Theorem [$\langle \omega, S\rangle$ satisfies Peano's axioms]

(1) $\emptyset \in \omega$

(2) $(\forall n\in\omega)(S(n)\in\omega)$ and $(\forall n\in\omega)(n\neq\emptyset \to (\exists m)(n=S(m)))$

(3) $(\forall m,n\in\omega)(m\neq n \to S(m)\neq S(n))$

(4) $(\forall X\subseteq\omega)[(\emptyset\in X \wedge (\forall n\in X)(S(n)\in X)) \to X=\omega]$

Arithmetic

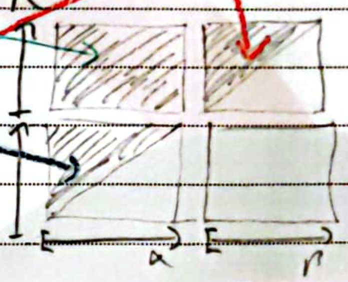
$\alpha+\beta = \mathrm{type}(\alpha\times\{0\} \cup \beta\times\{1\}, R)$

$R = [(\alpha\times\{0\})\times(\beta\times\{1\})] \cup$

$\{\langle\langle\xi,0\rangle,\langle\eta,0\rangle\rangle : \xi<\eta<\alpha\} \cup$

$\{\langle\langle\xi,1\rangle,\langle\eta,1\rangle\rangle : \xi<\eta<\beta\}$

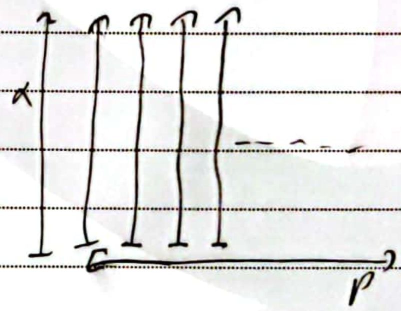
"α before β both with their own order"



$\alpha\cdot\beta = \mathrm{type}(\beta\times\alpha, R)$ R lexicographic order

$\langle\xi,\eta\rangle R \langle\gamma,\delta\rangle$ iff $(\xi<\gamma) \vee (\xi=\gamma \wedge \eta<\delta)$

"β copies of α
the copies according to β
each copy normally"



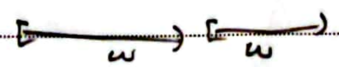
Note associative but not commutative

[Cantor: "do not expect new addition to have the same properties as the old one"]

Standard examples

$1+\omega = \omega$ $\omega+1 > \omega$

$2\cdot\omega = \omega$ $\omega\cdot 2 = \omega+\omega > \omega$



PROPERTIES

- $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$
- $\alpha + 0 = \alpha$
- $\alpha + 1 = S(\alpha)$
- $\alpha + S(\beta) = S(\alpha+\beta)$
- β LIMIT: $\alpha+\beta = \sup\{\alpha+\xi : \xi<\beta\}$

MAKES RECURSIVE COMPUTATION POSSIBLE

- $(\alpha\cdot\beta)\cdot\gamma = \alpha\cdot(\beta\cdot\gamma)$
- $\alpha\cdot 0 = 0$
- $\alpha\cdot 1 = \alpha$
- $\alpha\cdot S(\beta) = \alpha\cdot\beta+\alpha$
- β LIMIT: $\alpha\cdot\beta = \sup\{\alpha\cdot\xi : \xi<\beta\}$
- $\alpha\cdot(\beta+\gamma) = \alpha\cdot\beta+\alpha\cdot\gamma$.

MAKES RECURSIVE COMPUTATION POSSIBLE

FOR $n\in\omega$ WE LET A^n DENOTE THE SET OF FUNCTIONS FROM n INTO A

AND $A^{<\omega} = \bigcup\{A^n : n\in\omega\}$

EXISTENCE 1 WITH POWER SET

$A^n \subseteq \mathcal{P}(n\times A)$; USE SEPARATION

$A^{<\omega} \subseteq \mathcal{P}(\omega\times A)$; USE SEPARATION

EXISTENCE 2 WITH REPLACEMENT

WRITE $\varphi(n,y)$ BE

$(\forall s)(s\in y \leftrightarrow (s : n \to A))$ ← WRITTEN OUT IN FULL

- $\varphi(0,\emptyset)$ HOLDS SO $(\exists y)(\varphi(0,y))$ HOLDS
- IF $\varphi(n,y)$ HOLDS THEN BY REPLACEMENT

$F_s = \{ s\cup\{(n,a)\} : a\in A\}$

EXISTS FOR ALL $s\in y$ AS DOES $\{F_s : s\in y\}$

AND $z = \bigcup\{F_s : s\in y\}$

NOW PROVE $\varphi(n+1, z)$

SO $(\exists y)(\varphi(n,y)) \to (\exists y)(\varphi(n+1,y))$

SO BY INDUCTION

$$(\forall m \in \omega)(\exists y)\varphi(m,y)$$

AND EXTENSIONALITY: $(\forall m \in \omega)(\exists! y)\varphi(m,y)$

REPLACEMENT: $\{y : (\exists m \in \omega)\varphi(m,y)\}$ EXISTS

AND IT IS $\{A^m : m \in \omega\}$

NOTATION $s = \langle x_i : i \in m \rangle$

EXPRESSES s IS THE FUNCTION

WITH DOM $s = m$ AND $s(i) = x_i$ $i \in m$

CAVEAT $\langle x, y \rangle$ CAN MEAN

- $\{\{x\}, \{x, y\}\}$ AND
- $\{\langle 0, x\rangle, \langle 1, y\rangle\}$

CIRCUMSTANCES DICTATE WHAT WE MEAN.

INTERMEZZO

WE DEFINED $0 = \emptyset$, $1 = S(0)$, $2 = S(1)$, $3 = S(2)$, ...

BUT $0, 1, 2, 3, \ldots$ ALSO HAVE A MEANING ON THE META-LEVEL

QUINE-CORNER NOTATION $\ulcorner x \urcorner$ IS THE ANALOGON OF x IN THE THEORY

$\ulcorner 0 \urcorner = \emptyset$ AND $(\forall x)(x \notin \ulcorner 0 \urcorner)$ IS THE DEFINING AXIOM

OR $(\forall y)(y = \ulcorner 0 \urcorner \leftrightarrow (\forall x)(x \notin y))$

$\ulcorner 1 \urcorner = S(\ulcorner 0 \urcorner)$ OR $(\forall y)(y = \ulcorner 1 \urcorner \leftrightarrow y = S(\ulcorner 0 \urcorner))$

$\ulcorner 2 \urcorner = S(\ulcorner 1 \urcorner)$ OR $(\forall y)(y = \ulcorner 2 \urcorner \leftrightarrow y = S(\ulcorner 1 \urcorner))$

ETC

WRITE DOWN A FORMULA φ_n ONE FOR EACH OF OUR (META-) NATURAL NUMBERS

$\varphi_n(x)$ IS $(x$ IS A NATURAL NUMBER$) \wedge (\ulcorner n \urcorner \in x)$

OR JUST $(x \in \omega) \wedge (\ulcorner n \urcorner \in x)$

INTRODUCE A NEW CONSTANT: c.

The (meta-) set $\{\varphi_n(c) : n \in \omega\}$ is consistent:
Take finitely many, WLOG, $\varphi_0(c), \ldots, \varphi_n(c)$
then $\varphi_0(c) \wedge \varphi_1(c) \wedge \cdots \wedge \varphi_n(c)$ holds in ω
when we interpret c as $\ulcorner n+1 \urcorner$

Completeness: there is a model
that satisfies all $\varphi_n(c)$
So we have a set c there
such that
- $c \in \omega$ so $(\forall \beta \in c)(\exists \gamma)(\beta = S(\gamma))$
- $\ulcorner n \urcorner \in c$ for all (meta-) n.

So proving something for all $\ulcorner n \urcorner$ does
not suffice to prove it "for all $n \in \omega$"

End intermezzo.

General sequences:
$S = \langle S_\alpha : \alpha < \beta \rangle$
Functions with domain an ordinal
$t = \langle t_\gamma : \gamma < \delta \rangle$
$S^\frown t = \langle u_\eta : \eta < \beta + \delta \rangle$ $\begin{cases} u_\eta = S_\eta & \eta < \beta \\ u_{\beta+\eta} = t_\eta & \eta < \delta \end{cases}$

Induction and Recursion

Induction on On:
If C is a class and $C \subseteq On$ and $C \neq \emptyset$
then C has a least element
Proof: Take $\alpha \in C$ if $\alpha \cap C = \emptyset$ then $\alpha = \min C$
if $\alpha \cap C \neq \emptyset$ then $\min(\alpha \cap C) = \min C$.

OFFICIALLY C IS GIVEN BY A FORMULA

$$C(x, z_1, \dots, z_m)$$

"$C \subseteq ON$" : $(\forall x)(C(x, z_1, \dots, z_m) \to \mathrm{ORD}(x))$

"$C \neq \emptyset$" : $(\exists x)\, C(x, z_1, \dots, z_m)$

C HAS A LEAST ELEMENT:

$$(\exists x)(C(x, z_1, \dots, z_m) \wedge (\forall y)(C(y, z_1, \dots, z_m) \to x \leq y))$$

THE THEOREM SAYS

$$(\forall z_1)(\forall z_2)\dots(\forall z_m)[\, C \subseteq ON \wedge C \neq \emptyset \to C \text{ HAS A LEAST ELEMENT}]$$

OR BETTER

$$(\forall z_1)\dots(\forall z_m)\Big[\big[(\forall x)(C(x, z_1, \dots, z_m) \to \mathrm{ORD}(x)) \wedge (\exists x) C(x, z_1, \dots, z_m)\big] \to$$
$$\to \big((\exists x)(C(x, z_1, \dots, z_m) \wedge (\forall y)(C(y, z_1, \dots, z_m) \to x \leq y))\big)\Big]$$

RECURSION ON ON.

IF $F : V \to V$ IS A FUNCTION THEN
THERE IS A UNIQUE $G : ON \to V$
SUCH THAT $(\forall \alpha)(G(\alpha) = F(G \restriction \alpha))$.

UNIQUENESS: $(\forall \alpha)(G_1(\alpha) = G_2(\alpha))$ FOLLOWS BY INDUCTION
[GIVEN TWO CANDIDATES G_1 AND G_2]

EXISTENCE: USE APPROXIMATIONS

A δ-APPROXIMATION IS A FUNCTION g WITH DOMAIN δ AND SUCH THAT

$$(\forall \alpha \in \delta)(g(\alpha) = F(g \restriction \alpha))$$

- FOR EVERY δ THERE IS AT MOST ONE δ-APPROXIMATION
- IF g IS A δ-APPR AND h AN ε-APPR. THEN $g \restriction (\delta \cap \varepsilon) = h \restriction (\delta \cap \varepsilon)$
- FOR EVERY δ THERE IS A δ-APPROXIMATION
  - $\delta = 0$ $\quad g = \emptyset$
  - $\delta = \gamma + 1$ GIVEN h WITH DOMAIN γ LET $g = h \cup \{\langle \gamma, F(h) \rangle\}$.

δ A LIMIT BY REPLACEMENT WE HAVE
$\langle g_\gamma : \gamma < \delta \rangle$ WITH g_γ A γ-APPR.
LET $g = \bigcup_{\gamma<\delta} g_\gamma$

- FINALLY
$G(\alpha) = x$ IFF THERE IS AN $\alpha+1$-APPR g
SUCH THAT $g(\alpha) = x$.

EXERCISE FORMULATE THIS IN TERMS OF FORMULAS

LOOK BACK TO ADDITION AND MULTIPLICATION OF ORDINALS AND DEFINE THEM RECURSIVELY.

ADDITION:
$$F(\alpha, x) = \begin{cases} \alpha & \text{IF } x = \emptyset \\ S(x(\gamma)) & \text{IF } x \text{ IS A FUNCTION WITH DOMAIN } \gamma+1 \text{ AND } \gamma \text{ IS AN ORD} \\ \bigcup\{x_\eta : \eta < \beta\} & \text{IF } x \text{ IS A FUNCTION WITH DOMAIN } \beta \text{ AND } \beta \text{ IS A LIMIT.} \\ \emptyset & \text{OTHERWISE} \end{cases}$$

THIS GIVES $G(\alpha, \beta)$ SUCH THAT
$$G(\alpha, \beta) = F(\alpha, G\restriction\beta) \quad \text{FOR ALL } \beta$$
$$G(\alpha, 0) = F(\alpha, 0) = \alpha$$
$$G(\alpha, \gamma+1) = F(\alpha, G\restriction(\gamma+1)) = S(G(\gamma)) = G(\gamma)+1$$
$$G(\alpha, \beta) = F(\alpha, G\restriction\beta) = \bigcup\{G(\eta) : \eta < \beta\}$$
BY INDUCTION $G(\alpha, \beta) = \alpha + \beta$.

EXERCISE DO THIS FOR MULTIPLICATION

EXPONENTIATION OF ORDINALS

- $\alpha^0 = 1$
- $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$
- $\alpha^\beta = \sup\{\alpha^\gamma : \gamma < \beta\}$ β LIMIT.

EXERCISE: SHOW THAT α^β IS THE ORDER TYPE OF $F(\beta,\alpha) = \{f \in {}^\beta\alpha : \{\xi : f(\xi) \neq 0\}$ IS FINITE$\}$.

? x IS FINITE: IF THERE ARE $n \in \omega$ AND A BIJECTION $f : n \to x$.

NORMAL FUNCTIONS:

$F : ON \to ON$ IS NORMAL

IF IT IS - ORDER-PRESERVING: $\alpha < \beta \to F(\alpha) < F(\beta)$

- CONTINUOUS: $F(\beta) = \sup\{F(\gamma) : \gamma < \beta\}$ IF β IS A LIMIT

SO $\beta \mapsto \alpha + \beta$; $\beta \mapsto \alpha \cdot \beta$; $\beta \mapsto \alpha^\beta$ ARE ALL NORMAL

EXERCISE IF F IS NORMAL THEN

- $\beta \le F(\beta)$ FOR ALL β
- FOR EVERY β THERE IS A $\gamma \ge \beta$ SUCH THAT $F(\gamma) = \gamma$.

EXERCISE: FOR EVERY α THERE IS A β SUCH THAT $\alpha + \gamma = \gamma$ FOR ALL $\gamma \ge \beta$.

EXERCISE PROVE
- $\alpha \le \beta \to (\exists! \delta)(\alpha + \delta = \beta)$
- $\alpha \le \beta \wedge \alpha \neq 0 \to \exists! \delta \exists! \xi (\xi < \alpha \wedge \beta = \alpha \cdot \delta + \xi)$

EXERCISE PROVE THAT THE FOLLOWING ARE EQUIVALENT FOR AN ORDINAL α: (1) $(\forall \beta < \alpha)(\beta + \alpha = \alpha)$ (2) $(\forall \beta, \gamma < \alpha)(\beta + \gamma < \alpha)$ (3) $(\exists \delta)(\alpha = \omega^\delta)$