SET THEORY 2021-12-20

THE POWER SET AXIOM IN $M[G]$.
[MORE DIFFICULT THAN I INDICATED]

NICE EXERCISE [SHOULD HAVE BEEN IN GI]
IF $\tau \in M^{\mathbb{P}}$ AND $\operatorname{dom} \tau \subseteq \{\check{n} : n \in \omega\}$
THEN $\sigma = \{\langle \check{n}, p\rangle : (\forall q \in \mathbb{P})(\langle \check{n}, q\rangle \in \tau \to p \perp q)\}$
IS A NAME AND $\sigma_G = \omega \setminus \tau_G$.

TO DO THIS FOR ARBITRARY τ REQUIRES $\Vdash$
AS LAST WEEK:
MAKE $\tau' = \{\langle \check{n}, p\rangle : p \Vdash \check{n} \in \tau\}$
THEN $\tau_G = \tau'_G$
AND σ AS ABOVE FOR τ' IS A NAME FOR $\omega \setminus \tau_G$.

READ "CHAPTER 4" POSTED IN WEEK 12.
IN PARTICULAR THEOREM 3.23.
PAGES 62-66

LET $M \subseteq N$ BOTH MODELS OF ZFC
ASSUME WE HAVE $a \in M$ AND $\sigma \in N$
WITH $\sigma \subseteq a$.
TAKE ALL POSSIBLE RELATIONS $R \in M$
WITH $\operatorname{DOM} R \subseteq a$
LET $\operatorname{OB}(\sigma, M) = \{R[\sigma] : R \in M$ A RELATION, $\operatorname{DOM} R \subseteq a\}$
IF $\operatorname{OB}(\sigma, M)$ IS CLOSED UNDER DIFFERENCES
THEN σ IS AN M-GENERIC FILTER
WITH RESPECT TO A SUITABLE PARTIAL ORDER ON a.

GENERAL CASE

LET $\sigma \in M^{\mathbb{P}}$ WE MAKE $\varrho \in M^{\mathbb{P}}$ SUCH THAT
$$(\forall x \in M[G])(x \subseteq \sigma_G \to x \in \varrho_G)$$
LET $S = \sigma^M(\operatorname{dom}(\sigma) \times \mathbb{P})$
$= \{\tau \in M^{\mathbb{P}} : \operatorname{dom} \tau \subseteq \operatorname{dom} \sigma\}$
AND $\varrho = S \times \{\mathbb{1}\}$
LET $\mu \in M^{\mathbb{P}}$ WITH $\mu_G \subseteq \sigma_G$
MAKE $\tau = \{\langle \pi, p\rangle : \pi \in \operatorname{dom} \sigma \wedge p \Vdash \pi \in \mu\}$
THEN $\tau \in S$ AND $\tau_G \in \varrho_G$
WE SHOW $\mu_G = \tau_G$.

- $\mathcal{M}_G \subseteq \tau_G$:
  IF $x \in \mathcal{M}_G$ THEN $x = \pi_G$ FOR SOME $\pi \in \operatorname{dom} \tau$
  SO, AS $\pi_G \in \mathcal{M}_G$ TAKE $p \in G$ SUCH THAT $p \Vdash \pi \in \mathcal{M}$
  SO $\langle \pi, p \rangle \in \tau$ AND $\pi_G \in \tau_G$ OR $x \in \tau_G$.
- $\tau_G \subseteq \mathcal{M}_G$;
  IF $x \in \tau_G$ THEN THERE IS $p \in G$ AND $\pi \in \operatorname{dom} \tau$
  SUCH THAT $x = \pi_G$ AND $\langle \pi, p \rangle \in \tau$
  BUT THEN $p \Vdash \pi \in \mathcal{M}$
  AND SO $x = \pi_G \in \mathcal{M}_G$.

ALMOST DISJOINT FAMILIES

SIERPIŃSKI 1928

- THERE IS A FAMILY $\mathcal{A}$ OF SUBSETS OF ω
  SUCH THAT
  - $|\mathcal{A}| = 2^{\aleph_0}$
  - IF $A \in \mathcal{A}$ THEN $|A| = \aleph_0$
  - IF $A, B \in \mathcal{A}$ AND $A \neq B$
    THEN $|A \cap B| < \aleph_0$

PROOF CHOOSE FOR EVERY IRRATIONAL x
A SEQUENCE S_x IN $\mathbb{Q}$ THAT
CONVERGES TO x
- $|S_x| = \aleph_0$ FOR ALL x (RATHER $|\operatorname{ran} S_x| = \aleph_0$)
- $S_x \cap S_y$ IS FINITE IF $x \neq y$
- $|\mathbb{R} \setminus \mathbb{Q}| = 2^{\aleph_0}$

- [ALSO TARSKI]
  IF CH THEN THERE IS A FAMILY $\mathcal{A}$
  OF SUBSETS OF ω_1 SUCH THAT
  - IF $A \in \mathcal{A}$ THEN $|A| = \aleph_1$
  - $|\mathcal{A}| = 2^{\aleph_1}$
  - IF $A, B \in \mathcal{A}$ AND $A \neq B$ THEN $|A \cap B| < \aleph_1$

PROOF. $T = \bigcup_{\alpha < \omega_1} 2^{\alpha}$ HAS CARDINALITY $2^{\aleph_0}$
- FOR $x \in 2^{\omega_1}$ THE SET
  $B_x = \{ x \restriction \alpha : \alpha \in \omega_1 \}$
  HAS CARDINALITY $\aleph_1$
- IF $x \neq y$ THEN $B_x \cap B_y$ IS COUNTABLE.

BAUMGARTNER 1979

WE CANNOT PROVE THE RESULT FOR ω_1 IN ZFC. ASSUME M SATISFIES GCH

LET $\mathbb{P} = Fn(\omega_3 \times \omega_1, 2)$; IN $M[G]$: $2^{\aleph_0} = 2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$

ASSUME THAT IN $M[G]$ THERE IS A SEQUENCE $\langle A_\alpha : \alpha \in \omega_3 \rangle$ OF SUBSETS OF ω_1 SUCH THAT
- $|A_\alpha| = \aleph_1$
- $\alpha \neq \beta \rightarrow |A_\alpha \cap A_\beta| \leq \aleph_0$.

TAKE $p \in G$ AND $\varphi \in M^{\mathbb{P}}$ SUCH THAT

$\langle A_\alpha : \alpha \in \omega_3 \rangle = \varphi_G$ AND

$p \Vdash$ "$\varphi : \check{\omega}_3 \rightarrow \mathcal{P}(\check{\omega}_1)$; $|\varphi(\alpha)| = \aleph_1$; $\alpha \neq \beta \rightarrow |\varphi(\alpha) \cap \varphi(\beta)| \leq \aleph_0$ "

CLAIM: IF $\alpha < \beta$ THEN THERE IS $\gamma(\alpha, \beta) \in \omega_1$ SUCH THAT $p \Vdash$ "$\varphi(\check{\alpha}) \cap \varphi(\check{\beta}) \subseteq \gamma(\check{\alpha}, \check{\beta})$"

NOTE $p \Vdash$ "$\sup(\varphi(\check{\alpha}) \cap \varphi(\check{\beta})) \in \omega_1$"

LET $F_{\alpha,\beta} = \{ \gamma : (\exists q \leq p)(q \Vdash \sup(\varphi(\check{\alpha}) \cap \varphi(\check{\beta})) = \gamma) \}$.

THEN $F_{\alpha,\beta}$ IS COUNTABLE

IF $\gamma \neq \delta$ IN F WITH q_γ AND q_δ BELOW p THEN $q_\gamma \perp q_\delta$

NOW USE THAT ANTICHAINS ARE COUNTABLE

WE DEFINE $\underline{\gamma(\alpha, \beta)} = \sup F_{\alpha,\beta} + 1$

WE HAVE $\gamma : [\omega_3]^2 \longrightarrow \omega_1$ IN M

NOW: $\aleph_3 = (2^{\aleph_1})^+$ SO BY THE ERDÖS–RADO THEOREM $\aleph_3 \longrightarrow (\aleph_2)^2_{\aleph_1}$

WE GET $H \subseteq \omega_3$ AND $\gamma \in \omega_1$ SUCH THAT $|H| = \aleph_2$ AND IF $\alpha < \beta$ IN H THEN $p \Vdash$ "$\varphi(\check{\alpha}) \cap \varphi(\check{\beta}) \subseteq \gamma$"

OR: $p \Vdash \{ \varphi(\alpha) \setminus \gamma : \alpha \in \check{H} \}$ IS PAIRWISE DISJOINT

CONTRADICTION

$\aleph_2$ PAIRWISE DISJOINT SUBSETS OF ω_1 IS IMPOSSIBLE.

NOTE THIS ONLY USED THE CCC OF $\mathbb{P}$ TO KEEP ALMOST DISJOINT FAMILIES SMALL

RECTANGLES

X A SET

$\mathcal{R} = \{A \times B : A, B \subseteq X\}$

THE RECTANGLES IN $X \times X$.

$\mathcal{S}$ THE σ-ALGEBRA GENERATED BY $\mathcal{R}$

σ-ALGEBRA:
- $X \times X \in \mathcal{S}$
- $S \in \mathcal{S} \to (X \times X) \setminus S \in \mathcal{S}$
- $\{S_i : i \in \omega\} \subseteq \mathcal{S} \to \bigcup_{i \in \omega} S_i \in \mathcal{S}$

"GENERATED BY": THE SMALLEST ONE THAT CONTAINS $\mathcal{R}$.

- $X = \omega$: $\mathcal{S} = \mathcal{P}(\omega \times \omega)$

  EVERY POINT $\{\langle m, n \rangle\}$ IS A RECTANGLE

- $X = \omega_1$: $\mathcal{S} = \mathcal{P}(\omega_1 \times \omega_1)$

  LEMMA: IF $S \subseteq \omega_1 \times \omega_1$ THEN THERE ARE $\{x_\alpha : \alpha \in \omega_1\}$ AND $\{y_\alpha : \alpha \in \omega_1\}$ IN $\mathcal{P}(\omega)$ SUCH THAT $\langle \alpha, \beta \rangle \in S$ IFF $x_\alpha \cap y_\beta$ IS INFINITE

  $A_n = \{\alpha : n \in x_\alpha\}$ AND $B_m = \{\beta : m \in y_\beta\}$

  THEN
  $$S = \bigcap_{m \in \omega} \bigcup_{n \geqslant m} (A_n \times B_n)$$

- $X = \mathbb{R}$: $\mathcal{S} = \mathcal{P}(\mathbb{R} \times \mathbb{R})$ IF CH HOLDS

- $X = \omega_2$: KUNEN: NOT NECESSARILY

  ASSUME M SATISFIES CH

  $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$ $\quad \kappa \geqslant \aleph_2$

  IF G IS M-GENERIC ON $\mathbb{P}$ THEN IN $M[G]$ THE SET
  $$W = \{\langle \alpha, \beta \rangle : \alpha \in \beta \in \omega_2\}$$
  IS NOT IN $\mathcal{S}$.

  - IF $W \in \mathcal{S}$ THEN THERE IS A COUNTABLE FAMILY $\{A_n \times B_n : n \in \omega\}$ OF RECTANGLES SUCH THAT W IS IN THE σ-ALGEBRA GENERATED BY $\{A_n \times B_n : n \in \omega\}$.

  [GENERAL RESULT:
  $$\sigma(\mathcal{R}) = \bigcup \{\sigma(\mathcal{A}) : \mathcal{A} \in [\mathcal{R}]^{\leq \aleph_0}\}$$
  ]

- $\sigma(\{A_n \times B_n : n \in \omega\setminus\{0\}\})$ IS BUILT UP IN ω_1 STAGES

0 $\mathcal{S}_0 \{A_n \times B_n : n \in \omega\setminus\{0\}\}$

1 $\mathcal{S}_1 = \{\bigcup_{n \in x} A_n \times B_n : x \subseteq \omega\setminus\{0\}\} \cup \mathcal{S}_0$

2 $\mathcal{S}_2 = \mathcal{S}_1 \cup \{(X \times X)\setminus S : S \in \mathcal{S}_1\}$

α LIMIT $\quad \mathcal{S}_\alpha = \bigcup_{\beta \in \alpha} \mathcal{S}_\beta$

α EVEN $\quad \mathcal{S}_{\alpha+1} = \mathcal{S}_\alpha \cup \{\bigcup \mathcal{S}' : \mathcal{S}' \subseteq \mathcal{S} \text{ COUNTABLE}\}$

α ODD $\quad \mathcal{S}_{\alpha+1} = \mathcal{S}_\alpha \cup \{(X \times X)\setminus S : S \in \mathcal{S}_\alpha\}$.

$\sigma(\{A_n \times B_n : n \in \omega\}) = \bigcup_{\alpha \in \omega_1} \mathcal{S}_\alpha$

- EVERY MEMBER IS CODED BY A FUNCTION $r : \omega \to \omega$

0: $r(0) = \underline{0} \quad r(i) = 1 \Leftrightarrow i = n$

$S_r = A_n \times B_n$

1 $\quad r(0) = \underline{1} \quad r(i) \in \{0,1\}$

GIVES $S_r = \bigcup\{A_n \times B_n : r(n) = 1\}$

2 $\quad$ GIVEN $S_r \in \mathcal{S}_1$

PUT $t(0) = 2$; $t(i) = r(i-1) \quad i \geq 1$

$S_t = (X \times X)\setminus S_r$

DO THIS ALWAYS IF α IS ODD

$S_r \in \mathcal{S}_\alpha \rightsquigarrow t = \langle 2, r(0), r(1), \ldots \rangle$

$S_t = (X \times X)\setminus S_r$

3 $\quad$ GIVEN $\langle S_{r_i} : i \in \omega \rangle$

FIXED FOR ALWAYS $\longrightarrow$ TAKE BIJECTION $b : \omega\setminus\{0\} \leftrightarrow \omega \times \omega\setminus\{0\}$

$r(0) = 3$

$r(i) = r_n(m) \quad$ IF $b(i) = \langle n, m \rangle$

ALWAYS IF α IS EVEN

$\langle r_i : i \in \omega \rangle \rightsquigarrow r = \langle 3, r_{b(1)_1}(b(1)_2), \ldots \rangle$

SO IF $W \in S$ THEN WE HAVE (SIMPLIFY A BIT)

$r : \omega \to \omega$ AND $\langle A_n \times B_n : n \in \omega \rangle$

SUCH THAT

$W = S_r(A_n \times B_n : n \in \omega)$

WE GET NAMES ς, σ AND τ, AND $p \in G$

SUCH THAT $\quad p \Vdash \check{W} = S_\varsigma(\sigma(n) \times \tau(n) : n \in \omega)$

SO $\quad p \Vdash \langle \check{\alpha}, \check{\beta} \rangle \in S_\varsigma(\sigma(n) \times \tau(n) : n \in \omega)$

FOR ALL α, β WITH $\alpha \leq \beta$.

WE MAKE THE NAMES NICER VIA FORCING

SO

- $g = \bigcup\{\{\langle \check m,\check n\rangle\}\times X_{m,n} : \langle m,n\rangle\in\omega^2\}$

  $X_{m,n}$ A MAXIMAL ANTICHAIN IN $\{q\le p : q\Vdash g(\check m)=\check n\}$

- $\sigma_m = \bigcup\{\{\check\alpha\}\times Y_{m,\alpha} : \alpha\in\omega_2\}$
- $\tau_m = \bigcup\{\{\check\alpha\}\times Z_{m,\alpha} : \alpha\in\omega_2\}$

  $Y_{m,\alpha}$ MAXIMAL ANTICHAIN IN $\{q\le p : q\Vdash \check\alpha\in\sigma(\check m)\}$

  $Z_{m,\alpha}$ MAXIMAL ANTICHAIN IN $\{q\le p : q\Vdash \check\alpha\in\tau(\check m)\}$

So $p\Vdash \check W = S_g(\sigma_m\times\tau_m : m\in\omega)$

FOR $\alpha\in\omega_2$ LET

$$C_\alpha = \bigcup\{\mathrm{dom}\, q : q\in \textstyle\bigcup_{\langle m,n\rangle} X_{m,n}\}\cup \bigcup\{\mathrm{dom}\,q : q\in\bigcup_m Y_{m,\alpha}\}\cup\bigcup\{\mathrm{dom}\,q : q\in\bigcup_m Z_{m,\alpha}\}.$$

SINCE ANTICHAINS ARE COUNTABLE EACH SET C_α IS COUNTABLE

APPLY THE Δ-SYSTEM LEMMA FOR COUNTABLE SETS TO FIND $H\subseteq\omega_2$ AND $R\subseteq\kappa\times\omega$ SUCH THAT $|H|=\aleph_2$ AND IF $\alpha<\beta$ IN H THEN $C_\alpha\cap C_\beta\subseteq R$.

NOTE $\bigcup\{\mathrm{dom}\,q : q\in\bigcup_{\langle m,n\rangle}X_{m,n}\}\subseteq R$

FOR EACH $\alpha\in H$ LET $\langle h(\alpha,n) : n\in\omega\rangle$ BE A BIJECTION BETWEEN ω AND C_α

COPY THE STRUCTURE R, $\langle Y_{m,\alpha} : m\in\omega\rangle$, $\langle Z_{m,\alpha} : m\in\omega\rangle$, $\langle X_{m,n} : \langle m,n\rangle\in\omega\rangle$ TO ω VIA THIS BIJECTION

$$\mathcal{R}_\alpha = \{n : h(\alpha,n)\in R\}$$
$$\mathcal{X}_{\alpha,m,n} = \{q\in Fn(\omega,2) : \{\langle h(\alpha,n),q(n)\rangle : n\in\mathrm{dom}\,q\}\in X_{m,n}\}$$
$$\mathcal{Y}_{\alpha,m} = \{q\in Fn(\omega,2) : \{\langle h(\alpha,n),q(n)\rangle : n\in\mathrm{dom}\,q\}\in Y_{\alpha,m}\}$$
$$\mathcal{Z}_{\alpha,m} = \{q\in Fn(\omega,2) : \underbrace{\{\langle h(\alpha,n),q(n)\rangle : n\in\mathrm{dom}\,q\}}_{\text{WE COULD CALL THIS } h_\alpha(q)}\in Z_{\alpha,m}\}$$

THERE ARE AT MOST $2^{\aleph_0}$ SUCH STRUCTURES

$$\langle\mathcal{R},\mathcal{X},\mathcal{Y},\mathcal{Z}\rangle$$

SO TAKE $\alpha<\beta$ IN H SUCH THAT

$$\mathcal{R}_\alpha=\mathcal{R}_\beta$$
$$\mathcal{X}_{\alpha,m,n}=\mathcal{X}_{\beta,m,n}$$
$$\mathcal{Y}_{\alpha,m}=\mathcal{Y}_{\beta,m}$$
$$\mathcal{Z}_{\alpha,m}=\mathcal{Z}_{\beta,m}$$

AND $h(\alpha,n)=h(\beta,n)$ IF $n\in\mathcal{R}_\alpha=\mathcal{R}_\beta$

DEFINE $\pi : X \times \omega \to X \times \omega$ BY

$$\pi(h(\alpha), n) = h(\beta, n)$$
$$\pi(h(\beta, n)) = h(\alpha, n)$$
$$\pi(x) = x \quad \text{OTHERWISE}$$

EXTEND π TO $Fn(X \times \omega, 2)$

$$\pi(p)(x) =, p(\pi(x))$$

NOTE $\pi^{-1} = \pi$ ON $X \times \omega$ SO $\pi^{-1} = \pi$ ON $Fn(X \times \omega, 2)$

FOR ALL n π INTERCHANGES $Y_{\alpha, m}$ AND $Y_{\beta, m}$)
AND $Z_{\alpha, n}$ AND $Z_{\beta, m}$

FOR $r \le p$ THIS MEANS

$$r \Vdash \check{\alpha} \in \sigma_m \text{ IFF } \pi(r) \Vdash \check{\beta} \in \sigma_m$$

AND $r \Vdash \check{\beta} \in \sigma_n$ IFF $\pi(r) \Vdash \check{\alpha} \in \sigma_m$

BUT π ACTS ON NAMES TOO

$$\pi^*(\tau) = \{ \langle \pi^*(\sigma), \pi(p) \rangle : \langle \sigma, p \rangle \in \tau \}$$

NOTE THAT $\pi^*(\check{x}) = \check{x}$ FOR ALL x

IN OUR CASE

σ_m HAS $\{\check{\alpha}\} \times Y_{m,\alpha}$
$\{\check{\beta}\} \times Y_{m,\beta}$

$\pi^*(\sigma_m)$ HAS $\{\check{\alpha}\} \times Y_{m,\beta}$
$\{\check{\beta}\} \times Y_{m,\alpha}$

τ_m HAS $\{\check{\alpha}\} \times Z_{m,\alpha}$
$\{\check{\beta}\} \times Z_{m,\beta}$

$\pi^*(\sigma_m)$ HAS $\{\check{\alpha}\} \times Z_{m,\beta}$
$\{\check{\beta}\} \times Z_{m,\alpha}$

SO
$$\pi(r) \Vdash \check{\beta} \in \sigma_m \text{ IFF } r \Vdash \check{\beta} \in \pi^*(\sigma_m)$$
$$\pi(r) \Vdash \check{\alpha} \in \sigma_m \text{ IFF } r \Vdash \check{\alpha} \in \pi^*(\sigma_m)$$
$$\pi(r) \Vdash \check{\beta} \in \tau_m \text{ IFF } r \Vdash \check{\beta} \in \pi^*(\tau_m)$$
$$\pi(r) \Vdash \check{\alpha} \in \tau_m \text{ IFF } r \Vdash \check{\alpha} \in \pi^*(\tau_m)$$

CONCLUSION $r \Vdash \check{\alpha} \in \sigma_m$ IFF $r \Vdash \check{\beta} \in \pi^*(\sigma_m)$

ETC

IT FOLLOWS THAT

$$p \Vdash \langle \alpha, \beta \rangle^{\vee} \in S_g(\sigma_m \times \tau_n : n \in \check{\omega})$$
$$\text{IFF } \pi(p) \Vdash \langle \alpha, \beta \rangle^{\vee} \in S_{\pi(g)}(\pi^*(\sigma_m) \times \pi^*(\tau_m) : n \in \check{\omega})$$

$\pi(p) = p$
$\pi^*(g) = g$
$\rightarrow$ IFF $p \Vdash \langle \alpha, \beta \rangle^{\vee} \in S_r(\pi^*(\sigma_m) \times \pi^*(\tau_m) : n \in \check{\omega})$

IFF $p \Vdash \langle \beta, \alpha \rangle^{\vee} \in S_r(\sigma_m \times \tau_m : n \in \check{\omega})$

CONTRADICTION

Souslin's Problem

A linear order without jumps or gaps in which every pairwise disjoint family of intervals is at most denumerable, must it be a normal linear continuum?

"Normal linear continuum": interval in $\mathbb{R}$.

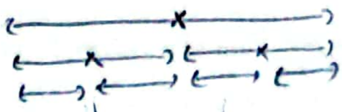
A counterexample is called a Souslin line

A dense linear order in which every pairwise disjoint family of intervals is countable but with no countable dense set

Answer: undecidable:
There are models with and models without Souslin lines

Jech: Chapter 9.
There is a Souslin line iff there is a Souslin tree

Souslin tree:
A partially ordered set $(T, \leq)$
such that:
- $|T| = \aleph_1$
- If $x \in T$ then $\{y : y < x\}$ is well-ordered
- If $A \subseteq T$ is such that $x \not\leq y \wedge y \not\leq x$ when $x \neq$ in A then A is countable
- If $C \subseteq T$ is a chain then C is countable

Line to tree L


- The intervals are the members of T
- The tree grows (upside down)

Tree to line: L is the set of maximal chains (the branches) of T ordered (suitably) lexicographically.

TENNENBAUM [1968] A MODEL WITH A SOUSLIN TREE

THE PARTIAL ORDER $\mathbb{P}$ THAT WE USE CONSISTS OF PAIRS $p = \langle F_p, \leq_p \rangle$

- F_p A FINITE SUBSET OF ω_1
- $\leq_p$ A PARTIAL ORDER OF F_p
  SUCH THAT IF $x \leq_p z$ AND $y \leq_p z$
  THEN $x \leq_p y$ OR $y \leq_p z$
  (IF $z \in F_p$ THEN $\{x \in F_p : x \leq_p z\}$ IS LINEARLY ORDERED BY $\leq_p$)
- IF $x <_p y$ THEN $x \in y$

ORDERING OF $\mathbb{P}$:

$p \leq q$ IF $F_p \supseteq F_q$ AND
$\leq_p \cap (F_q \times F_q) = \leq_q$

DENSE SET $D_\alpha = \{ p \in \mathbb{P} : \alpha \in F_p \}$

GIVEN q WITH $\alpha \notin F_q$. LET $F_p = F_q \cup \{\alpha\}$
$\leq_p = \leq_q \cup \{\langle \alpha, \alpha \rangle\}$
(α INCOMPARABLE WITH ALL $x \in F_q$)

IF G IS Π-GENERIC ON $\mathbb{P}$

THEN - $\bigcup \{F_p : p \in G\} = \omega_1$
- $\bigcup \{\leq_p : p \in G\}$ IS A PARTIAL ORDER: $\leq_G$
  IF $x \in \omega_1$ THEN $\{y : y \leq_G x\}$
  IS LINEARLY ORDERED BY $\leq_G$
  AND WELL-ORDERED BECAUSE
  $x <_G y$ IMPLIES $x \in y$.

SO $(\omega_1^M, \leq_G)$ IS A TREE

LEMMA

LET A BE AN UNCOUNTABLE SUBSET OF $\mathbb{P}$ AND LET $\langle x_p : p \in A \rangle$ BE A CHOICE FUNCTION FOR $\{F_p : p \in A\}$ SUCH THAT $\{x_p : p \in A\}$ IS UNCOUNTABLE

THEN THERE ARE p AND q IN A AND r IN $\mathbb{P}$
SUCH THAT $r \leq p, q$ AND
$x_p <_r x_q$ OR $x_q <_r x_p$
(DEPENDING ON $x_p \in x_q$ OR $x_q \in x_p$)

IN PARTICULAR: ALL ANTICHAINS IN $\mathbb{P}$ ARE COUNTABLE. SAY $x_p = \max F_p$ (MAX IN ω_1)

STEP 1 — WITHOUT LOSS OF GENERALITY THE MAP $\langle x_p : p \in A \rangle$ IS INJECTIVE

STEP 2 — WITHOUT LOSS OF GENERALITY THERE IS $k \in \omega$ SUCH THAT

$$|F_p| = k \quad \text{FOR ALL } p \in A$$

STEP 3 — APPLY THE Δ-SYSTEM LEMMA TO FIND A FINITE R AND $B \subseteq A$ UNCOUNTABLE SUCH THAT

$$F_p \cap F_q = R \quad \text{IF } p \neq q \text{ IN } B.$$

STEP 4 — IF $\langle \alpha_{p,i} : i \in k \rangle$ IS THE MONOTONE ENUMERATION OF F_p THEN

- $\leq_p$ GIVES A PARTIAL ORDER $\trianglelefteq_p$ ON k:

$$i \trianglelefteq_p j \quad \text{IFF} \quad \alpha_{p,i} \leq_p \alpha_{p,j}$$

- x_p HAS A NUMBER i_p
- $R = \{\alpha_{p,i} : i \in I_p\}$

THERE IS A TRIPLE $\langle \trianglelefteq, i, I \rangle$ SUCH THAT $C = \{p \in B : \trianglelefteq_p = \trianglelefteq,\ i_p = i,\ I_p = I\}$ IS UNCOUNTABLE

[BECAUSE $\min(F_p \setminus R) \leq \max R$ FOR ONLY COUNTABLY MANY p WE HAVE $I_p = |R| < k$]

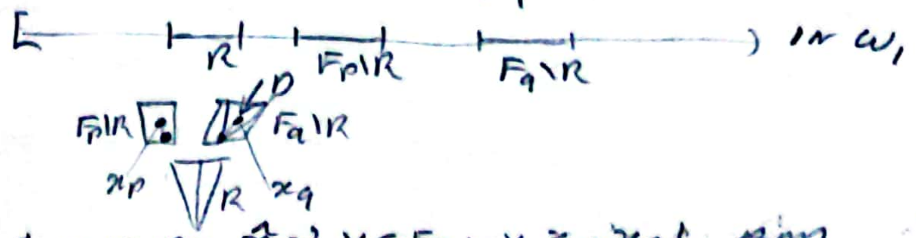
[R IS FINITE SO $i_p \geq R$ FOR $p \in C$]

SO $R = \{\alpha_{p,i} : i \in \ell\}$ $\quad \ell = |R|$

$i \geq \ell \quad x_p = \alpha_{p,i}$

STEP 5 — TAKE $p, q \in C$ SUCH THAT

$$\max F_p < \min(F_q \setminus R)$$



LOOK AT $\hat{x}_p = \{y \in F_p : y \leq_p x_p\}$ AND

$$\hat{x}_q = \{y \in F_q : y \leq_q x_q\}$$

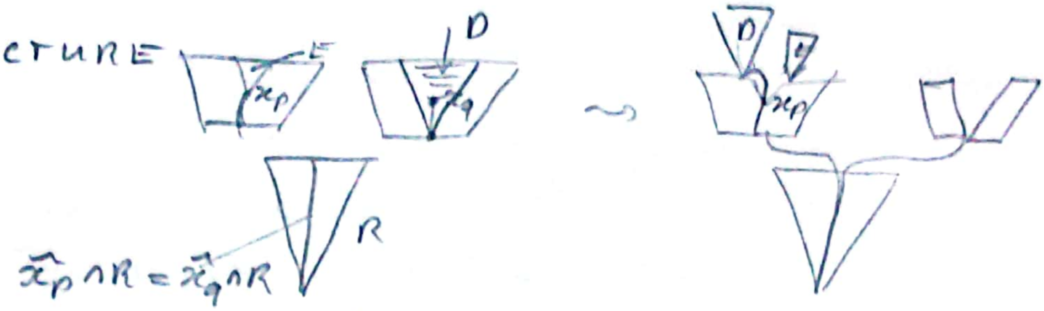
NOTE $\quad \hat{x}_p \cap R = \hat{x}_q \cap R$ BY ISOMORPHISM

LET $D = \{y \in F_q : \exists x \in \hat{x}_q \setminus R)(x \leq_p y)\}$

LET $F_r = F_p \cup F_q$

$$\leq_r = \leq_p \cup \leq_q \cup (\hat{x}_p \setminus R \times D)$$

PICTURE

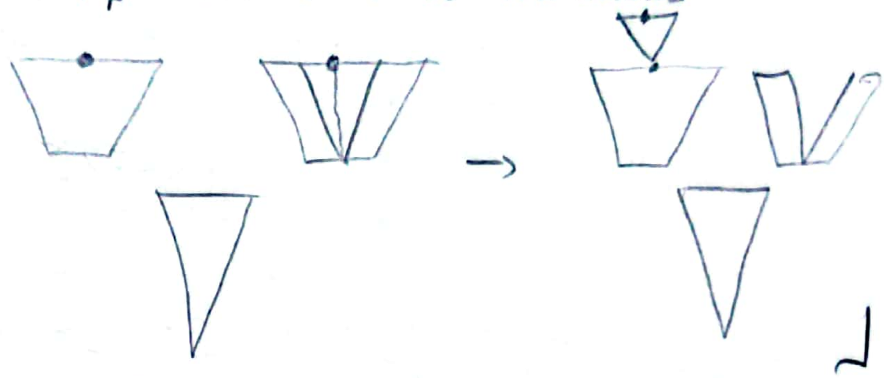


SO IF $x \in \hat{x}_p \setminus R$ AND $y \in D$ THEN $x \le_r y$

IN PARTICULAR $x_p < x_q$

SO $r = (F_r, \le_r)$ IS AS REQUIRED

[IF $x_p = \max F_p$ THEN THIS REDUCES TO



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SO AS ANTICHAINS IN $\mathbb{P}$ ARE COUNTABLE WE HAVE $cf^M \alpha = cf^{M[G]} \alpha$ FOR ALL ORDINALS IN M.

LET τ AND p BE SUCH THAT

$p \Vdash \tau$ IS UNCOUNTABLE

LET $B = \{\beta : (\exists q \le p)(q \Vdash \check{\beta} \in \tau)\}$

THEN $(\forall \alpha \in \omega_1)(\exists \beta \in B)(\beta > \alpha)$

TAKE $p_\beta \le p$ FOR $\beta \in B$ SUCH THAT

$p_\beta \Vdash \check{\beta} \in \tau$ AND $\beta \in F_{p_\beta}$

APPLY THE LEMMA TO $\{p_\beta : \beta \in B\}$

AND $x_\beta = \max\{\alpha \in F_{p_\beta} : p_\beta \Vdash \check{\alpha} \in \tau\}$

FIND $\beta, \gamma \in B$ AND r SUCH THAT

- $\beta \in \gamma$
- $r \le p_\beta, p_\gamma$
- $\beta \le_r \gamma$

WE SEE $r \Vdash \check{\beta} \le_{\dot{r}} \check{\gamma}$.

SO IF $A \subseteq \omega_1$ IS UNCOUNTABLE THEN THERE ARE $\beta \ne \gamma \in A$ WITH $\beta \le_G \gamma$ OR $\gamma \le_G \beta$

IF WE HAVE $\alpha <_G \beta$ IN $(\omega_1, \leq_G)$

THEN THERE IS $\gamma \in \omega_1$ SUCH

THAT - $\alpha, \beta \in \gamma$

- $\alpha <_G \gamma$

- $\beta \not\leq_G \gamma$ AND [CERTAINLY] $\gamma \not\leq_G \beta$

TAKE $p \in G$ SUCH THAT $p \Vdash \alpha <_G \beta$

AND $\alpha, \beta \in F_p$ SO $\alpha <_p \beta$

TAKE γ IN ω_1 ABOVE α AND β

LET $F_q = F_p \cup \{\gamma\}$

$\leq_q = \leq_q \cup \{\langle x, \gamma\rangle : x \leq_p \alpha\}$

$\cup \{\langle \gamma, \gamma\rangle\}$

SO $\beta \not\leq_q \gamma$ AND $\gamma \not\leq_q \beta$

THEN $q \Vdash \check{\alpha} <_{\dot{T}} \check{\beta} \wedge \check{\alpha} <_{\dot{T}} \check{\gamma} \wedge (\check{\beta} \perp \check{\gamma})$

THE SET OF SUCH q IS DENSE BELOW p.

SO IF $C \subseteq \omega_1$ IS A CHAIN WITH RESPECT TO $\leq_G$

THEN FOR $\alpha \in C$ THERE IS $\gamma_\alpha \in \omega_1$

SUCH THAT $\alpha <_G \gamma_\alpha$ AND

$\beta \not\leq_G \gamma_\alpha$ IF $\beta \in C$ AND $\alpha <_G \beta$

BUT THEN $\gamma_\alpha \not\leq \gamma_\delta \wedge \gamma_\delta \not\leq \gamma_\alpha$ WHENEVER WE HAVE $\alpha, \delta \in C$

SO C IS COUNTABLE.

A MODEL WITHOUT SOUSLIN TREES IS HARDER TO MAKE IT REQUIRES <u>ITERATED</u> FORCING: KILL THE SOUSLIN TREES ONE AT A TIME --------

DEVLIN AND JOHNSBRÅTEN "THE SOUSLIN PROBLEM"

"..., WE SIMPLY ARRANGE IT SO THAT THE SOUSLIN TREES CONCERNED WILL BE INCLUDED IN SOME FINAL MASS MURDER, ..."

STATIONARY SUBSETS OF ω_1.

HOMEWORK #9 EXERCISE (32)

LET $S \subseteq \omega_1$ BE STATIONARY

LET $\mathbb{P}_S = \{ p : p$ IS CLOSED IN ω_1, p IS COUNTABLE, $p \subseteq S \}$

IF $p \in \mathbb{P}_S$ AND $\alpha > \min(p)$ THEN $\sup(p \cap \alpha) \in p$

BECAUSE p IS COUNTABLE THIS ALSO HOLDS FOR $\alpha > \sup(p)$ SO $\sup(p) \in p$ AND p HAS A MAXIMUM.

ORDER $p \leq q$ IFF p IS AN END-EXTENSION OF q, THAT IS, $q = p \cap [0, \max q]$

- $D_\alpha = \{ p \in \mathbb{P}_S : \max(p) > \alpha \}$ IS DENSE

  IF $\max(q) \leq \alpha$ TAKE $p = q \cup \{\alpha+1\}$

  THEN $p \leq q$ AND $p \in D_\alpha$

- IF G IS M-GENERIC THEN $E = \bigcup G$ IS CLOSED AND COFINAL IN ω_1^M

- WE SHOW $\omega_1^M = \omega_1^{M[G]}$

  IN FACT, WE SHOW: IF $X \in M$ AND $f : \omega_1^M \to X$ IS IN $M[G]$ THEN $f \in M$.

  HOW ASSUME $p \Vdash \varphi : \check{\omega} \to \check{X}$

  WE FIND $q \leq p$ AND $g \in M$ SUCH THAT $q \Vdash \varphi = \check{g}$.

  PROBLEM: LAST WEEK'S STRATEGY MAY NOT WORK; THERE CAN BE SEQUENCES $\langle p_n : n \in \omega \rangle$ IN $\mathbb{P}$ WITH $p_{n+1} \leq p_n$ FOR ALL n FOR WHICH THERE IS NO $r \in \mathbb{P}$ SUCH THAT $r \leq p_n$ FOR ALL n.

  IF S IS NOT CUB TAKE $\alpha \in \omega_1 \setminus S$ SUCH THAT $\alpha = \sup(S \cap \alpha)$

  TAKE $\langle \alpha_n : n \in \omega \rangle$ INCREASING AND COFINAL IN $S \cap \alpha$.

  PUT $p_n = \{ \alpha_i : i \leq n \}$.

SO, ..., NOW WHAT?

FOR $n \in \omega$ LET $D_n = \{q \leq p : (\exists x \in X)(q \Vdash \varphi(\check{n}) = \check{x})\}$

NOTE D_n IS DENSE BELOW p [CLEAR]

AND OPEN [IF $q \in D_n$ AND $r \leq q$ THEN $r \in D_n$]

WE SHOW $\bigcap_{n \in \omega} D_n$ IS DENSE BELOW p, (AND OPEN)

[OPEN IS CLEAR : $q \in \bigcap_n D_n \wedge r \leq q \to r \in \bigcap_n D_n$]

FOR THEN IF $q \in \bigcap_{n \in \omega} D_n$ THEN FOR EVERY n THERE IS AN $x \in X$ SUCH THAT $q \Vdash \varphi(\check{n}) = \check{x}$ AND THAT x IS UNIQUE BECAUSE φ IS FORCED (BY p) TO BE A MAP.

WE GET $g = \langle x_n : n \in \omega \rangle$ SUCH THAT

$$q \Vdash \varphi(\check{n}) = \check{x}_n$$

FOR ALL n ; SO $q \Vdash \varphi = \check{g}$

LET $q_0 \leq p$ AND $\alpha_0 = \max(q_0) + 1$

WE BUILD A SEQUENCE $\langle A_\gamma : \gamma \in \omega_1 \rangle$ OF COUNTABLE SUBSETS OF $\mathbb{P}$ TOGETHER WITH AN INCREASING AND CONTINUOUS SEQUENCE $\langle \alpha_\gamma : \gamma \in \omega_1 \rangle$ AS FOLLOWS.

- $A_0 = \{q_0\}$ WE HAVE α_0 ALREADY
- GIVEN A_γ AND α_γ

  CHOOSE FOR EVERY $q \in A_\gamma$ AN ELEMENT $q_n \in D_n$ SUCH THAT

  $q_{\gamma,n} \leq q$ AND $\max q_{\gamma,n} > \alpha_\gamma$

  [FIRST $r \leq q$ WITH $r \in D_n$ AND THEN APPLY EXERCISE (32) TO EXTEND r TO $q_{\gamma,n}$ WITH $\max q_{\gamma,n} > \alpha_\gamma$]

  LET $A_{\gamma+1} = A_\gamma \cup \{q_{\gamma,n} : q \in A_\gamma; n \in \omega\}$

  $\alpha_{\gamma+1} = \sup\{\max(q) : q \in A_{\gamma+1}\}$
- $A_\gamma = \bigcup_{\beta \in \gamma} A_\beta$ AND $\alpha_\gamma = \sup_{\beta \in \gamma} \alpha_\beta$ IF γ IS A LIMIT.

THE SET $\{\alpha_\gamma : \gamma \in \omega_1\}$ IS CLOSED AND UNBOUNDED [GROUP INTERACTION # 8 (5)]

EVEN $\{\gamma : \alpha_\gamma = \gamma\}$ IS CLOSED AND UNBOUNDED [GI # 8 (6)]

TAKE A LIMIT ORDINAL γ SUCH THAT $\alpha_\gamma \in S$ AND $\langle \gamma_n : n \in \omega \rangle$ INCREASING AND COFINAL IN γ

TAKE $q_1 \leq q_0$ WITH $q_1 \in B_{\gamma_0}$ AND $q_1 \in D_0$

$q_2 \leq q_1$ WITH $q_2 \in B_{\gamma_1}$ AND $q_2 \in D_1$ AND $\max q_2 > \alpha_{\gamma_0}$

$q_{n+2} \leq q_{n+1}$ WITH $q_{n+2} \in B_{\gamma_{n+1}}$ AND $q_{n+2} \in D_{n+1}$ AND $\max q_{n+2} > \alpha_{\gamma_n}$

NOW OBSERVE THAT ALWAYS

$$\max q_{n+1} \leq \alpha_{\gamma_n} < \max q_{n+2}$$

LET $q = \{\alpha_\gamma\} \cup \bigcup_{n\in\omega} q_n$

- q IS CLOSED
- $q \subseteq S$

SO $q \leq q_n$ FOR ALL n

AND $q \in \bigcap_{n\in\omega} D_n$.

SO NOT ALL DECREASING SEQUENCES HAVE LOWER BOUNDS BUT ENOUGH OF THEM DO