TU Delft

mastermath

Set Theory

2022/2023 1st Semester

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Name:

University:

Student ID:

General comments.

1. **The time for this exam is 3 hours (180 minutes).**
2. **There are 110 points in the exam: if a student obtains x points, the exam grade will be $\frac{x+10}{12}$.**
3. **Please mark the answers to the questions in Question I on this sheet by crosses.**
4. **Make sure that you have your name, university and student ID on each of the sheets you are handing in.**
5. **If you have any questions, please indicate this silently and someone will come to you. Answers to questions that are relevant for everyone will be announced publicly.**
6. **No talking during the exam.**
7. **Cell phones must be switched off and stowed.**

Question I.	(45 points)	**Question IV.**	(20 points)
Question II.	(10 points)	**Question V.**	(15 points)
Question III.	(20 points)	***TOTAL***	(110 points)
		GRADE	

(45) Question I

Every multiple-choice question below has exactly one correct answer (3 points each):

Week 1. Consider Zermelo's set Z_0 (the smallest set that contains $\varnothing$ and that is closed under $a \mapsto \{a\}$) Define $f : \omega \to Z_0$ by $f(0) = \varnothing$ and $f(n+1) = \{f(n)\}$. The first $n \in \omega$ for which $f(n) \neq n$ is
☐ **A**: 2.
☐ **B**: 4.
☐ **C**: 1.
☐ **D**: 3.

Week 2. Our definition of the ordered pair is $\langle x, y\rangle = \bigl\{\{x\}, \{x, y\}\bigr\}$. What is the relationship between the ordered pair $\langle 1, 2\rangle$ and the ordinal 3?
☐ **A**: $\langle 1, 2\rangle \subset 3$ (proper subset).
☐ **B**: $3 \subset \langle 1, 2\rangle$ (proper subset).
☐ **C**: They are incomparable.
☐ **D**: $3 = \langle 1, 2\rangle$.

Week 3. The definition of $\operatorname{dom} R$ makes sense (formally) for arbitrary sets. Using this formal definition $\operatorname{dom}\omega$ is equal to
☐ **A**: ω.
☐ **B**: $\{0, 1\}$.
☐ **C**: $\{0\}$.
☐ **D**: $\varnothing$.

Week 4. Zermelo's proof of the well-ordering theorem starts, given a set X, with a choice function γ for the family $\mathcal{P}(X) \setminus \{\varnothing\}$. The resulting well-order $\prec$ satisfies:
☐ **A**: The order-type of $(X, \prec)$ is equal to the cardinal number of X.
☐ **B**: The order-type of $(X, \prec)$ is a singular ordinal.
☐ **C**: For all $x \in X$ we have $x = \gamma\bigl(\{y : x \preccurlyeq y\}\bigr)$.
☐ **D**: For all $A \in \mathcal{P}(X) \setminus \{\varnothing\}$ we have $\min A = \gamma(A)$.

Week 5. What is $(\omega^{2023} + 2022) \cdot (\omega^{2022} + 2023)$ (ordinal arithmetic)?
☐ **A**: $\omega^{4045} + \omega^{2023} \cdot 2023 + 2022$.
☐ **B**: $\omega^{4045} + \omega^{2022} \cdot 2022 + 2023$.
☐ **C**: $\omega^{4045} + \omega^{2022} + \omega^{2023} \cdot 2023 + 4090506$.
☐ **D**: $\omega^{4045} + 4090506$.

Week 6. One of the following statements ***cannot*** be proved in ZF without the Axiom of Choice. Which one?
☐ **A**: If $f : \omega \to \mathbb{R}$ is a map then f is not surjective.
☐ **B**: There is an injection from ω_1 into $\mathcal{P}(\omega)$.
☐ **C**: There is a bijection between $[0, 1]$ and $[0, 1]^\omega$.
☐ **D**: There is a surjection from $\mathcal{P}(\omega)$ to ω_1.

Week 7. Which of the following statements is ***not*** forbidden by the Axiom of Foundation?
☐ **A**: There is a sequence $\langle x_n : n \in \omega\rangle$ of sets such that $x_{n+1} \in x_n$ for all n.
☐ **B**: There is an x such that $x \in x$.
☐ **C**: There is a sequence $\langle x_n : n \in \omega\rangle$ of sets such that $x_{2n+1} \in x_{2n}$ for all n.
☐ **D**: There are x and y such that $x \in y$ and $y \in x$.

Week 8. Assume that $2^{\aleph_n} = \aleph_{\omega+n+2023}$ for all $n \geqslant 2022$. Then the value of $2^{\aleph_\omega}$ is
☐ **A**: $\aleph_{\omega+\omega}$
☐ **B**: $\aleph_{\omega+\omega}^{\aleph_0}$.
☐ **C**: $\aleph_{\omega+2023}$.
☐ **D**: Not determined on the basis of the information provided.

Week 9. Define a set-mapping F on the ordinal ω_ω as follows: if $\alpha < \omega_0$ then $F(\alpha) = \alpha$, and if $\omega_n \leqslant \alpha < \omega_{n+1}$ then $F(\alpha) = \alpha \setminus \omega_n$. The maximum cardinality of a free set for this mapping is

☐ **A**: $\aleph_{2023}$.
☐ **B**: 1.
☐ **C**: $\aleph_0$.
☐ **D**: $\aleph_\omega$.

Week 10. Let $\langle q_n : n \in \omega \rangle$ be an enumeration of $\mathbb{Q}$, the set of rational numbers. Define a $F[\mathbb{Q}]^2 \to \{0,1\}$ by

$$F\big(\{q_m, q_n\}\big) = \begin{cases} 1 & \text{if } m \in n \Leftrightarrow q_m < q_n \\ 0 & \text{if } m \in n \Leftrightarrow q_m > q_n \end{cases}$$

Which of ***one*** of the following sets is ***definitely not*** homogeneous for F (independent of the chosen enumeration)?

☐ **A**: $\mathbb{N}$.
☐ **B**: $\{2^{-n} : n \in \mathbb{N}\}$.
☐ **C**: $\{-3^n : n \in \mathbb{N}\}$.
☐ **D**: $\{(-2)^{-n} : n \in \mathbb{N}\}$.

Week 11. Let $T = \{s \in \omega^{<\omega} : (\forall i \in \operatorname{dom} s)(s(i) \leqslant i)\}$. The cardinality of the set of branches of T is

☐ **A**: $\aleph_1$.
☐ **B**: $2^{\aleph_0}$.
☐ **C**: $\aleph_0$.
☐ **D**: 0.

Week 12. The informal definition of $\boldsymbol{L}$ in week 12 was really informal because

☐ **A**: The operation $\operatorname{Def}(M)$ only worked on the meta-level.
☐ **B**: We could not prove the Comprehension/Separation schema.
☐ **C**: Not every formula is a Δ_0-formula.
☐ **D**: Not every property is absolute.

Week 13. Let M be a countable transitive model of $\mathsf{ZF} - \mathsf{P}$. Which of the following notions ***is not*** upward absolute between M and $\boldsymbol{V}$?

☐ **A**: x is a natural number
☐ **B**: x is countable.
☐ **C**: x is an ordinal
☐ **D**: x is uncountable.

Week 14. Let δ be a limit ordinal larger than ω and let $M \prec L_\delta$ be an elementary substructure. Which of the following statements is ***definitely true*** about M?

☐ **A**: M is isomorphic to L_β for some limit ordinal β.
☐ **B**: M is isomorphic to L_β for some successor ordinal β.
☐ **C**: $M = L_\beta$ for some limit ordinal β.
☐ **D**: M is transitive.

Week 15. Let $(S, <)$ be the Souslin tree constructed in week 15. What is special about S?

☐ **A**: S is the $<_{\boldsymbol{L}}$-first Souslin tree.
☐ **B**: S is definable in L_{ω_2}.
☐ **C**: All antichains of S are finite.
☐ **D**: It is also an Aronszajn tree.

(10) Question II

In this problem we do not assume the Axiom of Choice.
By definition a set X is finite if there are $n \in \omega$ and a bijection $f : n \to X$.
Another notion of finiteness was proposed by Dedekind: X is DD-finite if there is a map $f : X \to X$ with the property that if $Y \subseteq X$ and $f[Y] \subseteq Y$ then $Y = X$ or $Y = \varnothing$.

(5) (i) Prove that every finite set is DD-finite.

(5) (ii) Prove that every DD-finite set is finite. *Hint*: Fix $a \in X$ and define $g : \omega \to X$ by $g(0) = f(a)$ and $g(n+1) = f(g(n))$. Consider $Y = g[\omega]$.

(20) Question III

Let $[\omega_1]^{<\omega}$ denote the family of finite subsets of ω_1, and let F be the subtree of the tree $\omega_1^{<\omega}$ that consists of all *strictly decreasing* sequences.
On F we define

$$s \lhd t \text{ if } \begin{cases} s \subset t & \text{(proper initial segment), or} \\ s(i) < t(i) & \text{where } i = \min\{j : s(j) \neq t(j)\} \end{cases}$$

(5) (i) Prove that $f : F \to [\omega_1]^{<\omega}$, defined by $f(s) = \operatorname{ran} s$, is a bijection.

(5) (ii) Prove that $\lhd$ is a well-order of F.

Let $\prec$ be the well-order on $[\omega_1]^{<\omega}$ induced by $\lhd$ and f.

(5) (iii) Prove: if $\alpha \in \omega_1$ then $[\alpha]^{<\omega} = \{a : a \prec \{\alpha\}\}$.

(5) (iv) Calculate the order-types of $[\omega]^{<\omega}$, $[\omega+1]^{<\omega}$, and $[\omega_1]^{<\omega}$ with respect to the order $\prec$.

(20) Question IV

(10) (i) Let $\kappa = (2^{\aleph_0})^+$ and let $\{A_\alpha : \alpha < \kappa\}$ be a family of countable subsets of κ. Prove that there are a countable subset R of κ and a stationary subset S of κ such that $A_\alpha \cap A_\beta = R$ whenever $\alpha, \beta \in S$ and $\alpha \neq \beta$. *Hint*: Let $T = \{\alpha < \kappa : \operatorname{cf} \alpha > \aleph_0\}$ and consider $f : T \to \kappa$ defined by $f(\alpha) = \sup(A_\alpha \cap \alpha)$.

(10) (ii) Prove the first non-trivial case of Ramsey's theorem:

$$\aleph_0 \to (\aleph_0)^2_2.$$

(15) Question V

One way to define the set $\mathbb{Z}$ of integers is via the following equivalence relation on ω^2: we let $\langle k, l\rangle \equiv \langle m, n\rangle$ iff $k+n = m+l$, and we let $\mathbb{Z}$ be the set of $\equiv$-equivalence classes (the equivalence class of $\langle k, l\rangle$ represents "$k - l$").

(5) (i) Calculate $\operatorname{rank}(\mathbb{Z})$, the rank of $\mathbb{Z}$ in the hierarchy $\langle V_\alpha : \alpha \in \boldsymbol{On}\rangle$.

(5) (ii) Prove that $\mathbb{Z} \subseteq L_{\omega+1}$, that is, every $\equiv$-equivalence class is a definable subset of L_ω.

(5) (iii) Calculate $\rho(\mathbb{Z})$, the rank of $\mathbb{Z}$ in the hierarchy $\langle L_\alpha : \alpha \in \boldsymbol{On}\rangle$.