

TO CONTINUE FROM LAST WEEK:

WHAT IS DEFINITE?

FRAENKEL: USE FUNCTIONS $\varphi(x)$

AN OBJECT $\varphi(x)$ IS FORMED FROM A VARIABLE OBJECT x AND POSSIBLY FROM FURTHER GIVEN OBJECTS (CONSTANTS) BY MEANS OF A PRESCRIBED APPLICATION, ONLY A FINITE NUMBER OF TIMES (OF COURSE) OF

AXIOMS II - VI: EXAMPLE:

$$\varphi(x) = \{\{\{x\}, \{\emptyset\}, \varphi(x) \cup \{\emptyset\}\}\}$$

NEW AXIOM III

IF Π IS GIVEN AND TWO FUNCTIONS φ AND ψ , IN THAT ORDER, THEN Π HAS A SUBSET $\Pi_{\varphi, \psi}$ (OR A SUBSET $\Pi_{\varphi, \psi}$) CONSISTING OF THE ELEMENTS OF Π FOR WHICH $\varphi(x) \in \psi(x)$ (OR $\varphi(x) \notin \psi(x)$).

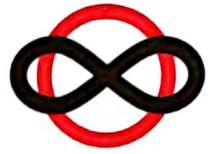
SKOLEM

FIVE BASIC OPERATIONS OF MATHEMATICAL LOGIC

- 1_∧ CONJUNCTION, DENOTED BY \wedge (•)
- 2_∨ DISJUNCTION, DENOTED BY \vee (+)
- 3_¬ NEGATION, DENOTED BY \neg (̄)
- 4_∀ UNIVERSAL QUANTIFICATION: \forall (Π)
- 5_∃ EXISTENTIAL QUANTIFICATION: \exists (Σ)

"IT IS WELL KNOWN THAT WE NEED ONLY THREE"

DEFINITE PROPOSITIONS: A FINITE EXPRESSION CONSTRUCTED FROM ELEMENTARY PROPOSITIONS OF THE FORM $a \in b$ OR $a = b$ BY MEANS OF THE FIVE OPERATIONS MENTIONED.



PRESENT-DAY FORMULATION: WE HAVE A LANGUAGE

1. SYMBOLS SPECIFIC FOR SET THEORY

$\in, =$; $\cap, \cup, \setminus, \rightarrow, \leftrightarrow$; \forall, \exists ; $(,)$;
 $\mathcal{P}, \mathcal{P}^2, \mathcal{P}^3, \dots$ → GENERAL FOR LOGIC

2. FORMATION RULES FOR WELL-FORMED FORMULAS

- $\mathcal{U}_i \in \mathcal{U}_j$, $\mathcal{U}_i = \mathcal{U}_j$ (FOR ALL i, j IN \mathbb{N})
- IF φ AND ψ ARE WELL-FORMED THEN SO ARE $(\varphi) \wedge (\psi)$, $(\varphi) \vee (\psi)$, $\neg(\varphi)$, $(\varphi) \rightarrow (\psi)$, AND $(\varphi) \leftrightarrow (\psi)$.
- IF φ IS WELL-FORMED THEN SO ARE $(\forall \mathcal{U}_i)(\varphi)$ AND $(\exists \mathcal{U}_i)(\varphi)$ (FOR ALL i IN \mathbb{N})

3. FREE/BOUND VARIABLES

$(\exists x)(x \in y) \wedge (\exists y)(y = z) ; \varphi(y, z)$
 x BOUND
 y FIRST FREE SECOND BOUND
 z FREE

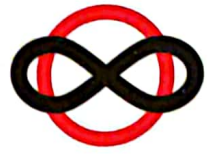
YOU CAN SUBSTITUTE FOR FREE VARIABLES
 BUT NOT FOR BOUND VARIABLES.

AXIOM 0: SET EXISTENCE $(\exists x)(x = x)$
 (IN SOME LOGICAL BOOKS THIS IS ALWAYS TRUE)

AXIOM 1: EXTENSIONALITY
 $(\forall x)(\forall y)(\forall z)(z \in x \leftrightarrow z \in y) \rightarrow x = y$

NOTE $x = y \rightarrow (\forall u)(u \in x \leftrightarrow u \in y)$

IS A LOGICAL AXIOM/THEOREM (DEPENDING ON THE BOOK)



[WE FOLLOW KUNEN'S NUMBERING]

AXIOM 3: COMPREHENSION / SEPARATION SCHEME

FOR EACH FORMULA φ WITH y NOT A FREE VARIABLE THE FOLLOWING IS AN AXIOM

$$(\exists y)(\forall x)(x \in y \leftrightarrow (x \in z \wedge \varphi))$$

OR BETTER: ITS UNIVERSAL CLOSURE.

EXAMPLES φ IS $x \in z$
 $\neg \varphi$ IS $\neg(x \in z)$

SO

φ GIVES $(\forall z)(\forall w)(\exists y)(\forall x)(x \in y \leftrightarrow (x \in z \wedge x \in w))$
 $\neg \varphi$ GIVES $(\forall z)(\forall w)(\exists y)(\forall x)(x \in y \leftrightarrow (x \in z \wedge x \notin w))$

THEOREM $(\exists y)(\forall x)(x \neq y)$

PROOF

$$(\exists z)(z = z)$$

AXIOM 0

$$(\forall z)(\exists y)(\forall x)(x \in y \leftrightarrow (x \in z \wedge x \neq x)) \quad \text{Ax 3}$$

$$(\forall x)(x = x)$$

LOGICAL

$$(\exists y)(\forall x)(x \neq y) \quad \dots \text{ LOGICAL STEPS}$$

ALSO IF $(\forall x)(x \neq y)$ AND $(\forall x)(x \neq z)$ THEN $y = z$

BY EXTENSIONALITY

SO ABBREVIATION $(\exists! y)(\forall x)(x \neq y)$

WE EXTEND THE LANGUAGE BY ADDING

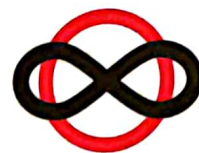
A CONSTANT \emptyset , AND ADD

$$(\forall x)(x \neq \emptyset)$$

AS AN AXIOM

WELL-KNOWN RESULT FROM MATHEMATICAL LOGIC:

WE CANNOT PROVE MORE THAN WE
 ALREADY COULD.



A MODEL $\{\emptyset\}$ OR \bullet
ARE MODELS OF AXIOMS 0, 1, AND 3

SO, WE CANNOT PROVE MUCH ON THEIR BASIS.

AXIOM 4: PAIRING

$$(\forall x)(\forall y)(\exists z)(x \in z \wedge y \in z)$$

THEOREM

$$(\forall x)(\forall y)(\exists z)(\forall u)(u \in z \leftrightarrow (u = x \vee u = y))$$

PROOF

APPLY AXIOM 4 TO GIVEN x AND y TO GET z
SUCH THAT $x \in z \wedge y \in z$

$$\text{SO } (\forall u)(u = x \vee u = y \rightarrow u \in z)$$

LET φ BE $u = x \vee u = y$

APPLY AX3: $(\exists w)(\forall u)(u \in w \leftrightarrow (u \in z \wedge (u = x \vee u = y)))$.

LOGICAL STEPS: $(\exists w)(\forall u)(u \in w \leftrightarrow (u = x \vee u = y))$

BY EXTENSIONALITY THIS w IS UNIQUE.

WE INTRODUCE NEW NOTATION $\{x, y\}$

$$\text{AND } (\forall x)(\forall y)(\forall u)(u \in \{x, y\} \leftrightarrow (u = x \vee u = y))$$

AS AN AXIOM

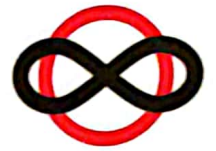
AGAIN: WE CANNOT PROVE MORE THAN
WE ALREADY COULD.

$$\text{FROM } (\forall x)(\forall y)(\exists z)(x \in z \wedge y \in z)$$

WE GET, BY LOGICAL STEPS, $(\forall x)(\exists z)(x \in z)$

THIS GIVES US ZERMELO OTHER ELEMENTARY
SET: $\{x\}$.

NOTE: FORMALLY $\{x\}$ AND $\{x, y\}$ ARE FUNCTION
SYMBOLS



NOTE $\{x, y\} = \{y, x\}$ (UNORDERED PAIR)

DEFINITION $\langle x, y \rangle = \{\{x\}, \{x, y\}\}$ (KURATOWSKI)

EXERCISE: $\langle x, y \rangle = \langle u, v \rangle \Leftrightarrow (x = u \wedge y = v)$

AXIOM 5: UNION

$(\forall x)(\exists z)(\forall y)(\forall u)((u \in y \wedge y \in x) \rightarrow u \in z)$

APPLY AXIOM 3 TO $(\exists y)(y \in x \wedge u \in y)$

TO GET w SUCH THAT

$u \in w \Leftrightarrow (u \in z) \vee (\exists y)(y \in x \wedge u \in y)$

By EXTENSIONALITY w IS UNIQUE

SO WE INTRODUCE A NEW FUNCTION

SYMBOL: \cup WITH AXIOM
 $(\forall x)(\forall u)(u \in x \Leftrightarrow (\exists y)(y \in x \wedge u \in y))$

PRODUCT SETS

GIVEN TWO SETS X AND Y DOES

$X \times Y = \{\langle u, v \rangle : u \in X \wedge v \in Y\}$

EXIST?

THAT IS, IS THIS PROVABLE?

$(\forall x)(\forall y)(\exists z)(\forall w)((w \in z \Leftrightarrow (\exists u)(\exists v)(u \in x \wedge v \in y \wedge w = \langle u, v \rangle))$

NOT YET

- WE DON'T HAVE A SET TO APPLY SEPARATION TO.

- DOES UNION HELP?

FIX $u \in X$ $\{\langle u, v \rangle : v \in Y\} = \cup \{\{\langle u, v \rangle\} : v \in Y\}$

BUT IS $\{\{\langle u, v \rangle\} : v \in Y\}$ A SET?

ALSO $X \times Y = \cup \{\{\langle u, v \rangle : v \in Y\} : u \in X\}$

BUT IS $\{\{\langle u, v \rangle : v \in Y\} : u \in X\}$ A SET?



IF THERE IS ANY JUSTICE THEN
 $\{ \langle u, v \rangle : v \in Y \}$
 SHOULD BE A SET.

AXIOM 6 REPLACEMENT (FRÄENKEL, SKOLEM)

LET ϕ BE A FORMULA IN WHICH y
 IS NOT A FREE VARIABLE

THEN THIS (ITS UNIVERSAL CLOSURE)
 IS AN AXIOM

$$(\forall u \in X)(\exists w)(\phi(u, w) \rightarrow (\exists y)(\forall u)(u \in X \rightarrow (\exists w)(w \in y \wedge \phi(u, w))))$$

BACK TO $\mathcal{O} \times Y$.

- FIX $u \in X$: ϕ IS $w = \langle u, v \rangle$
 THEN $(\forall v \in Y)(\exists w)(\phi(u, w))$
 HENCE $(\exists z)(\forall v \in Y)(\exists w)(w \in z \wedge w = \langle u, v \rangle)$

SEPARATION GIVES US $\{ \langle u, v \rangle : v \in Y \}$

- NOTE THAT WE NOW HAVE

$$(\forall u \in X)(\exists z)(\forall w)(w \in z \rightarrow (\exists v \in Y)(w = \langle u, v \rangle))$$

(BY EXTENSIONALITY) ψ

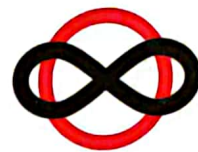
- WE GET

$$(\exists p)(\forall u \in X)(\exists z)(z \in p \wedge \psi)$$

SEPARATION GIVES US

$$q = \{ \{ \langle u, v \rangle : v \in Y \} : u \in X \}$$

- UNION GIVES US $\cup q$ AND THAT
 IS EXACTLY $\mathcal{O} \times Y$.



Axiom 7 Infinity

FIRST DEFINE $S(x) = x \cup \{x\}$.

$$(\exists x) (\emptyset \in x \wedge (\forall y)(y \in x \rightarrow S(y) \in x))$$

A SET AS IN AXIOM 7 IS CALLED INDUCTIVE

AS ZERMELO DID FOR HIS SET WE PROVE

THAT THERE IS A SMALLEST INDUCTIVE SET.

WRITE $I(x)$ FOR $\emptyset \in x \wedge (\forall y)(y \in x \rightarrow S(y) \in x)$

TAKE X SUCH THAT $I(X)$ HOLDS

DEFINE

$$N = \{x \in X : (\forall y)(I(y) \rightarrow x \in y)\}$$

FACT ABOUT INDUCTIVE SETS:

- IF A IS INDUCTIVE THEN SO IS

$$\{x \in A : x \in A\}$$

- x IS TRANSITIVE MEANS

IF $y \in x$ AND $z \in y$ THEN $z \in x$

- IF A IS INDUCTIVE THEN SO IS

$$\{x \in A : x \text{ IS TRANSITIVE}\}$$

- IF A IS INDUCTIVE THEN SO IS

$$\{x \in A : x \text{ IS TRANSITIVE AND } x \notin x\}$$

- IF A IS INDUCTIVE THEN SO IS

$$\{x \in A : (\forall y \in x)(\forall z \in x)(y \in z \vee y = z \vee z \in y)\}$$

- IF A IS INDUCTIVE THEN SO IS

$$\{x \in A : x \text{ IS TRANSITIVE AND}$$

$$(\forall z)(z \in x \wedge z \neq \emptyset \rightarrow (\exists u)(u \in z \wedge$$

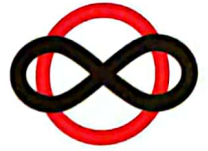
$$(\forall v)(v \in u \rightarrow v \in z))\}$$

"EVERY NONEMPTY SUBSET OF X HAS

A \in -MINIMAL ELEMENT

- IF A IS INDUCTIVE THEN SO IS

$$\{x \in A : x = \emptyset \vee (\exists y)(x = S(y))\}$$



DEDUCE

- \mathbb{N} IS TRANSITIVE

- IF $x, y \in \mathbb{N}$ THEN $x \leq y$ OR $x = y$ OR $y < x$

- IF $x \in \mathbb{N}$ THEN $x \neq \emptyset$

- IF $z \in \mathbb{N}$ AND $z \neq \emptyset$

THEN THERE IS A UCB THAT

IS ϵ -MINIMAL: IF $z \in u$ THEN $z \notin z$

- IF $A \subseteq \mathbb{N}$ AND $0 \in A$ AND $x \in A \rightarrow S(x) \in A$

THEN $A = \mathbb{N}$