To continue from last week:
What is definite?

Fraenkel: use functions $\varphi(x)$

An object $\varphi(x)$ is formed from a variable object x and possibly from further given objects (constants) by means of a prescribed application, only a finite number of times (of course) of Axioms II – VI: example

$$\varphi(x) = \{\{\{x\}, \{\emptyset\}\}, \mathcal{P}(x) \cup \{\{\emptyset\}\}\}$$

New Axiom III

If M is given and two functions φ and ψ, in that order, then M has a subset $M_{\varphi;\psi}$ (or a subset $M_{\varphi;\psi'}$) consisting of the elements of M for which $\varphi(x) \in \psi(x)$ (or $\varphi(x) \notin \psi(x)$).

Skolem

Five basic operations of Mathematical Logic

$1_\wedge$	Conjunction,	denoted by	$\wedge$	$(\cdot)$
$1_\vee$	Disjunction,	denoted by	$\vee$	$(+)$
2	Negation,	denoted by	$\neg$	$(\bar{\ })$
$3_\forall$	Universal Quantification:		$\forall$	(Π)
$3_\exists$	Existential Quantification		$\exists$	(Σ)

"It is well known that we need only three"

Definite proposition: a finite expression constructed from elementary propositions of the form $a \in b$ or $a = b$ by means of the five operations mentioned.

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PRESENT-DAY FORMULATION: WE HAVE A LANGUAGE

1. SYMBOLS SPECIFIC FOR SET THEORY

$\boxed{\in, =}$; $\wedge, \vee, \neg, \to, \leftrightarrow$; $\forall, \exists$; (,) ;
$v_1, v_2, v_3, \ldots$ $\longrightarrow$ GENERAL FOR LOGIC

2. FORMATION RULES FOR WELL-FORMED FORMULAS

- $v_i \in v_j$, $v_i = v_j$ (FOR ALL i, j IN $\mathbb{N}$)
- IF φ AND ψ ARE WELL-FORMED THEN SO ARE $(\varphi) \wedge (\psi)$, $(\varphi) \vee (\psi)$, $\neg(\varphi)$, $(\varphi) \to (\psi)$, AND $(\varphi) \leftrightarrow (\psi)$.
- IF φ IS WELL-FORMED THEN SO ARE $(\forall v_i)(\varphi)$ AND $(\exists v_i)(\varphi)$ (FOR ALL i IN $\mathbb{N}$)

3. FREE/BOUND VARIABLES

$(\exists x)(x \in y) \wedge (\exists y)(y = z)$: $\varphi(y, z)$.

x BOUND
y FIRST FREE SECOND BOUND
z FREE

YOU CAN SUBSTITUTE FOR FREE VARIABLES BUT NOT FOR BOUND VARIABLES.

AXIOM 0: SET EXISTENCE $(\exists x)(x = x)$
(IN SOME LOGICAL BOOKS THIS IS ALWAYS TRUE)

AXIOM 1: EXTENSIONALITY

$$(\forall x)(\forall y)\big((\forall z)(z \in x \leftrightarrow z \in y) \to x = y\big)$$

NOTE $x = y \to (\forall u)(u \in x \leftrightarrow u \in y)$
IS A LOGICAL AXIOM/THEOREM (DEPENDING ON THE BOOK)

[WE FOLLOW KUNEN'S NUMBERING]

AXIOM 3: COMPREHENSION / SEPARATION SCHEME

FOR EACH FORMULA φ WITH y NOT A FREE VARIABLE THE FOLLOWING IS AN AXIOM

$(\exists y)(\forall x)(x \in y \Leftrightarrow (x \in z \wedge \varphi))$

OR BETTER: ITS UNIVERSAL CLOSURE.

EXAMPLES φ IS $x \in w$

ψ IS $\neg(x \in w)$

SO

φ GIVES $(\forall z)(\forall w)(\exists y)(\forall x)(x \in y \Leftrightarrow (x \in z \wedge x \in w))$

ψ GIVES $(\forall z)(\forall w)(\exists y)(\forall x)(x \in y \Leftrightarrow (x \in z \wedge x \notin w))$

THEOREM $(\exists y)(\forall x)(x \notin y)$

PROOF $(\exists z)(z = z)$ AXIOM 0

$(\forall z)(\exists y)(\forall x)(x \in y \Leftrightarrow (x \in z \wedge x \neq x))$ AX 3

$(\forall x)(x = x)$ LOGICAL

$(\exists y)(\forall x)(x \notin y)$ --- LOGICAL STEPS

ALSO IF $(\forall x)(x \notin y)$ AND $(\forall x)(x \notin z)$ THEN $y = z$

BY EXTENSIONALITY

SO ABBREVIATION $(\exists! y)(\forall x)(x \notin y)$

WE EXTEND THE LANGUAGE BY ADDING A CONSTANT $\emptyset$, AND ADD

$(\forall x)(x \notin \emptyset)$

AS AN AXIOM

WELL-KNOWN RESULT FROM MATHEMATICAL LOGIC: WE CANNOT PROVE MORE THAN WE ALREADY COULD.

A MODEL $\{\emptyset\}$ OR .
ARE MODELS OF AXIOMS 0, 1, AND 3
SO, WE CANNOT PROVE MUCH ON THEIR BASIS.

AXIOM 4: PAIRING
$(\forall x)(\forall y)(\exists z)(x \in z \wedge y \in z)$

THEOREM
$(\forall x)(\forall y)(\exists z)(\forall u)(u \in z \leftrightarrow (u = x \vee u = y))$

PROOF
APPLY AXIOM 4 TO GIVEN x AND y TO GET z
SUCH THAT $x \in z \wedge y \in z$
SO $(\forall u)((u = x \vee u = y) \rightarrow u \in z)$
LET φ BE $u = x \vee u = y$
APPLY AX 3: $(\exists w)(\forall u)(u \in w \leftrightarrow (u \in z \wedge (u = x \vee u = y)))$.
LOGICAL STEPS: $(\exists w)(\forall u)(u \in w \leftrightarrow (u = x \vee u = y))$
BY EXTENSIONALITY THIS w IS UNIQUE.
WE INTRODUCE NEW NOTATION $\{x, y\}$
AND $(\forall x)(\forall y)(\forall u)(u \in \{x, y\} \leftrightarrow (u = x \vee u = y))$
AS AN AXIOM
AGAIN: WE CANNOT PROVE MORE THAN
WE ALREADY COULD.

FROM $(\forall x)(\forall x)(\exists z)(x \in z \wedge x \in z)$
WE GET, BY LOGICAL STEPS, $(\forall x)(\exists z)(x \in z)$
THIS GIVES US ZERMELO OTHER ELEMENTARY
SET : $\{x\}$.

NOTE: FORMALLY $\{x\}$ AND $\{x, y\}$ ARE FUNCTION
SYMBOLS

NOTE $\{x,y\} = \{y,x\}$ (UNORDERED PAIR)

DEFINITION $\langle x,y\rangle = \{\{x\},\{x,y\}\}$ (KURATOWSKI)

EXERCISE: - $\langle x,y\rangle = \langle u,v\rangle \leftrightarrow (x=u \wedge y=v)$

AXIOM 5 : UNION

$$(\forall x)(\exists z)(\forall y)(\forall u)((u\in y \wedge y\in x) \to u\in z)$$

APPLY AXIOM 3 TO $(\exists y)(y\in x \wedge u\in y)$

TO GET w SUCH THAT

$$u\in w \leftrightarrow ((u\in z) \wedge (\exists y)(y\in x \wedge u\in y))$$

BY EXTENSIONALITY w IS UNIQUE

SO WE INTRODUCE A NEW FUNCTION

SYMBOL : $\bigcup x$ WITH AXIOM

$$(\forall x)(\forall u)(u\in \bigcup x \leftrightarrow (\exists y)(y\in x \wedge u\in y))$$

PRODUCT SETS

GIVEN TWO SETS x AND y DOES

$$x\times y = \{\langle u,v\rangle : u\in x \wedge v\in y\}$$

EXIST?

THAT IS, IS THIS PROVABLE:

$$(\forall x)(\forall y)(\exists z)(\forall w)(w\in z \leftrightarrow (\exists u)(\exists v)(u\in x \wedge v\in y \wedge w=\langle u,v\rangle))$$

NOT YET

- WE DON'T HAVE A SET TO APPLY SEPARATION TO.
- DOES UNION HELP?

FIX $u\in x$: $\{\langle u,v\rangle : v\in y\} = \bigcup\{\{\langle u,v\rangle\} : v\in y\}$

BUT IS $\{\{\langle u,v\rangle\} : v\in y\}$ A SET?

ALSO $x\times y = \bigcup\{\{\langle u,v\rangle : v\in y\} : u\in x\}$

BUT IS $\{\{\langle u,v\rangle : v\in y\} : u\in x\}$ A SET?

IF THERE IS ANY JUSTICE THEN

$\{\langle u,v\rangle : v \in y\}$

SHOULD BE A SET.

Axiom 6 Replacement (Fraenkel, Skolem)

LET φ BE A FORMULA IN WHICH y IS NOT A FREE VARIABLE

THEN THIS (ITS UNIVERSAL CLOSURE) IS AN AXIOM

$$(\forall u \in x)(\exists! w)\,\varphi(u,w) \longrightarrow (\exists y)(\forall u)(u \in x \to (\exists w)(w \in y \wedge \varphi(u,w)))$$

BACK TO $x \times y$.

- FIX $u \in x$: φ IS $w = \langle u,v\rangle$

  THEN $(\forall v \in y)(\exists! w)\,\varphi(v,w)$

  HENCE $(\exists z)(\forall v \in y)(\exists w)(w \in z \wedge w = \langle u,v\rangle)$

  SEPARATION GIVES US $\{\langle u,v\rangle : v \in y\}$
- NOTE THAT WE NOW HAVE

  $(\forall u \in x)(\exists! z)\underbrace{(\forall w)(w \in z \leftrightarrow (\exists v \in y)(w = \langle u,v\rangle))}_{\psi}$

  (BY EXTENSIONALITY)
- WE GET

  $(\exists p)(\forall u \in x)(\exists z)(z \in p \wedge \psi)$

  SEPARATION GIVES US

  $q = \{\{\langle u,v\rangle : v \in y\} : u \in x\}$
- UNION GIVES US $\bigcup q$ AND THAT IS EXACTLY $x \times y$.

Axiom 7 Infinity

First define $S(x) = x \cup \{x\}$.

$$(\exists x)(\emptyset \in x \wedge (\forall y)(y \in x \to S(y) \in x))$$

A set as in Axiom 7 is called <u>inductive</u>

As Zermelo did for his set we prove that there is a <u>smallest</u> inductive set.

Write $I(x)$ for $\emptyset \in x \wedge (\forall y)(y \in x \to S(y) \in x)$

Take X such that $I(X)$ holds

Define

$$N = \{x \in X : (\forall y)(I(y) \to x \in y)\}$$

Fact about inductive sets:

- If A is inductive then so is $\{x \in A : x \subseteq A\}$
- x is transitive means
  if $y \in x$ and $z \in y$ then $z \in x$
- If A is inductive then so is $\{x \in A : x \text{ is transitive}\}$
- If A is inductive then so is $\{x \in A : x \text{ is transitive and } x \notin x\}$
- If A is inductive then so is $\{x \in A : (\forall y \in x)(\forall z \in x)(y \in z \vee y = z \vee z \in y)\}$
- If A is inductive then so is $\{x \in A : x \text{ is transitive and } (\forall z)(z \subseteq x \wedge z \neq \emptyset \to (\exists u)(u \in z \wedge (\forall v)(v \in u \to v \notin z)))\}$

  "Every nonempty subset of x has a $\in$-minimal element"
- If A is inductive then so is $\{x \in A : x = \emptyset \vee (\exists y)(x = S(y))\}$

DEDUCE

- N IS TRANSITIVE
- IF $x, y \in N$ THEN $x \in y$ OR $x = y$ OR $y \in x$
- IF $x \in N$ THEN $x \notin x$
- IF $Z \subseteq N$ AND $Z \neq 0$
  THEN THERE IS A $u \in Z$ THAT
  IS $\in$-MINIMAL: IF $v \in u$ THEN $v \notin Z$
- IF $A \subseteq N$ AND $0 \in A$ AND $x \in A \to S(x) \in A$
  THEN $A = N$