Set Theory 2021-11-15

More combinatorics.

1. The Free Set Theorem
2. Ramsey's Theorem
3. The Erdős-Rado Theorem

1936 S. Ruziewicz [in our language]

Let κ be a cardinal and λ a cardinal smaller than κ.

Let $F: \kappa \longrightarrow [\kappa]^{<\lambda}$ be a map such that $\alpha \notin F(\alpha)$ for all α.

Must there be a <u>free</u> subset S for F of κ of cardinality κ?

S is free for F: if $\alpha, \beta \in S$ and $\alpha \neq \beta$ then $\alpha \notin F(\beta)$ (and $F(\beta) \not\ni \alpha$)

or $S \cap \bigcup\{F(\alpha) : \alpha \in S\} = \emptyset$.

In Homework #9 we saw the case "κ is regular"

A quick sketch of the proof:

- For all α we know $\sup F(\alpha) < \kappa$

  So there is a club set C such that

  $\forall \delta \in C \ \forall \alpha \in \delta \quad \sup F(\alpha) < \delta$ (so $F(\alpha) \subseteq \delta$)

  Trick: $C = \Delta_{\alpha < \kappa} [\sup F(\alpha) + 1, \kappa)$

- Assume λ is regular.

  Then $f: E^{\kappa}_{\lambda} \longrightarrow \kappa$ given by

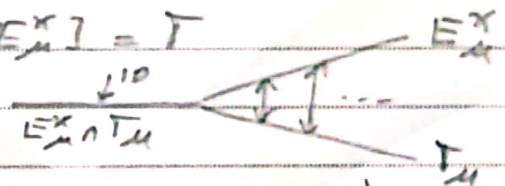
  $f(\alpha) = \sup(\alpha \cap F(\alpha))$;

  is regressive, hence constant on a stationary set T, with value γ

- Look at $S = T \cap C$;
  - S is stationary
  - S is free: if $\alpha < \beta$ in S then

    $\alpha \in (\gamma, \beta)$ and $F(\beta) \cap (\gamma, \beta) = \emptyset$; and $\sup F(\alpha) < \beta$

- If λ is singular observe that
  $\kappa = \bigcup\{T_\mu : \mu < \lambda,\ \mu \text{ a cardinal}\}$
  where $T_\mu = \{\alpha : |F(\alpha)| < \mu\}$
  so there is a μ such that
  T_μ is stationary and so
  T_ϑ is stationary if $\mu \leq \vartheta < \lambda$
  so we can assume μ is regular.
  - If $T_\mu \cap E^\kappa_\mu$ is stationary then follow the "λ regular" case
  - If not take a bijection $b : \kappa \to \kappa$ such that $b \circ b = \mathrm{id}$, $b[T] = E^\kappa_\mu$ and $b[E^\kappa_\mu] = T$



Look at $G : \kappa \to [\kappa]^{<\lambda}$ defined by $G(\alpha) = b[F(b(\alpha))]$.
Then $|G(\alpha)| < \mu$ when $\alpha \in E^\kappa_\mu$
so there is a free set S of cardinality κ

$$\begin{aligned}\emptyset = S \cap \bigcup\{G(\alpha) : \alpha \in S\} &= S \cap \bigcup\{b[F(b(\alpha))] : \alpha \in S\} \\ &= S \cap b[\bigcup\{F(b(\alpha)) : \alpha \in S\}] \\ &= S \cap b[\bigcup\{F(\beta) : \beta \in b[S]\}]\end{aligned}$$

But then $b[S] \cap \bigcup\{F(\beta) : \beta \in b[S]\} = \emptyset$.

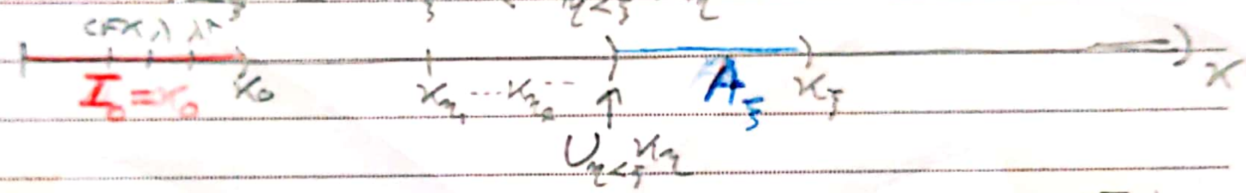
Now the case where κ is singular
This will take more work.
So, we have a singular cardinal κ,
a cardinal λ below κ and
a map $F : \kappa \to [\kappa]^{<\lambda}$
We need to make some preparations.

Let $\langle \kappa_\xi : \xi < cf\,\kappa \rangle$ be a strictly increasing and cofinal sequence of cardinals in κ. We assume

- $cf\,\kappa < \lambda$ [if we can prove it for large λ ....]
- $\lambda^+ < \kappa_0$
- each κ_ξ is regular [replace each κ_ξ by κ_ξ^+ if necessary]

Now split κ into intervals

$$I_\xi = \kappa_\xi \setminus \bigcup_{\eta<\xi} \kappa_\eta$$



By regularity of the κ_ξ we have $|I_\xi| = \kappa_\xi$

We need an analogue of the subset C of the regular case.

Define $\langle A_\xi : \xi < cf\,\kappa \rangle$ such that

1. $\bigcup_{\xi < cf\kappa} A_\xi = \kappa$; $\eta < \xi \to A_\eta \subseteq A_\xi$
2. $|A_\xi| = \kappa_\xi$
3. if $\alpha \in A_\xi$ then $F(\alpha) \subseteq A_\xi$

- Assume we have $\langle A_\eta : \eta < \xi \rangle$

Let $A_{\xi,0} = I_\xi \cup \bigcup_{\eta<\xi} A_\eta$

$A_{\xi,n+1} = A_{\xi,n} \cup \bigcup \{F(x) : x \in A_{\xi,n}\}$

and $A_\xi = \bigcup_{n\in\omega} A_{\xi,n}$.

1 and 3 are clear

2: • $I_\xi \subseteq A_\xi$ so $|A_\xi| \geq \kappa_\xi$

• $|A_{\xi,0}| \leq \kappa_\xi + \sum_{\eta<\xi} \kappa_\eta \leq |\xi| \cdot \kappa_\xi = \kappa_\xi$

$|A_{\xi,n+1}| \leq \kappa_\xi + \kappa_\xi \cdot \lambda = \kappa_\xi$

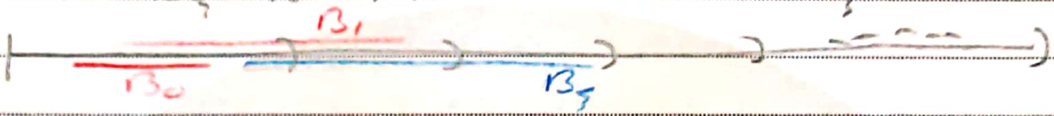
• $|A_\xi| \leq \kappa_\xi \cdot \aleph_0 = \kappa_\xi$.

APPLY THE REGULAR CASE WE GET

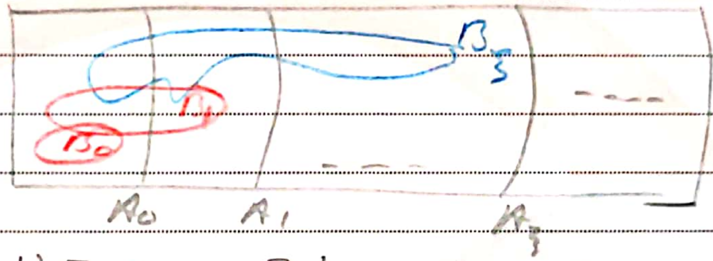
$\langle B_\xi : \xi < cf\,\kappa \rangle$

SUCH THAT

- $B_\xi \subseteq A_\xi$ AND $|B_\xi| = \kappa_\xi$
- B_ξ IS FREE FOR $F \upharpoonright A_\xi$.



OR MAYBE



SO $B_\xi \cap \bigcup\{F(\alpha) : \alpha \in B_\xi\} = \emptyset$. FOR ALL ξ

WE MAKE THE B_ξ A BIT SMALLER

WE DEFINE $\langle C_\xi : \xi < cf\,\kappa \rangle$ WITH

- $C_\xi \subseteq B_\xi$ AND $|C_\xi| = |B_\xi| = \kappa_\xi$

SUCH THAT

- IF $\eta \leq \xi$ AND $\alpha \in C_\eta$ THEN $F(\alpha) \cap C_\xi = \emptyset$

  [ONLY $\eta < \xi$ NEEDS WORK

  IF $\alpha \in C_\eta$ THEN $F(\alpha) \cap B_\xi = \emptyset$]

- GIVEN $\langle C_\eta : \eta < \xi \rangle$ LET

  $H_\xi = \bigcup_{\eta < \xi} C_\eta$

  SO $|H_\xi| \leq \Sigma_{\eta < \xi} \kappa_\eta < \kappa_\xi$

  AND $|\bigcup\{F(\alpha) : \alpha \in H_\xi\}| \leq |H_\xi| \cdot \lambda < \kappa_\xi = |B_\xi|$

  LET $C_\xi = B_\xi \setminus \bigcup\{F(\alpha) : \alpha \in H_\xi\}$

[THIS LOOKS LIKE THE CUB SET C]

BUT, WE PROBABLY STILL HAVE MANY

CASES WHERE $\eta < \xi$, $\alpha \in C_\xi$

AND $F(\alpha) \cap C_\eta \neq \emptyset$.

HERE COMES THE CLEVER BIT:

WE CREATE YET MORE SEQUENCES

$\langle D_\xi : \xi < \mathrm{cf}\,\kappa \rangle$ AND

$\langle D_{\xi,\gamma} : \gamma < \lambda^+ \rangle$ FOR EACH $\xi < \mathrm{cf}\,\kappa$

SUCH THAT

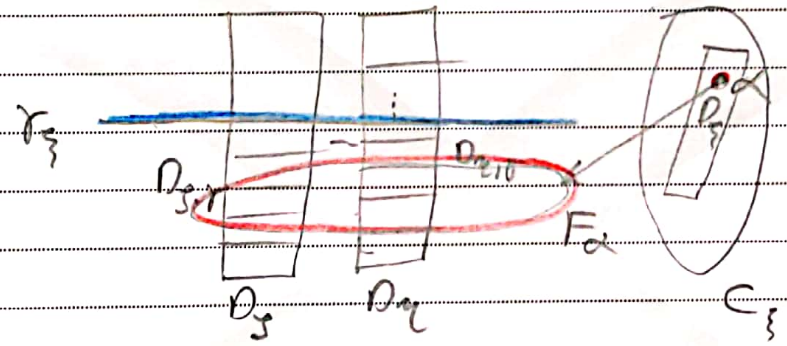
- $D_\xi \subseteq C_\xi$; $D_\xi = \bigcup_{\gamma < \lambda^+} D_{\xi,\gamma}$; $D_{\xi,\gamma} \cap D_{\xi,\delta} = \emptyset$ $(\gamma \neq \delta)$
- $|D_\xi| = |D_{\xi,\gamma}| = \kappa_\xi$ FOR ALL γ
- FOR EVERY $\xi < \gamma$ THERE IS $\gamma_\xi < \lambda^+$

  SUCH THAT

  FOR ALL $\alpha \in D_\xi$ AND ALL $\eta < \xi$

  IF $\gamma_\xi < \gamma < \lambda^+$

  THEN $F(\alpha) \cap D_{\eta,\gamma} = \emptyset$



GIVEN $\langle D_\eta : \eta < \xi \rangle$ AND $\langle D_{\eta,\gamma} : \eta < \xi ; \gamma < \lambda^+ \rangle$

FIND D_ξ AS FOLLOWS:

FOR $\alpha \in C_\xi$ WE HAVE $|F(\alpha)| < \lambda$

AND SO FOR $\eta < \xi$ THERE IS $\gamma_{\alpha,\eta}$

SUCH THAT $F(\alpha) \cap \bigcup_{\gamma > \gamma_{\alpha,\eta}} D_{\eta,\gamma} = \emptyset$

ALSO $\xi < \mathrm{cf}\,\kappa < \lambda^+$ SO THERE IS ONE γ_α

SUCH THAT FOR ALL $\eta < \xi$ AND ALL $\gamma > \gamma_\alpha$

WE HAVE $F(\alpha) \cap D_{\eta,\gamma} = \emptyset$

NOW $C_\xi = \bigcup_{\gamma < \lambda^+} \{\alpha : \gamma_\alpha \leq \gamma\}$

TAKE $\delta_\xi < \lambda^+$ SUCH THAT

$D_\xi = \{\alpha \in C_\xi : \gamma_\alpha \leq \delta_\xi\}$

HAS CARDINALITY κ_ξ (κ_ξ IS REGULAR AND $\lambda^+ < \kappa_\xi$)

NEXT LET $\langle D_{\xi,\gamma} : \gamma < \lambda' \rangle$ BE ANY PARTITION OF D_ξ INTO PIECES OF CARDINALITY κ_ξ.

LOOK AT $\langle \delta_\xi : \xi < cf\kappa \rangle$.

USE THAT $cf\kappa < \lambda'$.

THERE IS A $\underline{\delta} < \lambda'$ SUCH THAT

$\delta_\xi < \delta$ FOR ALL $\xi < cf\kappa$

FOR ALL $\xi < cf\kappa$ WE SET

$$E_\xi = \bigcup_{\gamma > \delta} D_{\xi,\gamma}$$

NOW, IF $\eta < \xi$ AND $\alpha \in E_\xi$

THEN $\alpha \in D_\xi$ AND SO

$$F(\alpha) \cap E_\eta \subseteq F(\alpha) \cap \bigcup_{\gamma > \delta_\eta} D_{\eta,\gamma} = \emptyset$$

WE ALREADY KNOW

IF $\eta < \xi$ AND $\alpha \in E_\eta$

THEN $\alpha \in C_\eta$ AND SO $F(\alpha) \cap E_\xi = \emptyset$

FINALLY THEN

$$E = \bigcup_{\xi < cf\kappa} E_\xi$$

IS A FREE SET AND

$$|E| = \sup_{\xi < cf\kappa} |E_\xi| = \sup_{\xi < cf\kappa} \kappa_\xi = \kappa.$$

PARTITION CALCULUS

WE KNOW: IF WE COVER A LARGE SET WITH A SMALL NUMBER OF SETS THEN ONE OF THESE SETS IS RELATIVELY LARGE.

- IF "SMALL NUMBER" IS "FEWER THAN $cf\kappa$" THEN LARGE = RELATIVELY LARGE = SIZE κ
- IF "LARGE" IS "REGULAR" AND "SMALL NUMBER" IS "FEWER THAN κ" THEN "RELATIVELY LARGE" IS "STATIONARY".

OTHER INSTANCES

- RAMSEY'S THEOREM

IF $n, k < \omega$ AND $[\omega]^n = I_0 \cup \dots \cup I_{k-1}$, THEN THERE IS AN INFINITE $H \subseteq \omega$ SUCH THAT $[H]^n \subseteq I_i$ FOR SOME $i \in k$.

$n = 1$: $\omega = I_0 \cup \dots \cup I_{k-1}$ THEN SOME I_i IS INFINITE

$n = 2$: $[\omega]^2$ IS THE COMPLETE GRAPH ON ω

$[\omega]^2 = I_0 \cup \dots \cup I_{k-1}$ COLOURS THE EDGES

RAMSEY: THERE IS AN INFINITE H SUCH THAT ALL EDGES BETWEEN THE VERTICES IN H HAVE ONE AND THE SAME COLOUR.

H IS CALLED HOMOGENEOUS.

- SO WE HAVE THE CASE $n = 1$
- $n \to n+1$ LET $F : [\omega]^{n+1} \to k$ BE GIVEN

($\text{SO } I_i = \{ x \in [\omega]^{n+1} : F(x) = i \}$)

FOR $a \in \omega$ WE HAVE $F_a : [\omega \setminus \{a\}]^n \to k$

$$F(x) = F(x \cup \{a\})$$

LET $a_0 = 0$ TAKE i_0 AND $H_0 \subseteq \omega \setminus \{a_0\}$ INFINITE WITH F_{a_0} MAPS $[H_0]^n$ TO i_0

LET $a_1 = \min H_0$

TAKE i_1 AND $H_1 \subseteq H_0 \setminus \{a_0, a_1\}$ INFINITE WITH F_{a_1} MAPS $[H_1]^n$ TO i_1

⋮

$a_{j+1} = \min H_j$

TAKE i_{j+1} AND $H_{j+1} \subseteq H_j \setminus \{a_0, \dots, a_j\}$ INFINITE WITH $F_{a_{j+1}}$ MAPS $[H]^n_{j+1}$ TO i_{j+1}

⋮

LET $H = \{a_j : j \in \omega\}$

OBSERVE IF $x \in [\omega]^{n+1}$: $x = \{a_{j_0} < \dots < a_{j_n}\}$

THEN $\{a_{j_1}, \dots, a_{j_n}\} \in [H_{j_0}]^n$

AND SO $F(x) = i_{j_0}$:

THE VALUE OF $F(x)$ DEPENDS ON MIN x !

LET I BE INFINITE AND $i \in k$

BE SUCH THAT $j \in I \to i_j = i$

AND NOW

$$J = \{ a_j : j \in I \}$$

SATISFIES $F[[J]^{n+1}] = \{i\}$.

NOTATION $\aleph_0 \longrightarrow (\aleph_0)^n_k$ ← SIZE OF FINITE SET; ← NUMBER OF COLOURS

SIZE OF STARTING SET SIZE OF RESULTING SET

THIS OPENS UP POSSIBILITIES

CAN WE GET

$$\aleph_1 \to (\aleph_1)^n_k \; ?$$

NO, NOT EVEN

$$\aleph_1 \to (\aleph_1)^2_2$$

EVEN

$$2^{\aleph_0} \not\to (\aleph_1)^2_2 \qquad \text{[SIERPIŃSKI]}$$

LET $<$ BE THE NORMAL ORDER OF $\mathbb{R}$

AND LET $\lhd$ BE SOME WELL-ORDER OF $\mathbb{R}$

$$F(\{x,y\}) = \begin{cases} 0 & \text{IF } x<y \Leftrightarrow y \lhd x \\ 1 & \text{IF } x<y \Leftrightarrow x \lhd y \end{cases}$$

"TRUTH VALUE OF "$<$ AND $\lhd$ AGREE ON $\{x,y\}$""

IF H IS HOMOGENEOUS THEN H

IS WELL-ORDERED BY $<$ OR BY $>$

AND HENCE COUNTABLE

In general $2^\kappa \not\to (\kappa^+)^2_2$

Because $\{0,1\}^\kappa$ with lexicographic order does not have increasing or decreasing sequences of length κ^+.

We do have the Erdős-Rado theorem

$$(2^\kappa)^+ \longrightarrow (\kappa^+)^2_\kappa$$

$\longleftarrow$ even κ many! colours

Let $\lambda = (2^\kappa)^+$

Let $F: [\lambda]^2 \longrightarrow \kappa$ be given.

We define an auxiliary function

if $A \subseteq \lambda$ has cardinality κ (or less)

and if $f: A \to \kappa$ is such that

there is an $\alpha \in \lambda$ such that

- $f(\beta) = F(\{\alpha, \beta\})$ $\beta \in A$
- $A \subseteq \alpha$

Then we let $G(A,f)$ be the minimum such α. In all other cases $G(A,f) = 0$

Now define $\langle A_\xi : \xi \leq \kappa^+ \rangle$ as follows

- $A_0 = 2^\kappa$
- if A_ξ is given with $|A_\xi| = 2^\kappa$

  then we let

  $$A_{\xi+1} = A_\xi \cup \{G(A,f) : A \subseteq A_\xi; |A| \leq \kappa; f: A \to \kappa\}$$

  note $\{(A,f) : A \subseteq A_\xi ; |A| \leq \kappa; f: A \to \kappa\}$

  has cardinality (at most) 2^κ

  so $|A_{\xi+1}| = 2^\kappa$

- ξ limit : $A_\xi = \bigcup_{\eta < \xi} A_\eta$

  assuming $|A_\eta| \leq 2^\kappa$ for all η we get

  $|A_\xi| \leq 2^\kappa$.

Consider A_{κ^+}

- $|A_{\kappa^+}| = 2^{\kappa}$
- if $A \in [A_{\kappa^+}]^{\le \kappa}$ and $f : A \to \kappa$ then $G(A,f) \in A_{\kappa^+}$ [We did this before...]

Take $\alpha \in \lambda$ such that $A_{\kappa^+} \subseteq \alpha$

Define $\langle \beta_\xi : \xi < \kappa^+ \rangle$ in A_{κ^+} increasing

- $\beta_0 = 0$
- given $\langle \beta_\eta : \eta < \xi \rangle$
  let $A = \{\beta_\eta : \eta < \xi\}$
  and $f : A \to \kappa$ defined by $f(\beta_\eta) = F(\{\beta_\eta, \alpha\})$
  then $G(A,f) \neq \emptyset$ because "there is an α"!
  Let $\beta_\xi = G(A,f)$.

Let $I = \{\beta_\xi : \xi < \kappa^+\} \cup \{\alpha\}$

Note if $\eta < \xi$ then

$$F(\{\beta_\eta, \alpha\}) = F(\{\beta_\eta, \beta_\xi\})$$

Let $S \subseteq \kappa^+$ be of cardinality κ^+ such that $\eta \mapsto F(\{\beta_\eta, \alpha\})$ is constant on S with value δ.

Let $H = \{\beta_\eta : \eta \in S\} \cup \{\alpha\}$
then H is homogeneous with colour δ.

Alternative define $g : \lambda \to \lambda$ by

$$g(\alpha) = \sup\{G(A,f) : A \in [\alpha]^{\le\kappa};\ f : A \to \kappa\}$$

Let $C = \{\delta : (\forall \alpha < \delta)(g(\alpha) < \delta)\}$: C is cub

Let $\delta \in C$ the proof above goes through unchanged.