# EXERCISES SET THEORY (05)

2022/23

In the following exercises $\kappa$ always denotes a regular uncountable cardinal.

**1**. Prove: if $C$ is cub in $\kappa$ and $S$ is stationary in $\kappa$ then $C \cap S$ is stationary in $\kappa$. *Hint*: This is not overly difficult but a good exercise in writing down a solution so that another student can understand it without difficulty.

**2**. Remember that a map $f : \kappa \to \kappa$ is normal if it is strictly increasing and continuous.

a. Prove: if $C \subseteq \kappa$ is cub then the (unique) isomorphism $f : \kappa \to C$ is normal.

b. Prove: if $f : \kappa \to \kappa$ is normal then $\operatorname{ran} f$ is cub.

c. Prove: if $f : \kappa \to \kappa$ is normal then $\{\alpha : f(\alpha) = \alpha\}$ is cub.

d. Prove: if $f : \kappa \to \kappa$ is normal and $C \subseteq \kappa$ is cub then $f[C]$ is cub.

**3**. The details of the two disjoint stationary sets in $\omega_1$. Let $f : \omega_1 \to \mathbb{R}$ be injective. For every $x \in \mathbb{R}$ put $A_x = \{\alpha : f(\alpha) < x\}$ and $B_x = \{\alpha : f(\alpha) > x\}$.

a. Prove: for every $x$ at least one of $A_x$ and $B_x$ is stationary.

Let $I = \{x : A_x \text{ is non-stationary}\}$ and $E = \{x : B_x \text{ is non-stationary}\}$

b. Prove: if $x \in I$ and $y < x$ then $y \in I$, and symmetrically: if $x \in E$ and $x < y$ then $y \in E$.

c. Prove: $\bigcup_{x \in I} A_x$ and $\bigcup_{x \in E} B_x$ are both stationary. *Hint*: Consider $I \cap \mathbb{Q}$ and $E \cap \mathbb{Q}$.

d. Prove: $\sup I < \inf E$. (By convention $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.)

e. Let $x \in (\sup I, \inf E)$; show that $A_x$ and $B_x$ are both stationary.

**4**. Assume the Continuum Hypothesis ($2^{\aleph_0} = \aleph_1$) and prove the $\Delta$-system for countable subsets of $\omega_2$: if $\mathcal{A}$ is a family of countable subsets of $\omega_2$ and $|\mathcal{A}| = \aleph_2$ then $\mathcal{A}$ has a subfamily $\mathcal{B}$ of cardinality $\aleph_2$ that is a $\Delta$-system.

**5**. Let $\mathcal{A}$ be the following family of finite subsets of $\omega_\omega$:

$$\bigcup_{n<\omega} \bigl\{\{\omega_n, \alpha\} : \omega_n < \alpha < \omega_{n+1}\bigr\}$$

Show that $|\mathcal{A}| = \aleph_\omega$ and that every subfamily of $\mathcal{A}$ that is a $\Delta$-system has cardinality less than $\aleph_\omega$.

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*Date*: maandag 07-11-2022 at 22:28:28 (cet).