HOMEWORK SET THEORY (04) 2022-11-01

2022/23

Hand in next week by 14:00 on 2022-11-08, either by hand in class (on 2022-11-07 of course), or by uploading to the course page on `elo.mastermath.nl`.

Collaboration is not forbidden, encouraged even. You may also hand in joint work, provided each contributes equally to the solutions (honour system).

This homework is about well-founded sets and cardinal arithmetic.

1. In this problem we show that, assuming the Axiom of Foundation, the Axiom of Choice is equivalent to the statement that for every ordinal α the power set $\mathcal{P}(\alpha)$ can be well-ordered. From now on we assume the latter statement.

a. Prove that it suffices to show that for every ordinal α the set V_α can be well-ordered.

b. Verify that V_0 can be well-ordered.

c. Show: if V_α can be well-ordered then $V_{\alpha+1}$ can be well-ordered.

Now let α be a limit and assume that for all $\beta < \alpha$ the set V_β can be well-ordered. Let $\kappa = \aleph(V_\alpha)$ (Hartogs' aleph).

d. Show that $\mathcal{P}(\kappa \times \kappa)$ can be well-ordered.

e. Show: if $\beta < \alpha$ then there a subset R of $\kappa \times \kappa$ such that $(\operatorname{field} R, R)$ is isomorphic to $(V_\beta, \in)$, where $\operatorname{field} R = \operatorname{dom} R \cup \operatorname{ran} R$.

f. Prove: if R is as in the previous part and if $f : V_\beta \to \operatorname{field} R$ is an isomorphism, then the inverse g of f is given, recursively, by $g(x) = \{g(y) : y \, R \, x\}$.

g. Define a sequence of well-orders $\langle \prec_\beta : \beta < \alpha \rangle$, where $\prec_\beta$ well-orders V_β, as follows: fix a well-order $\lhd$ of $\mathcal{P}(\kappa \times \kappa)$ let R_β be the $\lhd$-first subset of $\kappa \times \kappa$ such that $(\operatorname{field} R_\beta, R_\beta)$ is isomorphic to $(V_\beta, \in)$ and show how to create $\prec_\beta$ from the well-order of R_β.

h. Define a well-order of V_α.

2. Prove the following statements

a. $\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}$.

b. If $2^{\aleph_1} = \aleph_2$ and $\aleph_\omega^{\aleph_0} > \aleph_{\omega_1}$ then $\aleph_{\omega_1}^{\aleph_1} = \aleph_\omega^{\aleph_0}$.

c. If $2^{\aleph_0} \geqslant \aleph_{\omega_1}$ then $\gimel(\aleph_\omega) = 2^{\aleph_0}$ and $\gimel(\aleph_{\omega_1}) = 2^{\aleph_1}$.

3. Prove: if β is such that $2^{\aleph_\alpha} = \aleph_{\alpha+\beta}$ for all α then $\beta < \omega$. Complete the following steps. Assume $\beta \geqslant \omega$.

a. Let α be minimal such that $\alpha + \beta > \beta$. Show that α is a limit.

b. Let $\kappa = \aleph_{\alpha+\alpha}$; show κ is singular.

c. Prove: $2^{\aleph_{\alpha+\xi}} = \aleph_{\alpha+\beta}$ whenever $\xi < \alpha$.

d. Calculate 2^κ and derive a contradiction.

Remark. It is consistent to have $2^{\aleph_\alpha} = \aleph_{\alpha+2}$ for *all* α.

Date: dinsdag 01-11-2022 at 15:16:05 (cet).