①

SET THEORY 2021-11-08

$\Omega = \omega_1$

Une fonction $f(x)$, définie sur un sous-ensemble A de $\mathcal{O}$ et à valeurs dans $\mathcal{O}$, sera dite *régressive* si $f(x) \leq x$ pour tout $x \in$ A, l'égalité étant exclue sauf pour $x = 1$, si $1 \in$ A. Un ensemble A de points de $\mathcal{O}$ est *stationnaire* s'il est non dénombrable et si pour toute fonction régressive $f(x)$ définie sur A, on a $\lim\limits_{x \succ \Omega,\, x \in A} f(x) < \Omega$; autrement dit s'il existe au moins un point a tel que $\overset{-1}{f}(a)$ soit non dénombrable. Dans tous les autres cas, A est *non stationnaire*.

Théorème I. — *Une condition nécessaire et suffisante pour qu'un ensemble soit stationnaire est que tout sous-ensemble fermé de son complément soit au plus dénombrable.*

EVERY CLOSED SET IN THE COMPLEMENT IS CTBLE

BLOCH 1953

REGRESSIVE FUNCTION

$f : \kappa \to \kappa$

OR $f : A \to \kappa \qquad A \subseteq \kappa$

κ REGULAR UNCOUNTABLE

$f(\alpha) < \alpha \qquad (\alpha \in A$ EXCEPT $\alpha = 0)$

Теорема. *Предположим, что для каждого порядкового числа $\alpha < \omega_1$ определено порядковое число $\mu(\alpha)$ под единственным условием $\mu(\alpha) < \alpha$ для любого $\alpha < \omega_1$. Тогда существует несчетное множество порядковых чисел $\alpha_1 < \alpha_2 < \ldots < \alpha_\lambda < \ldots, \lambda < \omega_1$, для которых $\mu(\alpha_1) = \mu(\alpha_2) = \ldots = \mu(\alpha_\lambda) = \ldots$*

ALEXANDROFF — URYSOHN 1929

NOTE $f : \omega_0 \to \omega_0 \qquad f(0) = 0$

$n \mapsto n-1 \qquad (n > 0)$

IS REGRESSIVE AND (ALMOST) INJECTIVE.

$A \subseteq \kappa$ IS CLOSED
IF FOR ALL $\alpha < \kappa$
IF $A \cap \alpha \neq \emptyset$ THEN
$\sup(A \cap \alpha) \in A$.

[THIS DEFINES A TOPOLOGY
ON κ:
O IS OPEN IFF FOR ALL $\alpha \in O$
THERE IS $\beta < \alpha$
SUCH THAT $(\beta, \alpha] \subseteq O$
THIS IS THE ORDER TOPOLOGY]

$C \subseteq \kappa$ IS CUB
(CLOSED AND UNBOUNDED)

IF IT IS CLOSED AND COFINAL
κ ITSELF $[\alpha, \kappa)$

$S \subseteq \kappa$ IS STATIONARY
IF $S \cap C \neq \emptyset$ FOR ALL
CUB SETS C.
[A LA BLOCH EVERY CLOSED
SET C WITH $C \cap S = \emptyset$
IS BOUNDED]

(1) IF C AND D ARE CUB THEN
SO IS $C \cap D$.
SO EVERY CUB SET IS STATIONARY

(a) $C \cap D$ IS CLOSED
TAKE α WITH $C \cap D \cap \alpha \neq \emptyset$
THEN $C \cap \alpha \neq \emptyset$ AND $D \cap \alpha \neq \emptyset$

LOOK AT $\beta = \sup(C \cap D \cap \alpha)$

NOTE

$\beta \supseteq C \cap \beta \supseteq C \cap D \cap \beta = C \cap D \cap \alpha$

$\beta = \sup(C \cap D \cap \beta)$ SO $\beta = \sup(C \cap \beta)$

SO $\beta \in C$

LIKEWISE $\beta \in D$

(b) UNBOUNDED:

TAKE $\alpha < \kappa$ FIND $\beta \in C \cap D$

ABOVE α.

$\alpha < \gamma_0 < \delta_0 < \gamma_1 < \delta_1 < \cdots$

$\cdots < \gamma_2 < \delta_2 < \cdots$

$\cdots < \gamma_3 < \delta_3 < \cdots$

$\gamma_0 = \min\{\gamma \in C : \gamma > \alpha\}$

$\delta_0 = \min\{\delta \in D : \delta > \gamma_0\}$

$\gamma_{n+1} = \min\{\gamma \in C : \gamma_1 > \delta_n\}$

$\delta_{n+1} = \min\{\delta \in D : \delta > \gamma_n\}$

LET $\beta = \sup\{\gamma_n : n \in \omega\} = \sup\{\delta_n : n \in \omega\}$

$\beta \in C \cap D$

MUCH BETTER:

LET $\langle C_\alpha : \alpha < \lambda \rangle$ BE A SEQUENCE OF CUB SETS IN κ WITH $\lambda < \kappa$.

THEN $\bigcap_{\alpha < \lambda} C_\alpha$ IS CUB AS WELL.

CLOSED CHECK THIS

$\beta < \kappa$

$$\begin{array}{ccccccccccc}
 & & \in C_0 & & \in C_1 & & \in C_2 & & \in C_\alpha & & \\
\beta & < & \gamma_{0,0} & < & \gamma_{1,0} & < & \gamma_{2,0} & \cdots < & \gamma_{\alpha,0} & < & \cdots \\
 & < & \gamma_{0,1} & < & \gamma_{1,1} & < & \gamma_{2,1} & < & \gamma_{\alpha,1} & < & \cdots \\
 & < & \gamma_{0,2} & < & \gamma_{1,2} & < & \gamma_{2,2} & < & \gamma_{\alpha,2} & < & \cdots
\end{array}$$

$\gamma_{0,1} > \gamma_{\alpha,0}$ (ALL α)

POSSIBLE! $\kappa > \lambda$

NOW TAKE δ TO BE

$$\delta = \sup_{n\in\omega} \gamma_{0,n} \qquad \delta \in C_0$$
$$= \sup_{n\in\omega} \gamma_{1,n} \qquad \delta \in C_1$$
$$= \sup \gamma_{\alpha,n} \quad --- \quad \delta \in C_\alpha$$
$$\vdots \qquad\qquad \vdots$$

$\mathrm{Cub}(\kappa) = \{A \subseteq \kappa :$ THERE IS A CUB C WITH $C \subseteq A\}$

THIS IS THE CUB FILTER
(CLUB FILTER)

$\emptyset$ NOT IN, κ IS IN
IF $A, B \in \mathrm{Cub}(\kappa)$ THEN $A \cap B \in \mathrm{Cub}(\kappa)$
IF $A \in \mathrm{Cub}(\kappa)$ AND $B \supseteq A$ THEN $B \in \mathrm{Cub}(\kappa)$

THIS FILTER IS κ-COMPLETE

CLOSED UNDER INTERSECTIONS
OF FEWER THAN κ MANY
MEMBERS.

$$\bigcap_{\alpha<\kappa} [\alpha, \kappa) = \emptyset$$

DIAGONAL INTERSECTION
LET $\langle C_\alpha : \alpha < \kappa \rangle$ BE A SEQUENCE
OF CUB SETS.

$$\Rightarrow \triangle_{\alpha<\kappa} C_\alpha = \{\delta \in \kappa : (\forall \alpha \in \delta)(\delta \in C_\alpha)\}$$

($\bigcap_{\alpha<\delta} C_\alpha$ IS CUB AND δ
IS IN IT!)

$\triangle_{\alpha<\kappa} C_\alpha$ IS $\underline{\underline{\text{CUB}}}$

CLOSED:

SUPPOSE $\delta = \sup(\delta \cap \Delta_{\alpha<\kappa} C_\alpha)$

MUST SHOW $\delta \in \Delta_{\alpha<\kappa} C_\alpha$

OR $(\forall \alpha < \delta)(\delta \in C_\alpha)$

LET $\gamma < \delta$ THEN ALSO

$\delta = \sup((\gamma, \delta) \cap \Delta_{\alpha<\kappa} C_\alpha)$

IF $\beta \in (\gamma, \delta) \cap \Delta_{\alpha<\kappa} C_\alpha$

THEN $\beta \in C_\gamma$

SO $\delta = \sup(C_\gamma \cap (\gamma, \delta))$

SO $\delta \in C_\gamma$

UNBOUNDED:

LET $\gamma < \kappa$

TAKE $\delta_0 = \min \bigcap_{\alpha<\gamma} C_\alpha \setminus (\gamma+1)$

TAKE $\delta_1 = \min \bigcap_{\alpha<\delta_0} C_\alpha \setminus (\delta_0+1)$

$\delta_{n+1} = \min \bigcap_{\alpha<\delta_n} C_\alpha \setminus (\delta_n+1)$

TAKE $\delta = \sup_n \delta_n$

- $\delta > \gamma$
- TAKE $\alpha < \delta$ THEN $\alpha < \delta_m$ FOR SOME m

  THEN $\{\delta_n : n > m\} \subseteq C_\alpha$

  SO $\delta = \sup_{n>m} \delta_n \in C_\alpha$

TRY TO COMPUTE

$\Delta_{\alpha<\kappa} [\alpha, \kappa)$, $\Delta_{\alpha<\kappa} [\alpha+1, \kappa)$,

$\Delta_{\alpha<\kappa} [\alpha+\omega, \kappa)$

$\Delta_{\alpha<\kappa} (\alpha+\omega, \kappa)$

FODOR'S PRESSING-DOWN LEMMA.

IF $S \subseteq \kappa$ IS STATIONARY AND $f : S \to \kappa$ IS REGRESSIVE THEN f IS CONSTANT ON A STATIONARY SET.

PROOF:

SUPPOSE NOT

THEN FOR EVERY $\alpha < \kappa$

$T_\alpha = \{\sigma \in S : f(\sigma) = \alpha\}$ IS NOT STATIONARY,

WE GET A SEQUENCE $\langle C_\alpha : \alpha < \kappa \rangle$ OF CUB SETS SUCH THAT

$$C_\alpha \cap T_\alpha = \emptyset \quad (\text{ALL } \alpha)$$

LET $D = \Delta_{\alpha<\kappa} C_\alpha$

D IS CUB, TAKE $\sigma \in D \cap S$.

SO $f(\sigma) < \sigma$: $\sigma \in T_{f(\sigma)}$

AND ALSO $\sigma \in C_{f(\sigma)}$

AND THERE IS OUR CONTRADICTION!

MAJOR APPLICATION: SILVER'S THEOREM

CHAPTER 8 pg6 (FF)

IF κ IS SINGULAR AND $\mathrm{cf}\,\kappa > \aleph_0$

AND $2^\lambda = \lambda^+$ FOR ALL $\lambda < \kappa$

THEN $2^\kappa = \kappa^+$.

BOOK: $\kappa = \aleph_{\omega_1}$

CONVERSE OF PDL

IF S IS COFINAL NOT STAT.

SAY $S \cap C = \emptyset$, C CUB

$f : S \to \kappa$ $\quad f(\alpha) = \max((C \cap \alpha) \cup \{0\})$

ONLY CONSTANT ON BOUNDED SETS.

ARE STATIONARY SETS IN $Cub(\kappa)$?

NO EASY FOR ω_2!

$E_0 = \{\alpha < \omega_2 : cf\alpha = \omega_0\}$
$E_1 = \{\alpha < \omega_2 : cf\alpha = \omega_1\}$

EASY FOR $\kappa \geq \omega_2$
IF $\lambda < \kappa$ IS REGULAR
THEN $E^{\kappa}_{\lambda} = \{\alpha < \kappa : cf(\alpha) = \lambda\}$
IS STATIONARY.

HOW ABOUT ω_1?
HOMEWORK (1): TWO DISJOINT
STAT. SETS.
NEEDS CHOICE

SOLOVAY
IF κ IS REGULAR UNCOUNTABLE
AND $S \subseteq \kappa$ IS STATIONARY
THEN S CAN BE SPLIT INTO
κ MANY STATIONARY SETS:

$$S = \bigcup_{\alpha < \kappa} T_\alpha$$

- T_α STATIONARY
- $T_\alpha \cap T_\beta = \emptyset \quad (\alpha \neq \beta)$

[κ A SUCCESSOR ULAM
SEE CHAPTER 10 "ULAM MATRIX"
p 131 – 132]

STEP 1 SUPPOSE $S \subseteq E^{\kappa}_{\omega}$

CHOOSE SEQUENCES

$s_\alpha = \langle \beta_{\alpha,n} : n\in\omega \rangle \; \lambda$

INCREASING COFINAL IN α ($\alpha \in S$)

CLAIM; THERE IS AN $m \in \omega$ λ

SUCH THAT FOR ALL $\gamma < \kappa$

THE SET $\{\alpha \in S : \beta_{\alpha,m} > \gamma\}$

IS STATIONARY.

$\gamma_0 + 1$

$\gamma_1 + 1$

$\vdots$

$\gamma + 1$

SUPPOSE NOT

TAKE FOR EVERY $m \in \omega$ λ

A CUB SET C_m AND $\eta_m < \kappa$

SUCH THAT

$$\alpha \in C_m \cap S \longrightarrow \beta_{\alpha,m} < \eta_m$$

NOW TAKE $\alpha \in S \cap \bigcap_{m\in\omega} C_m$

THEN $\beta_{\alpha,m} < \eta_m$

SO $\alpha \leq \underline{\sup_{m\in\omega} \eta_m}$

$$\alpha = \sup_{m\in\omega} \beta_{\alpha,m}$$

THIS WOULD MEAN THAT

$$S \cap \bigcap_{m\in\omega} C_m$$

IS BOUNDED ! CONTRADICTION

TAKE OUR m : $f : S \longrightarrow \kappa$

$\alpha \longmapsto \beta_{\alpha,m}$

TAKE $\gamma_0 < \kappa$ SUCH THAT

$T_0 = \{\alpha : f(\alpha) = \gamma_0\}$ IS STAT.

THERE IS $\gamma_1 > \gamma_0$ SUCH THAT

$T_1 = \{\alpha : f(\alpha) = \gamma_1\}$ IS STAT

$\vdots$

GIVEN $\gamma_0 < \gamma_1 < \cdots < \gamma_\xi < \cdots$ $\xi < \eta < \kappa$

LET $\gamma = \sup_{\xi < \eta} \gamma_\xi < \kappa$

NOW TAKE $\gamma_\eta > \gamma$ SUCH THAT $T_{\gamma_\eta} = \{\alpha \in S : f(\alpha) = \gamma_\eta\}$ IS ~~ST~~ STAT.

THE FAMILY $\{T_{\gamma_\eta} : \eta < \kappa\}$ IS AS REQUIRED.

STEP 2 SUPPOSE $S \subseteq E^{\kappa}_{\lambda}$ FOR SOME $\lambda < \kappa$.

SAME PROOF

∇
$\boxed{\begin{array}{l}\text{IF } S \text{ IS STATIONARY AND } S = \bigcup_{\alpha<\lambda} S_\alpha \text{ WHERE } \lambda < \kappa \\ \text{THEN SOME } S_\alpha \text{ IS STATIONARY} \\ \text{CONTRAPOSITIVE: THE UNION OF FEWER THAN } \kappa \text{ MANY } \underline{\text{NON-STATIONARY}} \text{ SETS} \\ \text{IS } \underline{\text{NON}} \text{ STATIONARY}\end{array}}$

$\mathcal{O}'$ désignant l'ensemble des nombres de deuxième espèce, une *suite distinguée* sur un sous-ensemble A de $\mathcal{O}'$ sera par définition une suite de fonctions régressives $f_n(x)(n \in \mathrm{N})$ définies sur A, telle que, pour tout $x \in \mathrm{A}$, $f_1(x) \leqq f_2(x) \leqq \ldots \leqq f_n(x) \leqq \ldots$, et $\lim_n f_n(x) = x$.

THÉORÈME II. — *Une suite distinguée étant donnée sur un sous-ensemble stationnaire* A *de* $\mathcal{O}'$, *il existe un entier* n_0 *tel que, pour* $n > n_0$, *et pour toute partition de* A *en deux ensembles* B_n *et* C_n *tels que* $f_n(x)$ *soit bornée supérieurement sur* B_n, *l'ensemble* C_n *soit stationnaire.*

La démonstration est omise faute de place.

En supprimant alors au besoin un nombre fini de termes dans la suite distinguée, nous pouvons supposer que $n_0 = i$. Considérons alors la fonction $f_1(x)$, et l'ensemble H des points de $\mathcal{O}$ dont l'image réciproque par $f_1(x)$ est non dénombrable. H est lui-même non dénombrable. En effet, si $\mathrm{K} = \overset{-1}{f_1}(\mathrm{H})$, et $\mathrm{L} = \mathrm{A} - \mathrm{K}$, l'ensemble L est non stationnaire; si alors H était dénombrable. $f_1(x)$ serait bornée supérieurement sur K et le théorème II serait en défaut. On peut alors numéroter les éléments h_i de H par ordre de grandeur croissante, l'indice i parcourant l'espace $\mathcal{O}$ tout entier.

STEP 3 $\quad S \subseteq \{\alpha : cf\alpha < \alpha\}$

$\alpha \mapsto cf\alpha$ IS REGRESSIVE

HENCE CONSTANT ON A STAT SUBSET T, SAY VALUE λ

BACK AT STEP 2 WITH $T \subseteq E^{\kappa}_{\lambda}$.

STEP 4 $\quad S \subseteq \{\alpha : cf\alpha = \alpha\}$

κ IS MAHLO [THE REGULAR CARDINALS FORM A STAT. SUBSET OF κ !!!

USE THE SAME PROOF BUT WITH A FEW TWISTS.

LOOK AT

$T = \{\alpha \in S : S \cap \alpha$ NOT STATIONARY IN $\alpha\}$

CLAIM: T IS STATIONARY.

TAKE $C \subseteq \kappa$ A CUB

LET C' BE THE SET OF LIMIT POINTS

$\alpha \in C'$ IF $\alpha = \sup(\alpha \cap C)$

C' IS CUB

LET $\alpha = \min S \cap C'$ $\quad$ (SO $\alpha \in C$)

- $\alpha \in C'$ SO $C \cap \alpha$ IS CUB IN α
- $C' \cap S \cap \alpha = \emptyset$

AND $C \cap \alpha$ IS CUB IN α IMPLIES $C' \cap \alpha$ IS CUB IN α.

SO $S \cap \alpha$ IS NOT STAT IN α

WORK WITH T

- T STAT.
- $\alpha \in T$: α REG. UNCOUNTABLE

  $T \cap \alpha$ NOT STAT. IN α

$C_\alpha \subseteq \alpha$ CUB WITH $C_\alpha \cap (T \cap \alpha) = \emptyset$

ENUMERATE C_α: $\langle \beta_{\alpha,\xi} : \xi < \alpha \rangle$

THE SEQUENCES ARE NORMAL; INCREASING AND CONTINUOUS

η LIMIT: $\beta_{\alpha,\eta} = \sup_{\xi<\eta} \beta_{\alpha,\xi}$

CLAIM THERE IS A $\xi < \kappa$ SUCH THAT FOR ALL $\eta < \kappa$

$\{\alpha \in T : \beta_{\alpha,\xi} > \eta\}$

IS STATIONARY

[THEN THE PROOF FINISHES AS BEFORE]

SUPPOSE NOT

FOR EVERY ξ TAKE

C_ξ A CUB AND $\gamma_\xi < \kappa$

SUCH THAT

$$\alpha \in C_\xi \longrightarrow \beta_{\alpha,\xi} < \gamma_\xi$$

$$C = \underset{\xi<\kappa}{\Delta} C_\xi \qquad D = \{\delta : \xi < \delta \to \gamma_\xi < \delta\}$$

[GROUP INTERACTION]

TAKE

$\alpha < \delta$ BOTH IN $T \cap C \cap D$

IF $\xi < \alpha$: $\beta_{\alpha,\xi} < \gamma_\xi < \alpha$

$\beta_{\delta,\xi} < \gamma_\xi < \alpha \Leftarrow$

$\langle \beta_{\delta,\xi} : \xi < \alpha \rangle$ IS INCREASING OF ORDER TYPE α, AND BELOW α

HENCE $\beta_{\delta,\alpha} = \underline{\sup_{\xi<\alpha} \beta_{\delta,\xi}} = \alpha$

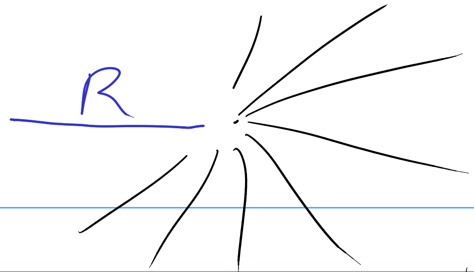
BUT $\alpha \in T$, $\beta_{\delta,\alpha} \in C_\delta$ $C_\delta \cap T = \emptyset$

THERE'S THE CONTRADICTION

THEOREM [SHANIN]

LET $\mathcal{A}$ BE AN UNCOUNTABLE FAMILY OF FINITE SETS.

THERE IS AN UNCOUNTABLE SUBFAMILY $\mathcal{B}$ AND THERE IS A SET R SUCH THAT IF $B_1, B_2 \in \mathcal{B}$ AND $B_1 \neq B_2$ THEN $B_1 \cap B_2 = R$



$\leftarrow$ Δ-SYSTEM

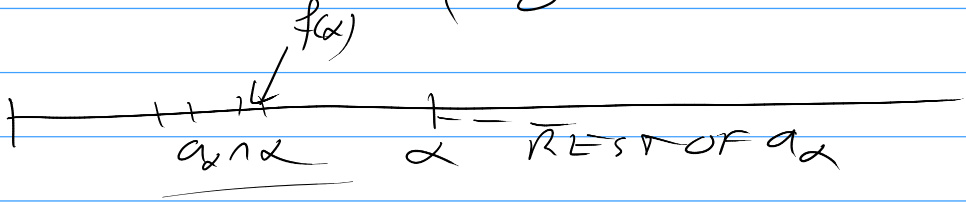
WE ASSUME $|\mathcal{A}| = \aleph_1$

AND $\bigcup\mathcal{A} \subseteq \omega_1$

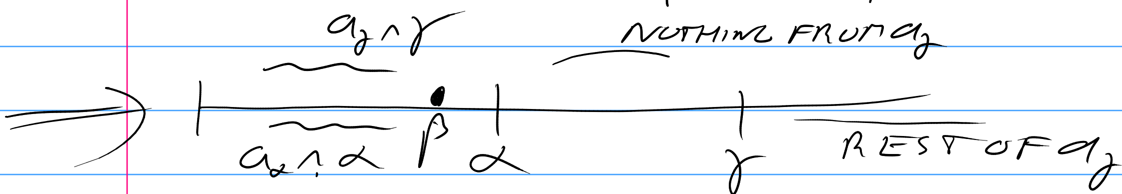
ENUMERATE $\mathcal{A}$ AS $\langle a_\alpha : \alpha \in \omega_1 \rangle$

REGRESSIVE FUNCTION:

$$f(\alpha) = \begin{cases} \max(a_\alpha \cap \alpha) & (a_\alpha \cap \alpha \neq \emptyset) \\ 0 & (a_\alpha \cap \alpha = \emptyset) \end{cases}$$



TAKE S STATIONARY AND β SUCH THAT $f(\alpha) = \beta$ $(\alpha \in S)$



FIRST

$[\beta+1]^{<\aleph_0}$ IS COUNTABLE

FOR EVERY x

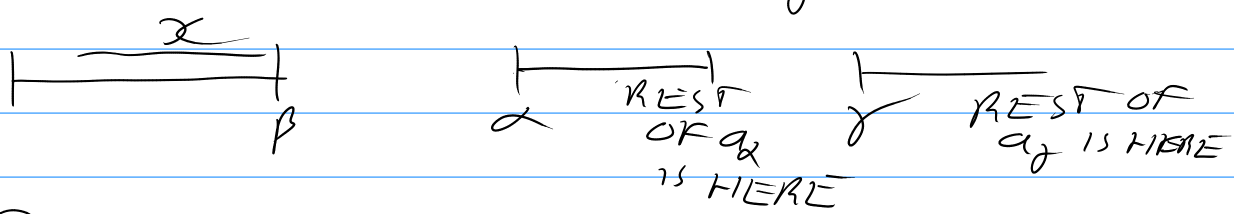
$$S_x = \{\alpha \in S : a_\alpha \cap \alpha = x\}$$

$$\bigcup_x S_x = S$$

SOME S_x IS STATIONARY!

$D = \{\delta : \alpha < \delta \to \max a_\alpha < \delta\}$

NOW LOOK AT $\alpha < \gamma$ IN $S \cap D$



$\mathcal{B} = \{a_\alpha : \alpha \in S \cap D\}$ IS AS REQUIRED WITH $R = x$

WHAT ABOUT COUNTABLE SETS?

$\mathcal{A} = \{ a_\alpha : \alpha < \omega_2 \}$

SAME PROOF WITH $E^{\omega_2}_{\omega_1}$

$$f(\alpha) = \sup(a_\alpha \cap \alpha)$$

PROBLEM:

WHAT IF $2^{\aleph_0} \geq \aleph_2$??

THE PROOF GOES THROUGH WITH $(2^{\aleph_0})^+$ INSTEAD OF $\aleph_2$.