SET THEORY

2021-12-20

LET $\sigma \in M^{\mathbb{P}}$ WE BUILD $\varrho \in M^{\mathbb{P}}$

$$S = \mathcal{P}^M(\mathrm{dom}(\sigma) \times \mathbb{P})$$
$$= \{\tau \in M^{\mathbb{P}} : \mathrm{dom}\,\tau \subseteq \mathrm{dom}\,\sigma\}$$
$$\varrho = S \times \{\mathbb{1}\}$$

- ASSUME $\mu \in M^{\mathbb{P}}$ AND $\mu_G \subseteq \sigma_G$

$$\boxed{\tau = \{\langle \pi, p\rangle : \pi \in \mathrm{dom}\,\sigma \text{ AND } p \Vdash \pi \in \mu\}}$$

$\tau \in S$ $\quad \tau_G \in S_G$

WE SHOW $\tau_G = \mu_G$.

– $\mu_G \subseteq \tau_G$: IF $x \in \mu_G$ THEN $x = \pi_G$ FOR SOME $\pi \in \mathrm{dom}\,\sigma$

$\pi_G \in \mu_G$ SO

TAKE $p \in G$ WITH $p \Vdash \pi \in \mu$

$\langle \pi, p\rangle \in \tau$

$x = \pi_G \in \tau_G$ THANKS TO p

– $\tau_G \subseteq \mu_G$

IF $x \in \tau_G$ THEN THERE ARE $p \in G$ AND $\pi \in \mathrm{dom}\,\tau$ WITH $x = \pi_G$ AND $\langle \pi, p\rangle \in \tau$

BUT $p \Vdash \pi \in \mu$

SO $x = \pi_G \in \mu_G$

ALMOST DISJOINT FAMILIES

- SIERPIŃSKI [1928]

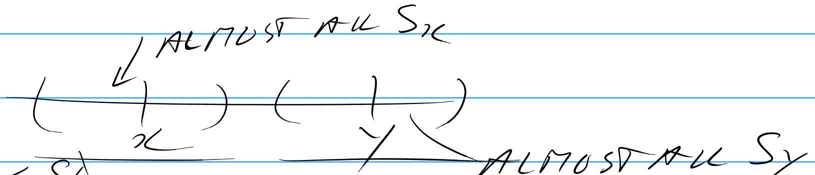
THERE IS A FAMILY $\mathcal{A} \subseteq \mathcal{P}(\omega)$ SUCH THAT
– IF $A \in \mathcal{A}$ THEN $|A| = \aleph_0$
– IF $A, B \in \mathcal{A}$ AND $A \neq B$ THEN $|A \cap B| < \aleph_0$

$-\ |\mathcal{A}| = 2^{\aleph_0}$

CHOOSE FOR $x \in \mathbb{R} \setminus \mathbb{Q}$ A SEQUENCE S_x OF RATIONAL NUMBERS THAT CONVERGES TO x

$-\ |S_x| = \aleph_0$

$-\ x \neq y$



$|S_x \cap S_y| < \aleph_0$

$-\ |\mathbb{R} \setminus \mathbb{Q}| = 2^{\aleph_0}$

[ALSO TARSKI]

IF CH THEN WE CAN DO SOMETHING SIMILAR ON ω_1 :

$-\ \mathcal{A} \subseteq \mathcal{P}(\omega_1) \quad |\mathcal{A}| = 2^{\aleph_1}$

$\cdot\ A \in \mathcal{A} \to |A| = \aleph_1$

$\cdot\ A \neq B \to |A \cap B| < \aleph_1$

$$T = \bigcup_{\alpha < \omega_1} 2^\alpha \qquad |2^\alpha| = 2^{\aleph_0} \ (\alpha \geq \omega)$$

$$|T| = \aleph_1$$

$x \in 2^{\omega_1} \rightsquigarrow B_x = \{x \restriction \alpha : \alpha < \omega_1\}$

$|B_x| = \aleph_1$

$x \neq y$ SAY $x(\alpha) \neq y(\alpha)$

THEN $B_x \cap B_y \subseteq \{x \restriction \beta : \beta \leq \alpha\}$

$\subseteq \{y \restriction \beta : \beta \leq \alpha\}$

NEXT HOUR :

CONSISTENCY OF

$ZFC + 2^{\aleph_0} > \aleph_2$

+ ALL ALMOST DISJOINT FAMILIES ON ω_1 HAVE CARD. $\leq \aleph_2$

[BAUMGARTNER, 1979]

START WITH GCH IN M.

$\mathbb{P} = Fn(\omega_3^M \times \omega, 2)$; IN $2^{\aleph_0} = 2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$

IN $M[G]$ NO ALMOST DISJOINT FAMILY OF CARD. $\aleph_3$

ASSUME $\langle A_\alpha : \alpha \in \omega_3 \rangle$ IS SUCH THAT
- $|A_\alpha| = \aleph_1$; $A_\alpha \subseteq \omega_1$
- $\alpha < \beta \rightarrow |A_\alpha \cap A_\beta| < \aleph_1$

TAKE $p \in G$ AND $\varphi \in M^{\mathbb{P}}$ SUCH THAT $\varphi_G = \langle A_\alpha : \alpha \in \omega_3 \rangle$

$$p \Vdash \text{"}\varphi : \check{\omega}_3 \rightarrow \mathcal{P}(\check{\omega}_1) \wedge \forall \alpha\, |\varphi(\alpha)| = \aleph_1 \wedge \alpha \neq \beta \rightarrow |\varphi(\alpha) \cap \varphi(\beta)| \leq \aleph_0\text{"}$$

CLAIM: IF $\alpha < \beta$ THEN THERE IS $\gamma(\alpha, \beta) < \omega_1$ SUCH THAT

$$p \Vdash \varphi(\check{\alpha}) \cap \varphi(\check{\beta}) \subseteq \gamma(\alpha,\beta)^\vee$$

WE KNOW

$$p \Vdash \sup(\varphi(\check{\alpha}) \cap \varphi(\check{\beta})) < \omega_1$$

$$F_{\alpha,\beta} = \{ \gamma \in \omega_1 : (\exists q \leq p)(q \Vdash \check{\gamma} = \sup \varphi(\check{\alpha}) \cap \varphi(\check{\beta})) \}$$

FOR $\gamma \in F_{\alpha,\beta}$ PICK q_γ AS IN DEFINITION

IF $\gamma \neq \delta$ THEN $q_\gamma \perp q_\delta$

WE NOW KNOW

$|F_{\alpha,\beta}| \leq \aleph_0$ BECAUSE ANTICHAINS IN $\mathbb{P}$ ARE COUNTABLE

$$\gamma(\alpha,\beta) = \sup F_{\alpha,\beta} + 1$$

WE HAVE

$$\gamma : [\omega_3]^2 \longrightarrow \omega_1 \qquad \underline{\underline{\text{IN } M}}$$

IN M:

$$\aleph_3 = (2^{\aleph_1})^+$$

ERDŐS-RADO $\quad \aleph_3 \longrightarrow (\aleph_2)^2_{\aleph_1}$

WE FIND ONE γ AND $H \subseteq \omega_3$ WITH
- $|H| = \aleph_2$
- $\gamma(\alpha, \beta) = \gamma \quad (\alpha, \beta \in H)$

SO

$$p \Vdash \varphi(\check{\alpha}) \cap \varphi(\check{\beta}) \subseteq \check{\gamma}$$

FOR $\alpha, \beta \in H$

IN G : $\varphi_G(\alpha) = A_\alpha$ IS UNCTBLE

SO $A_\alpha \setminus \gamma \neq \emptyset$

IF $\alpha \neq \beta$ IN H

THEN $(A_\alpha \setminus \gamma) \cap (A_\beta \setminus \gamma) = \emptyset$

IMPOSSIBLE, $\aleph_2$ MANY DISJOINT NONEMPTY SUBSETS OF ω_1.

CONTRADICTION

EXERCISE:

THERE IS ALWAYS AN ALMOST DISJOINT FAMILY OF CARD $\aleph_2$ ON ω_1

HINT WORK IN $\omega_1 \times \omega_1$

USE GRAPHS OF FUNCTIONS.

RECTANGLES

X A SET $A, B \subseteq X$

$A \times B$ IS A RECTANGLE IN $X \times X$

- $\mathcal{R}_X = \{A \times B : A, B \subseteq X\}$
- TAKE THE σ-ALGEBRA $\mathcal{S}_X$ GENERATED BY $\mathcal{R}_X$
  - $X \times X \in \mathcal{S}_X$
  - $S \in \mathcal{S}_X \to (X \times X) \setminus S \in \mathcal{S}_X$
  - $\{S_n\}_n \subseteq \mathcal{S}_X \to \bigcup_n S_n \in \mathcal{S}_X$

GENERATED BY: THE SMALLEST ONE THAT CONTAINS $\mathcal{R}_X$

?? $\mathcal{S}_X = \mathcal{P}(X \times X)$??

- YES $X = \omega$

  EVERY $\{\langle m, n \rangle\}$ IS A RECTANGLE
- YES $X = \omega_1$

  IF $S \subseteq \omega_1 \times \omega_1$ THEN YOU CAN MAKE $\{x_\alpha : \alpha \in \omega_1\}$ AND $\{y_\alpha : \alpha \in \omega_1\}$ SUBFAMILIES OF $\mathcal{P}(\omega)$

  $\langle \alpha, \beta \rangle \in S \Leftrightarrow x_\alpha \cap y_\beta$ INFINITE

$A_n = \{\alpha : n \in x_\alpha\}$
$B_n = \{\beta : n \in y_\beta\}$

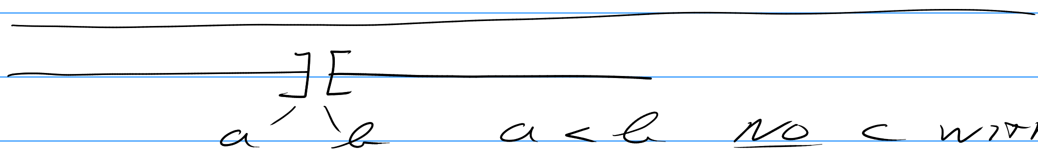
$$S = \bigcap_{m \in \omega} \left(\bigcup_{n \geq m} A_n \times B_n \right)$$

CH : YES IF $X = \mathbb{R}$

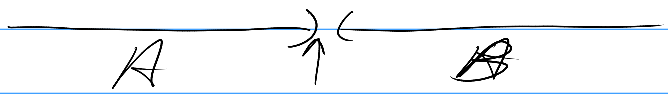
IF G IS M-GENERIC ON $Fn(\omega_2^{\kappa} \times \omega, 2)$
THEN NO FOR $X = \omega_2$ — ANY $\kappa \geq \omega_2$

SOUSLIN'S PROBLEM [1920]

IF L IS A LINEAR ORDER WITHOUT JUMPS OR GAPS IN WHICH EVERY PAIRWISE DISJOINT FAMILY OF OPEN INTERVALS IS COUNTABLE MUST L BE A NORMAL LINEAR CONTINUUM.

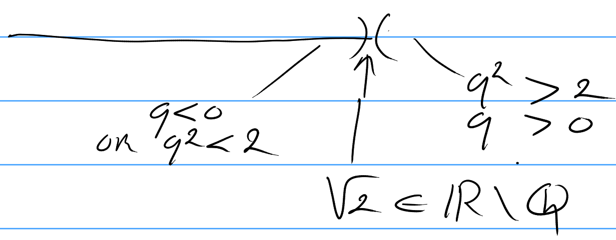


$a < b$ NO c WITH $a < c < b$



$a < b$ WHENEVER $a \in A$ $b \in B$
NO c WITH $a < c < b$ FOR ALL $a \in A$ AND $b \in B$

$\mathbb{Q}$



NORMAL LINEAR CONTINUUM: $\mathbb{R}$

1968 TENNENBAUM JECH — RELATIVELY EASY
CONSISTENT: NO

SOLOVAY TENNENBAUM
CONSISTENT YES ⟸ WAY MORE DIFFICULT
ITERATED FORCING.

Counter example: Souslin line

Jech: Chapter 9

There is a Sousline line
iff there is a Souslin tree

Souslin tree

- tree $(T, \leq)$ partial order

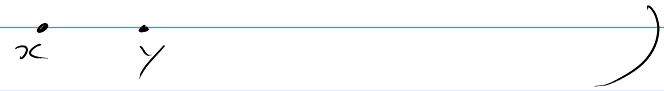
tree if $x \in T$ then $\{y : y < x\}$ is well-ordered

Souslin: - $|T| = \aleph_1$

- antichains are countable
- A is an antichain if

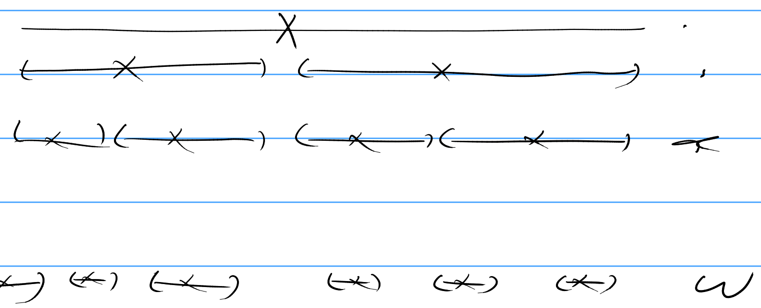
$x \not\leq y \wedge y \not\leq x$ if $x \neq y$

(incompatible in $(T, \geq)$



- all chains countable

Line to tree



Tree to line

The maximal chains (branches)
can be ordered to give a line

$\mathbb{P}$ consists of pairs $p = \langle F_p, \leq_p \rangle$

- F_p finite subset of ω_1
- $\leq_p$ a partial order of F_p

$\longrightarrow$ • $x \leq_p z \wedge y \leq_p z \rightarrow y \leq_p x \vee x \leq_p y$

$\longrightarrow$ • $x <_p y \rightarrow x \in y$

$p \le q \qquad F_p \supseteq F_q$

$\le_p \cap (F_q \times F_q) = \le_q$

- $D_\alpha = \{p : \alpha \in F_p\}$ IS DENSE $(\alpha \in \omega_1)$

  IF $\alpha \notin F_q$ MAKE $F_p = F_q \cup \{\alpha\}$

  $\le_p = \le_q \cup \{\langle \alpha, \alpha \rangle\}$

  α INCOMPARABLE WITH F_q

IF G IS M-GENERIC ON $\mathbb{P}$

THEN – $\bigcup \{F_p : p \in G\} = \omega_1$

– $\le_G = \bigcup \{\le_p : p \in G\}$

IS A PARTIAL ORDER OF ω_1

– $\{y : y \le_G x\}$ IS LINEARLY ORDERED

– $x <_G y$ IMPLIES $x \in y$

SO $<_G$ IS WELL-FOUNDED

LEMMA

LET $A \subseteq \mathbb{P}$ BE UNCOUNTABLE

$\langle x_p : p \in A \rangle$ A CHOICE FUNCTION FOR $\langle F_p : p \in A \rangle$

$\{x_p : p \in A\}$ IS UNCOUNTABLE.

THEN THERE ARE p AND q IN A AND $r \in \mathbb{P}$

SUCH THAT $r \le p \cup q$

$x_p <_r x_q$ OR $x_q <_r x_p$

DEPENDING ON $x_p \in x_q$

OR $x_q \in x_p$

IN PARTICULAR A IS NOT AN ANTICHAIN

($x_p = \max F_p$ (IN ω_1))

W.L.O.G.

1. $\langle x_p : p \in A \rangle$ IS INJECTIVE
2. THERE IS A $k \in \omega$ ST $|F_p| = k$ FOR ALL p

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3. APPLY THE Δ-SYSTEM LEMMA TO FIND R AND $B \subseteq A$

- $\mathcal{B}$ is uncountable
- $F_p \cap F_q = R \quad p \neq q$ in $\mathcal{B}$

(4) $\langle \alpha_{p,i} : i \in k \rangle$ monotone enumeration of F_p

- determines a partial order $\trianglelefteq_p$ on k via $i \trianglelefteq_p j$ iff $\alpha_{p,i} \leq_p \alpha_{p,j}$
- x_p has index i_p
- $R = \{\alpha_{p,i} : i \in I_p\}$

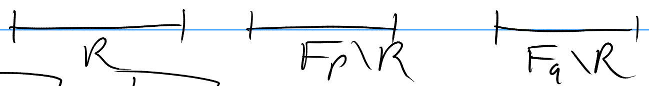
$\langle \trianglelefteq_p, i_p, I_p \rangle$ has only finitely many possibilities

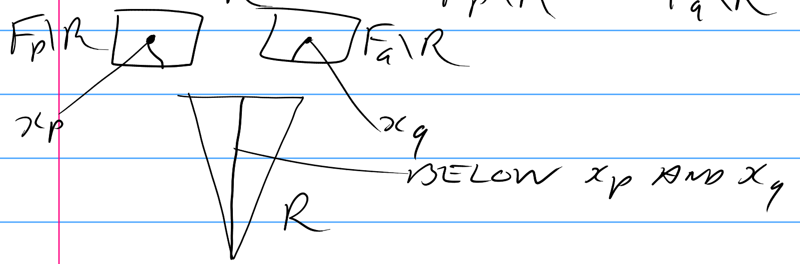
<u>ONE</u> triple $\langle \trianglelefteq, i, I \rangle$ and $C \subseteq \mathcal{B}$ uncountable st. $\langle \trianglelefteq_p, i_p, I_p \rangle = \langle \trianglelefteq, i, I \rangle \quad p \in C$

$\min\{F_p \setminus R : p \in C\}$ is uncountable

so $\max R < \min F_p \setminus R$ always

$\max F_p < \min F_q \setminus R$



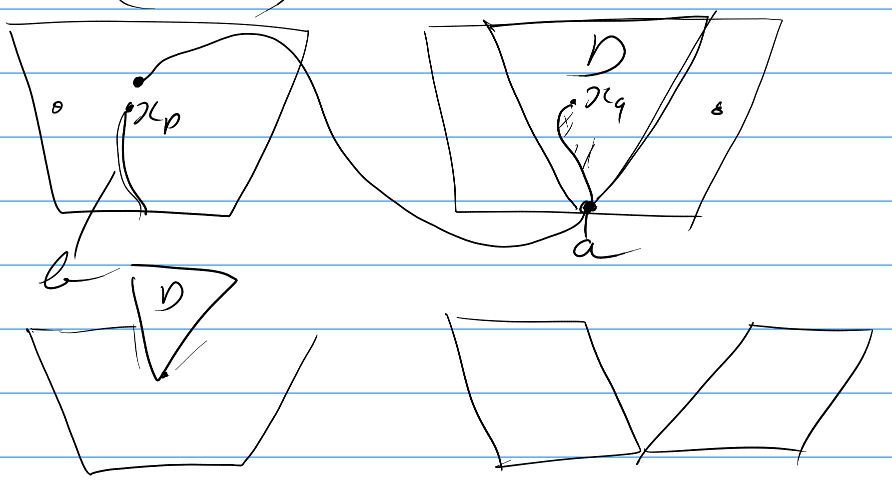


$F_r = F_p \cup F_q$

$\leq_r = \leq_p \cup \leq_q \cup (b \times D)$

$x_p \in x_q$

$\boxed{x_p <_r x_q}$



$p \Vdash \tau$ IS UNCOUNTABLE

$B = \{\beta : (\exists q \le p)(q \Vdash \beta \in \tau)\}$

IS UNCOUNTABLE

TAKE $p_\beta \le p$ FOR $\beta \in B$

MAY ASSUME $\beta \in F_{p_\beta}$

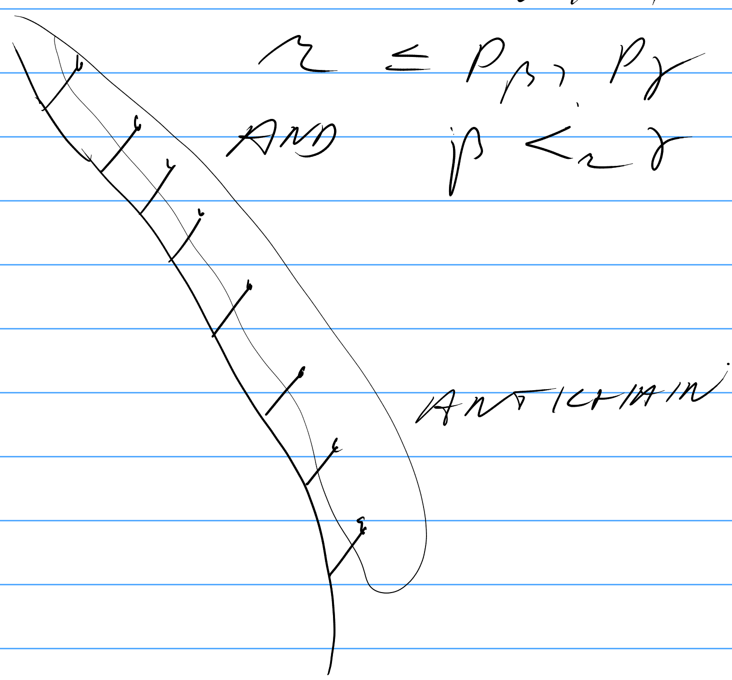
APPLY THE LEMMA

WE GET $\beta \neq \gamma$

AND r WITH $x_{p_\beta} = \beta$!

$r \le p_\beta, p_\gamma$

AND $\beta <_r \gamma$ SO $r \Vdash$ "$\beta < \gamma$"



SHOENFIELD: $M^P = M$!

LET $f : X \to Y$

NAME FOR f: $\dot{f}$, $\underset{\sim}{f}$, — BOLDFACE

NAME OF f

S STATIONARY IN ω_1

$\longrightarrow$ THERE IS A G

WITH IN $M[G]$ A CLUB SUBSET C

THAT IS IN S

HOMEWORK #9 EX (33)