SET THEORY 2021-11-29

TODAY IS A HISTORIC DAY

THE SITUATION THUS FAR.

GIVEN FINITELY MANY AXIOMS $\varphi_1, \ldots, \varphi_k$ OF ZFC, INCLUDING EXTENSIONALITY, PAIRING, UNION, POWER SET, INFINITY, AND REGULARITY,

WE FOUND

FIRST: (MANY) β SUCH THAT $\varphi_i^{V_\beta}$ FOR ALL i

SECOND: A COUNTABLE $A \subseteq V_\beta$ SUCH THAT φ_i^A FOR ALL i

THIRD: A COUNTABLE TRANSITIVE M THAT IS ISOMORPHIC WITH A. SO ALSO φ_i^M FOR ALL i.

OUR NEXT GOAL: EXTEND M TO A COUNTABLE TRANSITIVE N SUCH THAT φ_i^N FOR ALL i <u>AND</u> φ^N, WHERE φ IS A FORMULA LIKE CH OR $\neg$CH OR ---- MANY OTHER SET-THEORETIC STATEMENTS.

A SLIGHT COMPLICATION: TO GET φ_i^N FOR OUR GIVEN φ_i, WE SHALL NEED ψ^M FOR A LARGER SET OF FORMULAS ψ.

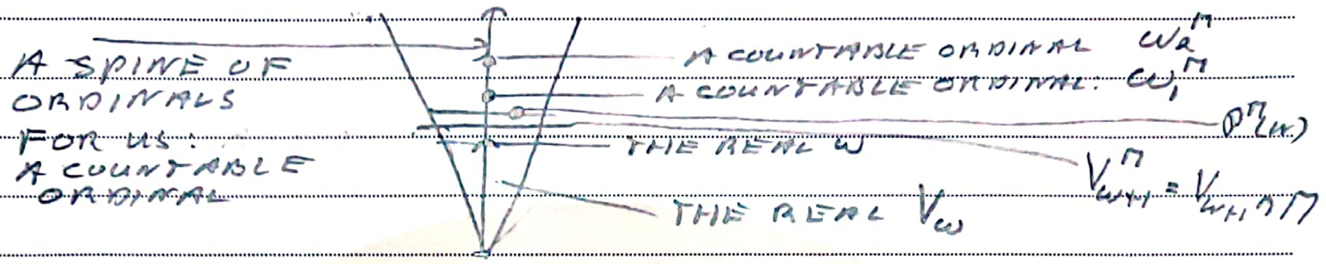
WE CHEAT (A BIT)

WE ASSUME THAT M SATISFIES <u>ALL</u> AXIOMS OF ZFC AND CONSTRUCT N SO THAT IT ALSO SATISFIES ALL AXIOMS OF ZFC.

WE CONCENTRATE ON THE TARGET FORMULA φ.

YOU CAN LATER 'AUDIT' THE PROOF AND EXTRACT A LIST OF THE FINITELY MANY AXIOMS YOU ACTUALLY USED.

WHAT DOES M LOOK LIKE?



M SATISFIES "THERE IS A FIRST ORDINAL α SUCH THAT THERE IS NO f THAT IS A SURJECTION FROM ω ONTO α"

$\omega_1^M = \min\{\alpha : (\forall f \in M)(\mathrm{Dom}\, f = \omega \to \alpha \notin \mathrm{Ran}\, f)\}$.

$\omega_2^M = $ —— " —— ω_1^M —— " ——

LAST WEEK'S GROUP INTERACTION

- $\mathcal{P}(\omega)$, $\mathbb{R}$, ... $\in A$
- M IS COUNTABLE AND TRANSITIVE
  SO $\mathcal{P}(\omega)$, $\mathbb{R} \notin M$
  BUT $\pi(\mathcal{P}(\omega))$, $\pi(\mathbb{R}) \in M$
- $\mathcal{P}^M(\omega) = \{x \in M : x \subseteq \omega\} = \mathcal{P}(\omega) \cap M$

WE EXTEND M HORIZONTALLY: $\mathrm{Ord}\, N$

- M AND N WILL HAVE THE SAME ORDINALS
- ASSUME WE WANT $\neg CH$ IN N; ASSUMING CH IN M
  EASY: TAKE $f : \omega_2^M \to \mathcal{P}(\omega)$ INJECTIVE
  <u>FROM OUTSIDE OF M</u>
  AND ADD IT TO M
  THAT IS: TAKE $N = M[f]$ SOMEHOW
- FOR EXAMPLE TAKE A β SUCH THAT
  $A \cup \{f\} \subseteq V_\beta$ AND V_β SATISFIES OUR φ_c ...
  APPLY LÖWENHEIM-SKOLEM TO GET $A^+ \supseteq A \cup \{f\}$
  AND MOSTOWSKI TO GET $N \supseteq M \cup \{f\}$.

What could possibly go wrong?

- f could have been a bijection between ω_2^M and $\mathbb{P}^M(\omega)$
  Cantor-Bernstein: also a bijection between ω_2^M and ω_1^M.
  So ω_2^M has become ω_1^M or worse...
- $f \restriction \omega$ could have been a bijection between ω and $\mathbb{P}^M(\omega)$
  we have lost ω_1^M and ω_2^M and have no idea about ω_1^N and ω_2^N
- $f[\omega_2^M]$ could contain ugly subsets of ω from outside of M.
  For example $a \subseteq \omega$ that gives, under a definable bijection $\omega \leftrightarrow \omega \times \omega$ a well-order in type $M \cap \mathrm{Ord}$
  The whole univere collapses!

We need to involve M more.
How? Approximate f from inside M.
How? There are many ways, but we use Cohen's original approach as (re)formulated by Schoenfield

We use finite approximations of f:
$\mathbb{P}$" is the set of finite functions p such that

- $\mathrm{dom}\, p \subseteq \omega_2^M \times \omega$
- $\mathrm{ran}\, p \subseteq \{0, 1\}$.

or better its version in M

$\mathbb{P}^M = \{p \in \mathbb{P} : \mathrm{dom}\, p \subseteq \omega_2^M \times \omega\}$

Notation $\underline{Fn(\omega_2 \times \omega, 2)}$

INTERPRETATION

IF $p \in \mathbb{P}$ AND $\langle \alpha, n \rangle \in \operatorname{dom} p$

THEN p TELLS US $\begin{cases} n \in f(\alpha) & \text{IF } p(\alpha, n) = 1 \\ n \notin f(\alpha) & \text{IF } p(\alpha, n) = 0 \end{cases}$

IF $\langle \alpha, n \rangle \notin \operatorname{dom} p$ THEN p KNOWS NOTHING ABOUT THIS.

NOTATION: $p \leq q$ IF $p \supseteq q$

"p IS STRONGER"
"p KNOWS MORE"
"p ALLOWS LESS VARIATION"
"p IS A BETTER APPROXIMATION"

HOW TO MAKE f OUT OF MEMBERS OF $\mathbb{P}$?

TAKE A SUITABLE SUBSET G AND ITS UNION $\bigcup G$

- IF $p, q \in G$ AND $\langle \alpha, n \rangle \in \operatorname{dom} p \cap \operatorname{dom} q$ THEN WE WANT $p(\alpha, n) = q(\alpha, n)$

  IN SHORT IF $p, q \in G$ THEN $p \cup q$ SHOULD BE A FUNCTION AGAIN:

  WE NEED p AND q TO BE COMPATIBLE

- IF $\langle \alpha, n \rangle \in \omega_2 \times \omega$ THEN $\langle \alpha, n \rangle \in \operatorname{dom} p$ FOR SOME $p \in G$.

- IF $\alpha \neq \beta$ THEN WE NEED $p \in G$ AND $n \in \omega$ SUCH THAT $\langle \alpha, n \rangle, \langle \beta, n \rangle \in \operatorname{dom} p$ AND $p(\alpha, n) \neq p(\beta, n)$.

- WE WANT G TO BE A FILTER:
  - IF $p, q \in G$ THEN THERE IS $r \in G$ SUCH THAT $r \leq p, q$ [HERE $r = p \cup q$]
  - IF $p \in G$ AND $p \leq q$ THEN $q \in G$.

SO, IF G IS A FILTER THEN $\bigcup G$ IS A FUNCTION

- TO GET $\operatorname{dom} \bigcup G = \omega_2 \times \omega$ FOR ALL $\langle \alpha, n \rangle \in \omega_2 \times \omega$ WE MUST HAVE

$$G \cap D_{\alpha, n} \neq \emptyset$$

WHERE $D_{\alpha, n} = \{ p \in \mathbb{P} : \langle \alpha, n \rangle \in \operatorname{dom} p \}$

THIS WE CAN DO, INSIDE M EVEN:

$$G = \{ p \in \mathbb{P} : \operatorname{ran} p \subseteq \{0\} \}$$

NOT VERY EXCITING.

- TO GET f INJECTIVE WE MUST HAVE FOR ALL $\alpha < \beta$ IN ω_2:

$$G \cap E_{\alpha, \beta} \neq \emptyset$$

WHERE $E_{\alpha, \beta} = \{ p : (\exists n \in \omega)(\langle \alpha, n \rangle, \langle \beta, n \rangle \in \operatorname{dom} p \wedge p(\alpha, n) \neq p(\beta, n)) \}$.

THIS WE CAN DO TOO - - - - -

WE DEFINE $\underline{f(\alpha) = \{ n : p(\alpha, n) = 1 \}}$ (OF COURSE)

— BUT NOT INSIDE M IF CH HOLDS IN M

OBSERVE THIS ABOUT $D_{\alpha,n}$ AND $E_{\alpha,\beta}$

- FOR ALL $p \in \mathbb{P}$ THERE IS $q \in \mathbb{P}$ SUCH THAT
  $q \le p$ AND $q \in D_{\alpha,m}$
- FOR ALL $p \in \mathbb{P}$ THERE IS $q \in \mathbb{P}$ SUCH THAT
  $q \le p$ AND $q \in E_{\alpha,\beta}$

NOW STEP OUTSIDE M AND NOTE THAT

$\{D_{\alpha,m} : \alpha \in \omega_2^M; m \in \omega\} \cup \{E_{\alpha,\beta} : \alpha \in \beta \in \omega_2^M\}$

IS <u>COUNTABLE</u>

ENUMERATE IT AS $\{D_m : m \in \omega\}$ AND
RECURSIVELY DEFINE $\langle p_m : m \in \omega \rangle$ IN $\mathbb{P}^M$
SUCH THAT
- $p_m \in D_m$ FOR ALL m
- $p_{m+1} \le p_m$ FOR ALL m

LET $G = \{p \in \mathbb{P} : (\exists m \in \omega)(p_m \le p)\}$

THEN G IS A FILTER AND
$G \cap D_{\alpha,m} \ne \emptyset$ AND $G \cap D_{\alpha,\beta} \ne \emptyset$
FOR ALL $\langle \alpha, m \rangle \in \omega_2^M \times \omega$ AND ALL $\alpha \in \beta \in \omega_2^M$

SO $\bigcup G : \omega_2^M \longrightarrow \mathcal{P}(\omega)$ IS INJECTIVE

WE WANT TO ADD (ADJOIN) G TO M TO
OBTAIN $N = M[G]$ IN SUCH A WAY THAT
- $\omega_1^N = \omega_1^M$, $\omega_2^N = \omega_2^M$
- $\operatorname{RAN} \bigcup G \subseteq \mathcal{P}^N(\omega)$

WE WILL ENSURE THAT $M[G]$ SATISFIES
ALL AXIOMS OF ZFC SO $\operatorname{RAN} \bigcup G \subseteq \mathcal{P}^N(\omega)$
WILL BE AUTOMATIC.

HOW TO PROVE $\omega_1^N = \omega_1^M$ AND $\omega_2^N = \omega_2^M$?

EASY: TAKE $f \in N$ SUCH THAT $f : \omega \to \omega_1^M$
$f : \omega_1^M \to \omega_2^M$
FOR $f \in N$
AND SHOW THAT IT
IS NOT SURJECTIVE.

NOT SO EASY: ACTUALLY DO THIS.

FIRST STEP: IMPROVE G: SINCE M IS
COUNTABLE <u>WE</u> CAN ENSURE
THAT $G \cap D \ne \emptyset$ FOR <u>EVERY</u> DENSE SET D
IN $\mathbb{P}$ THAT IS IN M.
- D IS DENSE: $(\forall p \in \mathbb{P})(\exists q \le p)(q \in D)$
- LET $\langle D_m : m \in \omega \rangle$ COUNT <u>ALL</u> DENSE SETS IN M

WE HAVE $X, Y \in M$ AND $f : X \to Y$ IN N

SO $f \subseteq X \times Y$

WE SHALL PROVE (NEXT TIME) THAT THERE IS A RELATION $R \subseteq \mathbb{P} \times (X \times Y)$

SUCH THAT
- $R \in M$
- $f = R[G] = \{\langle x, y\rangle : (\exists p \in G)(\langle p, \langle x, y\rangle\rangle \in R)\}$

AND EVEN
- FOR EVERY FILTER H THAT INTERSECTS EVERY DENSE SET $R[H]$ IS A MAP FROM X TO Y.

[EVERY FILTER THAT INTERSECTS EVERY DENSE SET THAT IS IN M IS CALLED M-GENERIC]

! ALSO EVERY $p \in \mathbb{P}$ CAN BE MEMBER OF AN M-GENERIC FILTER

MAIN CLAIM ABOUT R.

LET $x \in X$; LET $y_1 \neq y_2$ IN Y AND ASSUME THERE ARE $p_1, p_2 \in \mathbb{P}$

SUCH THAT $\langle p_1, \langle x, y_1\rangle\rangle$ AND $\langle p_2, \langle x, y_2\rangle\rangle$ ARE IN R.

THEN $p_1 \cup p_2$ IS NOT A FUNCTION

PROOF: IF IT WERE A FUNCTION THEN IT WOULD BE IN $\mathbb{P}$

HENCE THERE WOULD BE AN M-GENERIC FILTER H SUCH THAT $p_1 \cup p_2 \in H$

BUT THEN $\langle x, y_1\rangle, \langle x, y_2\rangle \in R[H]$

AND SO $R[H]$ WOULD NOT BE A MAP □

COROLLARY

LET $x \in X$ AND $I_x = \{y \in Y : (\exists p \in \mathbb{P})(\langle p, \langle x, y\rangle\rangle \in R)\}$

CHOOSE $p_y \in \mathbb{P}$ FOR $y \in I_x$ WITH $\langle p_y, \langle x, y\rangle\rangle \in R$

THEN $\{p_y : y \in I_x\}$ IS PAIRWISE INCOMPATIBLE.

WE PROVE:

IF $A \subseteq \mathbb{P}$ CONSISTS OF PAIRWISE INCOMPATIBLE ELEMENTS

THEN A IS COUNTABLE

COROLLARY

No map $f : \omega \to \omega_1^M$ and no map $f : \omega_1^M \to \omega_2^M$ that belongs to N is <u>surjective</u>

PROOF

From R <u>and only from R</u> we get the map $\alpha \mapsto I_\alpha$ that is <u>in M</u>

$[$from ω to $[\omega_1]^{\leq \aleph_0}$ or from ω_1 to $[\omega_2]^{\leq \aleph_1}]$

such that

$$(\forall \alpha \in \operatorname{dom} f)\left[(f(\alpha) \in I_\alpha) \text{ and } (I_\alpha \text{ is countable})\right]^M$$

In M we know

$$\bigcup_{\alpha \in \omega} I_\alpha \neq \omega_1 \quad \text{or} \quad \bigcup_{\alpha \in \omega_1} I_\alpha \neq \omega_2$$

And so in either case f is not surjective.

So let $A \subseteq \mathbb{P}$ be uncountable

we prove there are $p, q \in A$ such that $p \cup q$ is a function.

Consider $\{\operatorname{dom} p : p \in A\}$

This is a family of finite sets

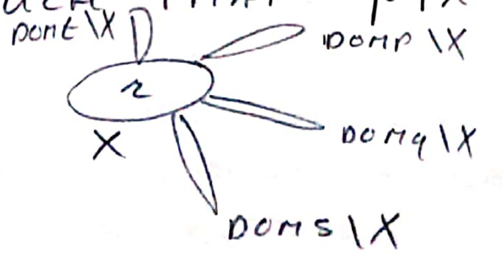
By the Δ-system Lemma

there are a finite set X and an uncountable subfamily B of A

such that $\operatorname{dom} p \cap \operatorname{dom} q = X$ if $p \neq q$ in B

Then there is one function $r : X \to \{0,1\}$ and an uncountable $C \subseteq B$

such that $p \restriction X = r$ for all $p \in C$



Then $p \cup q$ is a function whenever $p, q \in C$

So, provisionally, we have a countable transitive N that satisfies the/many axioms of ZFC <u>plus</u> an injective map from ω_2^N to $\mathcal{P}^N(\omega)$

We conclude that ZFC $+ \neg$CH is consistent.

Our To-Do-List

- Actually show how $M[G]$ is constructed from $M \cup \{G\}$
- Prove the existence of those relations R
- Prove that $M[G]$ can be made to satisfy as many axioms of ZFC as we may need. (As we may want).

Things to think about

1. What is the cardinality of $\mathcal{P}^N(\omega)$ in N?
2. Can we control the cardinality of $\mathcal{P}^N(\omega_1^M)$ in N?
3. How would you construct an N with a bijection from ω_1^N to $\mathcal{P}^N(\omega)$?

<u>Note</u> We used combinatorics <u>in M</u> to prove something combinatorial <u>in N</u>

Look at 1 what can we say?

Work with characteristic functions

As above char. function subset of ω in N is of the form $R[G]$ for some

$$R \subseteq \mathbb{P} \times (\omega \times 2)$$

Make R nicer:

$$R^+ = \{\langle p, \langle n, i\rangle\rangle : (\exists q \in \mathbb{P})(p \leq q \wedge \langle q, \langle n, i\rangle\rangle \in R)\}$$

Note $\langle n, i\rangle \in R[G] \Leftrightarrow \langle n, i\rangle \in R^+[G]$

$\Rightarrow$ clear $R \subseteq R^+$

$\Leftarrow$ if $\langle p, \langle n, i\rangle\rangle \in R^+$ and $p \in G$ then there is $q \geq p$ with $\langle q, \langle n, i\rangle\rangle \in R$ but then $q \in G$ so ---

Again we can ensure $R[H]$ is a function, for every generic H.

CLAIM IF $n \in \omega$

THEN $C_n = \{q : (\exists i \in 2)(\langle q, \langle n, i \rangle\rangle \in R^+)\}$

IS DENSE

PROOF LET $p \in \mathbb{P}$ AND LET G BE GENERIC WITH $p \in G$.

THEN $R[G]$ IS A FUNCTION SO

THERE IS $q \in G$ AND $i \in 2$

WITH $\langle q, \langle n, i \rangle\rangle \in R^+$

BUT THEN $p \cup q \leq p$ AND ALSO

$\langle p \cup q, \langle n, i \rangle\rangle \in R^+$.

INSIDE C_n TAKE A MAXIMAL SUBSET A_n THAT IS PAIRWISE INCOMPATIBLE [ZORN]

LET $S = \bigcup_{n \in \omega} \{\langle q, \langle n, i \rangle\rangle \in R^+ : q \in A_n\}$

CLAIM

$S[H] = R^+[H]$ FOR ALL GENERIC H.

- $S \subseteq R^+$ SO $S[H] \subseteq R^+[H]$
- ASSUME $\langle n, i \rangle \in R^+[H]$

  LET $q \in H$ BE SUCH THAT $\langle q, \langle n, i \rangle\rangle \in R^+$

  SO $q \in C_n$

  ALSO $\{p : (\exists a \in A_n)(p \leq a)\}$ IS DENSE:

  IF $r \in C_n$ THERE IS $a \in A_n$ SUCH THAT $r \cup a$ IS A FUNCTION

  BY MAXIMALITY; $r \cup a \leq a$.

  SO TAKE $r \in H$ AND $a \in A_n$ WITH $r \leq a$

  THEN $\langle q \cup r, \langle n, i \rangle\rangle \in R^+$

  AND $\langle a, \langle n, i \rangle\rangle \in S$:

SO NICE RELATION: $\langle A_n : n \in \omega \rangle$ ANTICHAINS

PARTITION $A_n = A_n^0 \cup A_n^1$

$S = \{\langle q, \langle n, i \rangle\rangle : q \in A_n^i ; n \in \omega; i \in 2\}$

- A_n's COUNTABLE ELEMENT OF $[\mathbb{P}]^{\leq \aleph_0}$
- SO $\aleph_2^{\aleph_0}$ ANTICHAINS

  $\aleph_2^{\aleph_0} \cdot 2^{\aleph_0}$ PARTITIONED ANTICHAINS

  $(\aleph_2^{\aleph_0} \cdot 2^{\aleph_0})^{\aleph_0}$ SEQUENCES

$\therefore$ AT MOST $\aleph_2^{\aleph_0}$ NEW SUBSETS OF ω.