \kappa$, $\rightarrow \kappa = \mu^+ : \kappa^\lambda = \mu^\lambda \cdot \kappa = \kappa$
  (b) *if* $\mathrm{cf}\,\kappa \le \lambda$ *then* $\kappa^\lambda = \kappa^{\mathrm{cf}\,\kappa}$.

$\mathrm{cf}\,\kappa > \lambda$ $\qquad \kappa^\lambda = \bigcup_{\alpha<\kappa} \alpha^\lambda$ $\quad {<\kappa}$

$$\kappa^\lambda = \sum_{\alpha<\kappa} |\alpha|^\lambda = \kappa$$

$\mathrm{cf}\,\kappa \le \lambda$ $\qquad \kappa^\lambda = \kappa^{\mathrm{cf}\,\kappa}$

$$\sup_{\alpha<\kappa} |\alpha|^\lambda = \kappa$$

Corollary 5.21. *For every κ and λ, the value of κ^λ is either 2^λ, or κ, or $\beth(\mu)$ for some μ such that $\mathrm{cf}\,\mu \le \lambda < \mu$.*

SMALLEST μ WITH $\mu^\lambda \ge \kappa$

SCH "SINGULAR CARDINAL HYPOTHESIS"

FOR EVERY SINGULAR κ
<u>IF</u> $2^{\mathrm{cf}\,\kappa} < \kappa$
<u>THEN</u> $\kappa^{\mathrm{cf}\,\kappa} = \kappa^+$

IT IS VERY HARD TO MAKE CONSISTENTLY FALSE