# HOMEWORK SET THEORY (03) 2022-10-17

2022/23

Hand in next week by 14:00 on 2022-10-24, either by hand in class, or by uploading to the course page on `elo.mastermath.nl`.

Collaboration is not forbidden, encouraged even. You may also hand in joint work, provided each contributes equally to the solutions (honour system).

This homework is about cardinals, their arithmetic, and finiteness.

**1**. This problem deals with the statement "if $A$ is an infinite set then $A \approx A \times A$"; let us call it $\mathbb{P}$.

a. Prove: the Axiom of Choice implies $\mathbb{P}$.

b. Prove: $\mathbb{P}$ implies the Well-Ordering Theorem. *Hint*: Let $X$ be infinite and let $A$ be the disjoint union of $X$ and $\aleph(X)$. From a bijection between $A \times A$ and $A$ you can *define* an injection from $X$ into $\aleph(X)$.

**2**. Prove: $(\forall n \in \omega)(n \not\approx n + 1)$.

**3**. There have been various attempts at defining finiteness.

- Dedekind defined $X$ to be infinite if there is an injective map $f : X \to X$ that is not surjective, *hence* we call $X$ *Dedekind-finite* if every injective $f : X \to X$ is surjective.
- Dually we can define $X$ to be $S$-finite if every surjective $f : X \to X$ is injective.
- Tarski defined $X$ to be finite if every nonempty family of subsets of $X$ has a maximal element, with respect to inclusion (call this $T$-finite).
- Kuratowski defined $X$ to be finite if $\mathcal{P}(X)$ is the only family of subsets of $X$ that contains $\emptyset$ and $\{x\}$ for all $x \in X$ and that is closed under taking unions, that is, $A \cup B$ belongs to the family if $A$ and $B$ do (call this $K$-finite).

Unless explicitly stated otherwise we do *not* assume the Axiom of Choice.

a. Prove that every finite set is $S$-finite.

b. Prove that every $S$-finite set is Dedekind-finite

c. Prove that the following are equivalent for a set $X$:

(1) $X$ is finite,
(2) $X$ is $T$-finite, and
(3) $X$ is $K$-finite.

d. Assume the Axiom of Choice and prove that Dedekind sets are finite.

*Date*: dinsdag 25-10-2022 at 15:30:58 (cest).