SET THEORY 2021-12-06

TODAY: QUITE TECHNICAL.

- CONSTRUCT $M[G]$ FROM M AND G.
- $g : \omega \to \omega_1^M$ $\quad h : \omega_1^M \to \omega_2^M$
  MAKE RELATIONS $\boxed{\text{IN } M}$
  TO HELP TO SHOW g, h NOT SURJ.
- $M[G]$ IS A MODEL OF (ENOUGH OF) ZFC

$\mathbb{P}$ A PARTIAL ORDER IN M

$Fn(\omega_2^M \times \omega, 2)$

$\leq$ REFLEXIVE, ANTISYMM, TRANS.

FOR OUR COMFORT:

$\mathbb{P}$ HAS A MAXIMUM $\mathbb{1}$

$Fn(\omega_2^M \times \omega, 2)$: $\mathbb{1} = \emptyset$

($p \leq q$ MEANS $p \supseteq q$)

p AND q ARE COMPATIBLE IF THERE IS AN r WITH $r \leq p, q$

$Fn(\omega_2^M \times \omega, 2)$ $\quad p \cup q$ IS A FUNCTION

$(x \in \operatorname{dom} p \wedge \operatorname{dom} q \to p(x) = q(x))$

INCOMPATIBLE IS NOT COMPATIBLE

NOTATION $\quad p \perp q$: $\exists x \in \operatorname{dom} p \wedge \operatorname{dom} q$
$p(x) \neq q(x)$.

FILTER ON $\mathbb{P}$: SUBSET $G \neq \emptyset$

SUCH THAT
- $p, q \in G \to \exists r \in G \ r \leq p, q$
- $p \in G,\ q \geq p \to q \in G$

($\text{SO } \mathbb{1} \in G$)

$D \subseteq \mathbb{P}$ IS DENSE IF

$\forall p \in \mathbb{P} \ \exists q \in D \ \ q \leq p$

A FILTER
G IS M-GENERIC ON $\mathbb{P}$
IF $G \cap D \neq \emptyset$ FOR EVERY DENSE D
THAT IS IN M.

M COUNTABLE IF $p \in \mathbb{P}$ THEN
THERE IS AN M-GENERIC G
SUCH THAT $p \in G$.

$\mathbb{P}$-NAME: A SET τ THAT IS
A RELATION AND SATISFIES:
IF $\langle \pi, p \rangle \in \tau$ THEN
π IS A $\mathbb{P}$-NAME AND $p \in \mathbb{P}$.
THIS USES RECURSION:
$x E y$ MEANS $x \in$ TRCL y
($\text{RANK } x < \text{RANK } y$)

$H(\tau) = 1$ IFF $\forall \langle \pi, p \rangle \in \tau$
$H(\pi) = 1 \wedge p \in \mathbb{P}$
0 IFF NOT

BEING A NAME IS ABSOLUTE
$\tau \in M$ $\mathbb{P} \in M$
τ IS A $\mathbb{P}$-NAME IFF $(\tau \text{ IS A } \mathbb{P}\text{-NAME})^M$

$\text{VAL}(\tau, G) = \{ \text{VAL}(\pi, G) : (\exists p \in G)(\langle \pi, p \rangle \in \tau) \}$
(OR τ_G) $\emptyset$

$M[G] = \{ \text{VAL}(\tau, G) : \tau \text{ IS A } \mathbb{P}\text{-NAME IN } M \}$

? $M \in M[G]$?
? $G \in M[G]$? $M^{\mathbb{P}}$ IS THE CLASS
OF $\mathbb{P}$-NAMES IN M

FOR $x \in M$
$\check{x} = \{ \langle \check{y}, \mathbb{1} \rangle : y \in x \}$ RECURSIVE
$\check{\emptyset} = \emptyset$ $\check{1} = \{ \langle \check{0}, \mathbb{1} \rangle \}$ $\check{\omega} = \{ \langle \check{n}, \mathbb{1} \rangle : n \in \omega \}$

$\mathrm{VAL}(\check{x}, G) = \{ \mathrm{VAL}(\check{y}, G) : y \in x \}$

$= \{ y : y \in x \}$

$= x$

SO $M \subseteq M[G]$!

$\Gamma = \{ \langle \check{p}, p \rangle : p \in \mathbb{P} \}$

$\mathrm{VAL}(\Gamma, G) = \{ \mathrm{VAL}(\check{p}, G) : p \in G \}$

$= \{ p : p \in G \}$

$= G$

$G \in M[G]$!

- $M[G]$ IS TRANSITIVE
- $\mathrm{RANK}(\tau_G) \leq \mathrm{RANK}(\tau)$ BY INDUCTION
  SO $\mathrm{On} \cap M[G] = \mathrm{On} \cap M$
- σ, τ NAMES : $\{ \langle \sigma, 1\!\!1 \rangle, \langle \tau, 1\!\!1 \rangle \} = \mathrm{up}(\sigma, \tau)$
  IS A NAME.
  $\mathrm{VAL}(\mathrm{up}(\sigma,\tau), G) = \{ \sigma_G, \tau_G \}$
  $M[G]$ SATISFIES PAIRING.
  $\mathrm{op}(\sigma,\tau) = \mathrm{up}(\mathrm{up}(\sigma,\sigma), \mathrm{up}(\sigma,\tau))$
  $\rightarrow \langle \sigma_G, \tau_G \rangle$

TRANSITIVITY : EXTENSIONALITY
REGULARITY

UNION; GIVEN τ LET

$\sigma = \bigcup \mathrm{dom}\, \tau$

$= \bigcup \{ \pi : (\exists p)(\langle \pi, p \rangle \in \tau) \}$

IN $M[G]$:

$\bigcup \tau_G \subseteq \sigma_G$ CHECK THIS !

GI : BUILD A σ SUCH THAT
EVEN $\bigcup \tau_G = \sigma_G$.

WE HAVE INFINITY TOO!

$\omega = \mathrm{VAL}(\check{\omega}, G) \in M[G]$.

$\mathrm{RANK}\, x = \sup \{ \mathrm{RANK}\, y + 1 : y \in x \}$

$\alpha \in \mathrm{On} \cap M \rightarrow V_\alpha^M = V_\alpha \cap M$

HENRI CARTAN 1937

1. Soit $\mathscr{E}$ un ensemble donné une fois pour toutes. Une famille **F** de sous-ensembles de $\mathscr{E}$ prend le nom de *filtre* (construit sur $\mathscr{E}$) si elle remplit les trois conditions suivantes :

F-I : la famille **F** n'est pas vide et ne contient pas le sous-ensemble vide ;

F-II : l'intersection de deux ensembles de **F** appartient à **F** ;

F-III : tout ensemble qui contient un ensemble de **F** appartient à **F**.

FORCING !

WORK WITH $Fn(\omega_2 \times \omega, 2)$

IS UG IN $M[G]$??

$$\Psi = \{ \langle \underbrace{\langle \langle \alpha, n \rangle, i \rangle^{\vee}}, p \rangle : \langle \alpha, n \rangle \in \mathrm{dom}\, p,\ p(\alpha, n) = i \}$$

THE 'CHECK' OF THAT ORDERED PAIR

$$\Psi_G = UG$$

MAKE A NAME FOR Φ

$$f : \omega_2 \longrightarrow \mathcal{P}(\omega)$$
$$\alpha \longmapsto \{ n : UG(\alpha, n) = 1 \}$$

MEMBERS OF $\mathbb{P}$: CONDITIONS

$$q = \{ \langle \langle \alpha, 10 \rangle, 1 \rangle, \langle \langle \alpha+1, 10 \rangle, 0 \rangle \}$$

IF $q \in G$ THEN $10 \in f(\alpha) \setminus f(\alpha+1)$ IN $M[G]$

$f = \Phi_G$

$$q \Vdash \check{10} \in \Phi(\check{\alpha}) \setminus \Phi(\check{\alpha+1})$$

DEFINITION OF $\Vdash$

IF φ IS A FORMULA AND $\tau_1, \ldots, \tau_k$ ARE NAMES

THEN $p \Vdash \varphi(\tau_1, \ldots, \tau_k)$ "p FORCES ..."

MEANS FOR ALL M-GENERIC G ON $\mathbb{P}$ WITH $p \in G$ WE HAVE

$$\varphi(\tau_{1,G}, \ldots, \tau_{k,G})^{M[G]}$$

$$M[G] \models \varphi(\tau_{1,G}, \ldots, \tau_{k,G})$$

REMEMBER $g : \omega \to \omega_1^M$

IT HAS A NAME γ

$g = \gamma_G$

$M \ni R_g = \{\langle p, \langle n, \alpha\rangle\rangle : n \in \omega, \alpha \in \omega_1, p \in \mathbb{P}, \underline{p \Vdash \gamma(\check{n}) = \check{\alpha}}\}$

① IS R_g IN M?

② DOES IT WORK? $R_g[G]$ A FUNCTION?

① THAT IS OUR NEXT JOB

② YES — ALMOST

WE DEFINE $\Vdash^*$

WITHOUT MENTIONING G'S

PROVE $p \Vdash \varphi$ IFF $(p \Vdash^* \varphi)^M$

IF THIS WORKS THEN

$R_g = \{\langle p, \langle n, \alpha\rangle\rangle : n \in \omega, \alpha \in \omega_1, p \in \mathbb{P}, p \Vdash^* \gamma(\check{n}) = \alpha\}$

IS DEFINED IN M

$p \Vdash^* \tau_1 = \tau_2$

(α) FOR ALL $\langle \pi_1, s_1\rangle \in \tau_1$

$$\{q \le p : q \le s_1 \longrightarrow \underbrace{(\exists \langle \pi_2, s_2\rangle \in \tau_2)}_{\text{ABSOLUTE}}(q \le s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$$

IS DENSE BELOW p.

$[\ \forall r \le p\ \exists q \le r\ \ q \in \ \ ]$

(β) INTERCHANGE 1 AND 2

RECURSION ON $\langle \pi_1, \pi_2\rangle\ E\ \langle \tau_1, \tau_2\rangle$

$: \pi_1 \in \mathrm{DOM}\,\tau_1, \pi_2 \in \mathrm{DOM}\,\tau_2$

$p \Vdash^* \tau_1 \in \tau_2$

MEANS $\{ q \leq p : (\exists \langle \pi, s \rangle \in \tau_2)(q \leq s \wedge q \Vdash^* \pi = \tau_1) \}$

IS DENSE BELOW p

CONJUNCTION:

$p \Vdash^* \varphi \wedge \psi$ IFF $p \Vdash^* \varphi$ AND $p \Vdash^* \psi$

NEGATION:

$p \Vdash^* \neg\varphi$ $\{ q \leq p : q \Vdash^* \varphi \} = \emptyset$

QUANTIFICATION

$p \Vdash^* (\exists x)\, \varphi(x, \tau_1, \ldots, \tau_n)$

IFF

$\{ q \leq p : (\exists \sigma)(q \Vdash^* \varphi(\sigma, \tau_1, \ldots, \tau_n)) \}$

IS DENSE BELOW p.

[WE WANT

$\varphi(\tau_{1,G}, \ldots, \tau_{n,G})^{M[G]}$ IFF

$(\exists p \in G)(p \Vdash^* \varphi(\tau_1, \ldots, \tau_n))$]

BIG THEOREM

M TRANSITIVE MODEL OF ZFC

$\mathbb{P}$ A PARTIAL ORDER IN M

$\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$

G AN M-GENERIC FILTER ON $\mathbb{P}$

$\varphi(x_1, \ldots, x_n)$ FORMULA

1 IF $p \in G$ AND $(p \Vdash^* \varphi(\tau_1, \ldots, \tau_n))^M$

THEN $\varphi(\tau_{1G}, \ldots, \tau_{nG})^{M[G]}$

2 IF $\varphi(\tau_{1G}, \ldots, \tau_{nG})^{M[G]}$

THEN THERE IS A $p \in G$ SUCH THAT

$p \Vdash^* \varphi(\tau_1, \ldots, \tau_n)$

FULL PROOF IN KUNEN'S BOOK 1980

JUST FOR $\tau_1 = \tau_2$

1. $p \Vdash^* \tau_1 = \tau_2$ AND $p \in G$

TO SHOW $\tau_{1G} \subseteq \tau_{2G}$ AND $\tau_{2G} \subseteq \tau_{1G}$.

TAKE $\langle \pi_1, s_1 \rangle \in \tau_1$ WITH $s_1 \in G$

NEED TO SHOW $\pi_{1G} \in \tau_{2G}$

TAKE $r \in G$ WITH $r \leq p, s_1$ $\quad r \Vdash^* \tau_1 = \tau_2$

$D = \{ q : q \perp r$ OR $q \leq s_1 \to \exists \langle \pi_2, s_2 \rangle \in \tau_2$
$q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$

TAKE $q \in G \cap D$: WLOG $q \leq s_1$

SO THERE IS $\langle \pi_2, s_2 \rangle \in \tau_2$

WITH $q \leq s_2$ AND $q \Vdash^* \pi_1 = \pi_2$

INDUCTION $\pi_{1G} = \pi_{2G}$

SO $\pi_{1G} \in \tau_{2G}$

2. D IS THE UNION OF

$D_1 = \{ r \in \mathbb{P} : r \Vdash^* \tau_1 = \tau_2 \}$

$D_2 = \{ r \in \mathbb{P} : (\exists \langle \pi_1, s_1 \rangle \in \tau_1)(r \leq s_1 \wedge$
$(\forall \langle \pi_2, s_2 \rangle \in \tau_2)(\forall q \in \mathbb{P})$
$(q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2 \to q \perp r)) \}$

$r \in G$: $\pi_{1G} \in \tau_{1G}$

IF q THINKS $\pi_{2G} \in \tau_{2G}$

q WOULD GIVE ~~$\pi_{2G} = \pi_{1G}$~~

THEN $q \in G$

D_3 LIKE D_2 BUT WITH 1 AND 2 INTERCHANGED

$D_1 \cup D_2 \cup D_3$ IS DENSE

IF $p \in \mathbb{P}$ THEN $p \Vdash^* \tau_1 = \tau_2$; $p \in D_1$

OR NOT : THEN THERE IS A $\langle \pi_1, s_1 \rangle$ WHERE

$\{ q : q \leq s_1 \to \exists \langle \pi_2, s_2 \rangle \in \tau_2 \; q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2 \}$

IS NOT DENSE BELOW p

WE HAVE $r \le p$ WITH NO SUCH q BELOW r

$\forall q \le r \; q \le s_1 \wedge \forall \langle \pi_2, s_2 \rangle \in \tau$
$\underline{(q \not\le s_2 \vee q \not\Vdash^* \pi_1 = \pi_2)} \}$

NOW USE CONTRAPOSITIVE

SO WE FIND $G \cap (D_1 \cup D_2 \cup D_3) \neq \varphi$

IF $\underline{p \in G \cap D_2}$ THEN WE

FIND $\langle \pi_1, s_1 \rangle \in \tau_1$ WITH $p \le s_1$

SO $s_1 \in G$ AND $\pi_{1G} \in \tau_{1G}$

IF $\pi_{1G} = \pi_{2G}$ FOR SOME $\langle \pi_2, s_2 \rangle \in \tau_2$

WITH $s_2 \in G$

THERE IS $q_0 \in G$ $\underbrace{q_0 \Vdash^* \pi_1 = \pi_2}_{\text{INDUCTION}}$

$\underline{q_0} \le \underline{p}, s_1, s_2$ SO $\underline{q_0 \perp p}$ CONTRADICTION

SO $G \cap D_1 \neq \varphi$ AND WE ARE DONE

COROLLARY

$\rightarrow$ $\underline{p \Vdash} \varphi(\tau_1, \dots, \tau_n)$ IFF $(\underline{p \Vdash^*} \varphi(\tau_1, \dots, \tau_n))^M$

$\rightarrow$ FOR ALL G WE GET

$\varphi^{M[G]}$ IFF $(\exists p \in G)(p \Vdash \varphi)$

WE TAKE $g: \omega \longrightarrow \omega_1^M$

WE HAVE $\gamma \in M^P$: $g = \gamma_G$

$(g$ IS A FUNCTION FROM ω TO $\omega_1^M)^{M[G]}$

THERE IS A $p \in G$ SUCH THAT

$(p \Vdash^* \gamma$ IS A FUNCTION FROM $\check{\omega}$ TO $\check{\omega}_1)^M$

R_g TWO PARTS

(1) $\boxed{\{ \langle q, \langle n, \alpha \rangle \rangle : q \le p, \cancel{q \Vdash \gamma(\check{n}) = \check{\alpha}} \}}$

(2) $\{ \langle q, \langle n, 0 \rangle \rangle : q \perp p,$ NEW

IF $p \in H$ THEN γ_H IS A FUNCTION

IF $p \notin H$ THEN γ_H IS THE ZERO FUNCTION.